MECHANICS APPLIED TO ENGINEERING

MECHANICS APPLIED TO ENGINEERING

BY

JOHN GOODMAN

WH. SCH., M.I.C.E., M.I.M.E.

PROFESSOR OF ENGINEERING IN THE UNIVERSITY OF LEEDS

With 741 Illustrations and Numerous Examples

NEW IMPRESSION

LONGMANS, GREEN AND CO.
39 PATERNOSTER ROW, LONDON
FOURTH AVENUE & 30TH STREET, NEW YORK
BOMBAY, CALCUTTA, AND MADRAS
1921

All rights reserved

PREFACE

THIS book has been written especially for Engineers and Students who already possess a fair knowledge of Elementary Mathematics and Theoretical Mechanics; it is intended to assist them to apply their knowledge to practical engineering problems.

Considerable pains have been taken to make each point clear without being unduly diffuse. However, while always aiming at conciseness, the short-cut methods in common use have often—and intentionally—been avoided, because they appeal less forcibly to the student, and do not bring home to him the principles involved so well as do the methods here adopted.

Some of the critics of the first edition expressed the opinion that Chapters I., II., III. might have been omitted or else considerably curtailed; others, however, commended the innovation of introducing Mensuration and Moment work into a book on Applied Mechanics, and this opinion has been endorsed by readers both in this country and in the United States. In addition to the value of the tables in these chapters for reference purposes, the worked-out results afford the student an opportunity of reviewing the methods adopted.

The Calculus has been introduced but sparingly, and then only in its most elementary form. That its application does not demand high mathematical skill is evident from the working out of the examples in the Mensuration and Moment chapters. For the benefit of the beginner, a very elementary sketch of the subject has been given in the Appendix; it is hoped that he will follow up this introduction by studying such works as those by Barker, Perry, Smith, Wansbrough, or others.

For the assistance of the occasional reader, all the symbols employed in the book have been separately indexed, with the exception of certain ones which only refer to the illustrations in their respective accompanying paragraphs.

In this (fourth) edition, some chapters have been considerably enlarged, viz. Mechanics; Dynamics of Machinery; Friction; Stress, Strain, and Elasticity; Hydraulic Motors and Machines; and Pumps. Several pages have also been added to many of the other chapters.

A most gratifying feature in connection with the publication of this book has been the number of complimentary letters received from all parts of the world, expressive of the help it has been to the writers; this opportunity is taken of thanking all correspondents both for their kind words and also for their trouble in pointing out errors and misprints. It is believed that the book is now fairly free from such imperfections, but the author will always be glad to have any pointed out that have escaped his notice, also to receive further suggestions. While remarking that the sale of the book has been very gratifying, he would particularly express his pleasure at its reception in the United States, where its success has been a matter of agreeable surprise.

The author would again express his indebtedness to all who kindly rendered him assistance with the earlier editions, notably Professor Hele-Shaw, F.R.S., Mr. A. H. Barker, B.Sc., Mr. Andrew Forbes, Mr. E. R. Verity, and Mr. J. W. Jukes. In preparing this edition, the author wishes to thank his old friend Mr. H. Rolfe for many suggestions and much help; also his assistant, Mr. R. H. Duncan, for the great care and pains he has taken in reading the proofs; and, lastly, the numerous correspondents (most of them personally unknown to him) who have sent in useful suggestions, but especially would he thank Professor Oliver B. Zimmerman, M.E., of the University of Wisconsin, for the "gearing" conception employed in the treatment of certain velocity problems in the chapter on "Mechanisms."

JOHN GOODMAN.

THE UNIVERSITY OF LEEDS,
August, 1904.

PREFACE TO EIGHTH EDITION

New Chapters on "Vibration" and "Gyroscopic Action" have been added to this Edition. Over a hundred new figures and many new paragraphs have been inserted. The sections dealing with the following subjects have been added or much enlarged—Cams, Toothed Gearing, Flywheels, Governors, Ball Bearings, Roller Bearings, Lubrication. Strength of Flat Plates, Guest's Law, Effect of Longitudinal Forces on Pipes under pressure. Reinforced Concrete Beams, Deflection of Beams due to Shear, Deflection of Tapered Beams, Whirling of Shafts, Hooks, Struts, Repeated Loading. Flow of Water down Steep Slopes, Flooding of Culverts, Time of Emptying Irregular Shaped Vessels, Continuous and Sinuous flow in Pipes, Water Hammer in Pipes, Cavitation in Centrifugal Pumps.

The mode of treatment continues on the same lines as before; simple, straightforward, easily remembered methods have been used as far as possible. A more elegant treatment might have been adopted in many instances, but unfortunately such a treatment often requires more mathematical knowledge than many readers possess, hence it is a "closed book" to the majority of engineers and draughtsmen, and even to many who have had a good mathematical training in their student days.

There are comparatively few Engineering problems in which the data are known to within, say, 5 per cent., hence it is a sheer waste of time for the Engineer in practice to use long, complex methods when simple, close approximations can be used in a fraction of the time. For higher branches of research work exact, rigid methods of treatment may be, and usually are, essential, but the number of Engineers who require to make use of such methods is very small.

Much of the work involved in writing and revising this

Edition has been performed under very great difficulties, in odd moments snatched from a very strenuous life, and but for the kind and highly valued assistance of Mr. R. H. Duncan in correcting proofs and indexing, this Edition could not have been completed in time for this Autumn's publication.

JOHN GOODMAN.

THE UNIVERSITY OF LEEDS,
August, 1914.

CONTENTS

CHAP.		PAGE
I.	INTRODUCTORY	1
II.	MENSURATION	20
III.	MOMENTS	50
IV.	RESOLUTION OF FORCES	106
V.	MECHANISMS	119
VI.	DYNAMICS OF THE STEAM-ENGINE	179
VII.	VIBRATION	259
VIII.	GYROSCOPIC ACTION	277
IX.	FRICTION	284
X.	STRESS, STRAIN, AND ELASTICITY	360
XI.	BEAMS	429
XII.	BENDING MOMENTS AND SHEAR FORCES	474
XIII.	DEFLECTION OF BEAMS	506
XIV.	COMBINED BENDING AND DIRECT STRESSES	538
XV.	STRUTS	550
XVI.	TORSION. GENERAL THEORY	571
XVII.	STRUCTURES	593
XVIII.	HYDRAULICS	637
XIX.	HYDRAULIC MOTORS AND MACHINES	691
XX.	PUMPS	738
	APPENDIX	781
	EXAMPLES	794
	INDEX	846

MECHANICS APPLIED TO ENGINEERING

CHAPTER I.

INTRODUCTORY.

THE province of science is to ascertain truth from sources far and wide, to classify the observations made, and finally to embody the whole in some brief statement or formula. If some branches of truth have been left untouched or unclassified, the formula will only represent a part of the truth; such is the cause of discrepancies between theory and practice.

A scientific treatment of a subject is only possible when our statements with regard to the facts and observations are made in definite terms; hence, in an attempt to treat such a subject as Applied Mechanics from a scientific standpoint, we must at the outset have some means of making definite statements as to *quantity*. This we shall do by simply stating how many arbitrarily chosen units are required to make up the quantity in question.

Units.

Mass (M).—Unit, one pound.

1 pound (lb.)	= 0·454 kilogramme.
1 kilogramme	= 2·2046 lbs.
1 hundredweight (cwt.)	= 50·8 kilos.
1 ton	= 1016 ,, (tonneau or Millier).
1 tonneau or Millier	= 0·984 ton.

Space (*s*).—Unit, one foot.

1 foot	= 0·305 metre.	1 mile	= 1609·3 metres.
1 metre	= 3·28 feet.	1 kilometre	= 1093·63 yards.
1 inch	= 25·4 millimetres.		= 0·621 mile.
1 millimetre	= 0·0394 inch.	1 sq. foot	= 0·0929 sq. metre.
1 yard	= 0·914 metre.	1 sq. metre	= 10·764 sq. feet.
1 metre	= 1·094 yards.	1 sq. inch	= 6·451 sq. cms.

1 sq. mm.	= 0·00155 sq. inch.
1 sq. cm.	= 0·155 sq. inch.
	= 0·001076 sq. feet.
1 sq. metre	= 10·764 sq. feet.
	= 1·196 sq. yards.
1 atmosphere	= 760 mm. of mercury.
	= 29·92 inches of mercury.
	= 33·90 feet of water.
	= 14·7 lbs. per. sq. inch.
	= 1·033 kg. per sq. cm.
1 lb. per sq. inch	= 0·0703 kg. per sq. cm.
	= 2·307 feet of water.
	= 2·036 inches of mercury.
	= 68970 dynes per sq. cm.
1 lb. per sq. foot	= 479 dynes per sq. cm.
1 kilo. per sq. cm.	= 14·223 lbs. per sq. inch.
1 cubic inch	= 16·387 c. cms.
1 cubic foot	= 0·0283 cubic metre.
1 cubic yard	= 0·7646 c. metre.
1 c. cm.	= 0·06103 c. inch.
1 c. metre	= 35·31 c. feet.

(See also pp. 4, 9, 10, 11, 19.)

Dimensions.—The relation which exists between any given complex unit and the fundamental units is termed the dimensions of the unit. As an example, see p. 20, Chapter II.

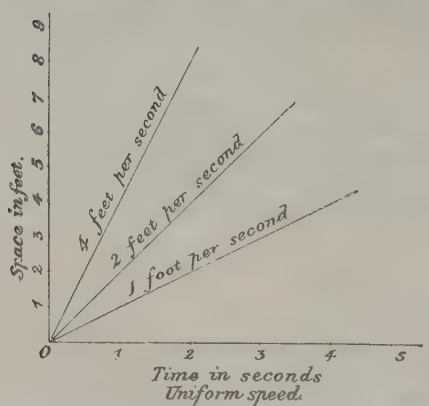


FIG. 1.

Speed.—When a body changes its position relatively to surrounding objects, it is said to be in motion. The rate at which a body changes its position when moving in a straight line is termed the speed of the body.

Uniform Speed.—A body is said to have uniform speed when it traverses equal spaces in equal intervals of time. The body is said to have unit speed when it traverses unit space in unit time.

$$\text{Speed (in feet per second)} = \frac{\text{space traversed (feet)}}{\text{time (seconds)}} = \frac{s}{t}$$

Varying Speed.—When a body does not traverse equal spaces in equal intervals of time, it is said to have a varying speed. The speed at any instant is the space traversed in an exceedingly short interval of time divided by that interval; the shorter the interval taken, the more nearly will the true speed be arrived at.

In Fig. 1 we have a diagram representing the distance travelled by a body moving with uniform speed, and in Fig. 2, varying speed. The speed at any instant, *a*, can be found by drawing a tangent to the curve as shown. From the slope of this tangent we see that, if the speed had been

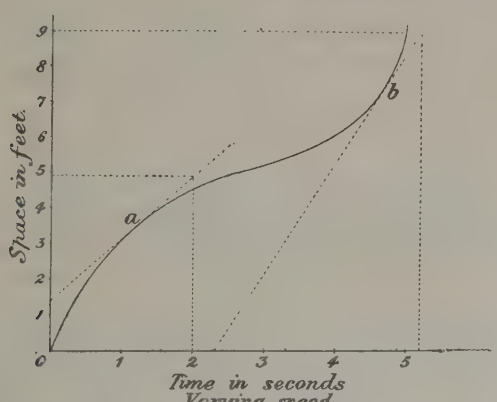


FIG. 2.

uniform, a space of $4{\cdot}9 - 1{\cdot}4 = 3{\cdot}5$ ft. would have been traversed in 2 secs., hence the speed at *a* is $\frac{3{\cdot}5}{2} = 1{\cdot}75$ ft. per second. Similarly, at *b* we find that 9 ft. would have been traversed in $5{\cdot}2 - 2{\cdot}3 = 2{\cdot}9$ secs., or the speed at *b* is $\frac{9}{2{\cdot}9} = 3{\cdot}1$ ft. per second. The same result will be obtained by taking any point on the tangent. For a fuller discussion of variable quantities, the reader is referred to either Perry's or Barker's Calculus.

Velocity (*v*).—The velocity of a body is the magnitude of its speed in any given direction; thus the velocity of a body may be changed by altering the *speed* with which it is moving, or by altering the *direction* in which it is moving. It does not

follow that if the speed of a body be uniform the velocity will be also. The idea of velocity embodies *direction* of motion, that of speed does not.

The speed of a point on a uniformly revolving wheel is constant, but the velocity is changing at every instant. Velocity and speed, however, have the same dimensions. The unit of velocity is usually taken as 1 foot per second.

$$\left.\begin{matrix}\text{Velocity in feet} \\ \text{per second}\end{matrix}\right\} = \frac{\text{space (feet) traversed in a given direction}}{\text{time (seconds)}}$$

$$v = \frac{s}{t}, \text{ or } s = vt$$

1 ft. per second = 0·305 metre per second
" " = 0·682 mile per hour
" " = 1·1 kilometre per hour
1 metre per second = 3·28 ft. per second
1 cm. per second = 0·0328 ft. per second
= 0·0224 miles per hour
1 mile per hour = 1·467 ft. per second
= 0·447 metre per second
1 kilometre " = 0·912 ft. "
= 0·278 metre "

Angular Velocity (ω), *or Velocity of Spin.*—Suppose a body to be spinning about an axis. The rate at which an angle is described by any line perpendicular to the axis is termed the angular velocity of the line or body, or the velocity of spin; the direction of spin must also be specified. When a body spins round in the direction of the hands of a watch, it is termed a + or positive spin; and in the reverse direction, a − or negative spin.

As in the case of linear velocity, angular velocity may be uniform or varying.

The unit of angular measure is a "radian;" that is, an angle subtending an arc equal in length to the radius. The length of a circular arc subtending an angle $\theta°$ is $2\pi r \times \frac{\theta°}{360°}$, where π is the ratio of the circumference to the diameter ($2r$) of a circle, and θ is the angle subtended (see p. 22).

Then, when the arc is equal to the radius, we have—

$$\frac{2\pi r\theta}{360} = r \qquad \theta = \frac{360}{2\pi} = 57{\cdot}296°$$

Thus, if a body be spinning in such a manner that a radius describes 100 degrees per second, its angular velocity is—

$$\omega = \frac{100}{57{\cdot}3} = 1{\cdot}75 \text{ radians per second}$$

It is frequently convenient to convert angular into linear velocities, and the converse. When one radian is described per second, the extremity of the radius vector describes every second a space equal to the radius, hence the space described in one second is $\omega r = v$, or $\omega = \frac{v}{r}$.

$$\text{Angular velocity in radians per sec.} = \frac{\text{linear velocity (ft. per sec.)}}{\text{radius (ft.)}}$$

The radius is a space quantity, hence—

$$\omega = \frac{s}{ts} = \frac{1}{t}$$

Thus an angular velocity is not affected by the unit of space adopted, and only depends on the time unit, but the time unit is one second in all systems of measurement, hence all angular measurements are the same for all systems of units—an important point in favour of using angular measure.

Acceleration (f_a) is the rate at which the velocity of a body increases in unit time—that is, if we take feet and seconds units, the acceleration is the number of feet per second that the velocity increases in one second; thus, unit acceleration is an increase of velocity of one foot per second per second. It should be noted that acceleration is the rate of change of *velocity*, and not merely change of speed. The speed of a body in certain cases does not change, yet there is an acceleration due to the change of direction (see p. 18).

As in the case of speed and velocity, acceleration may be either uniform or varying.

$$\left.\begin{array}{l}\text{Uniform acceleration in feet per sec. per sec.}\end{array}\right\} = \frac{\text{increase of velocity in ft. per sec. in a given time}}{\text{time in seconds}}$$

$$f_a = \frac{v_2 - v_1}{t} = \frac{v}{t}$$

$$\text{hence } v = f_a t, \text{ or } v_2 - v_1 = f_a t \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

where v_2 is the velocity at the end of the interval of time, and v_1 at the beginning, and v is the increase of velocity. In Fig. 3, the vertical distance of any point on any line *ab* from the base line shows the velocity of a body at the corresponding instant: it is straight because the acceleration is assumed constant, and therefore the velocity increases directly as the time. If the body start from rest, when v_1 is zero, the mean velocity over any interval of time will be $\frac{v_2}{2}$, and the space traversed in the interval will be the mean velocity × time, or—

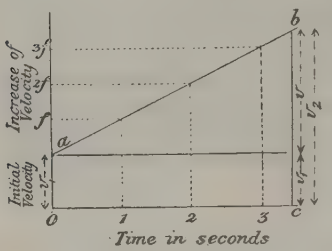


FIG. 3.

$$s = \frac{v_2}{2}t = \frac{f_a t^2}{2} \text{ (see equation i.)}$$

$$\text{and } f_a = \frac{2s}{t^2}$$

$$\text{Acceleration in feet per sec. per sec.} = \frac{\text{constant} \times \text{space (in } \textit{feet})}{(\text{time})^2 \text{ (in } \textit{seconds})}$$

When the body has an initial velocity v_1, the mean velocity during the time t is represented by the mean height of the figure *oabc*.

$$\text{Mean velocity} = \frac{v_2 + v_1}{2} = \frac{v + 2v_1}{2} = v_1 + \frac{f_a t}{2} \quad . \quad \text{(ii.)}$$

(see equation i.)

The space traversed in the time t—

$$s = \left(v_1 + \frac{f_a t}{2}\right)t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

which is represented in the diagram by the area of the diagram *oabc*. From equations i. and ii., we get—

$$\frac{v_2^2 - v_1^2}{2} = \left(v_1 + \frac{f_a t}{2}\right)f_a t$$

Substituting from iii., we get—

$$\frac{v_2^2 - v_1^2}{2} = \left(\frac{s}{t}\right)f_a t = f_a s$$

$$v_2^2 - v_1^2 = 2f_a s$$

$$\text{or } v_2^2 = v_1^2 + 2f_a s$$

When a body falls freely due to gravity, $f_a = g = 32{\cdot}2$ ft. per second per second, it is then usual to use the letter h, the height through which the body has fallen, instead of s.

When the body starts from rest, we have $v_1 = 0$, and $v_2 = v$; then by substitution from above, we have—

$$v^2 = 2gh$$
$$v = \sqrt{2gh} = 8{\cdot}02\sqrt{h} \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

Momentum (M_0).—If a body of mass M [1] move with a velocity v, the moving mass is said to possess *momentum*, or *quantity of motion*, $= Mv$.

Unit momentum is that of unit mass moving with unit velocity—

$$M_0 = Mv = \frac{Ms}{t}$$

Impulse.—Consider a ball of mass M travelling through space with a velocity v_1, and let it receive a fair blow in the line of motion (without causing it to spin) as it travels along, in such a manner that its velocity is suddenly increased from v_1 to v_2.

The momentum before the blow $= Mv_1$
„ after „ $= Mv_2$
The change of momentum due to the blow $= M(v_2 - v_1)$

The effect of the blow is termed an *impulse*, and is measured by the change of momentum.

Impulse = change of momentum $= M(v_2 - v_1)$

Force (F).—If the ball in the paragraph above had received a very large number of very small impulses instead of a single blow, its velocity would have been *gradually* changed, and we should have had—

The whole impulse per second = the change of momentum per second

When the impulses become infinitely rapid, the whole impulse per second is termed the *force* acting on the body. Hence the momentum may be changed gradually from M_1v_1 to M_2v_2 by a force acting for t seconds. Then—

[1] For a rational definition of *mass*, the reader is referred to Prof. Kar Pearson's "Grammar of Science," p. 357.

$$Ft = M(v_2 - v_1)$$

$$\text{where } F = \frac{\text{total change of momentum}}{\text{time}}$$

$$\text{But } \frac{v_2 - v_1}{t} = f_a \text{ (acceleration) (see p. 5)}$$

$$\text{hence } F = Mf_a = \frac{Mv}{t}$$

Hence the dimensions of this unit are—

$$F = \frac{Ms}{t^2}$$

Force = mass × acceleration
Unit force = unit mass × unit acceleration

Thus *unit force* is that force which, when acting on a mass of one $\left\{\begin{matrix}\text{pound}\\\text{gram}\end{matrix}\right\}$ for one second, will change its velocity by one $\left\{\begin{matrix}\text{foot}\\\text{centimetre}\end{matrix}\right\}$ per second, and is termed one $\left\{\begin{matrix}\text{poundal.}^1\\\text{dyne.}\end{matrix}\right.$

We are now in a position to appreciate the words of Newton—

Change of momentum is proportional to the impressed force, and takes place in the direction of the force ; . . . also, *a body will remain at rest, or, if in motion, will move with a uniform velocity in a straight line unless acted upon by some external force.*

Force simply describes *how* motion takes place, *not why* it takes place.

It does not follow, because the velocity of a body is not changing, or because it is at rest, that no forces are acting upon it ; for suppose the ball mentioned above had been acted upon by two equal and opposite forces at the same instant, the one would have tended to accelerate the body backwards (termed a negative acceleration, or retardation) just as much as the other tended to accelerate it forwards, with the result that the one would have just neutralized the other, and the velocity, and consequently the momentum, would have remained unchanged. We say then, in this case, that the positive acceleration is equal and opposite to the negative acceleration.

If a railway train be running at a constant velocity, it must not be imagined that no force is required to draw it ; the force exerted by the engine produces a positive acceleration, while

[1] The poundal unit is never used by engineers.

the friction on the axles, tyres, etc., produces an equal and opposite negative acceleration. If the velocity of the train be constant, the whole effort exerted by the engine is expended in overcoming the frictional resistance, or the negative acceleration. If the positive acceleration at any time exceeds the negative acceleration due to the friction, the positive or forward force exerted by the engine will still be equal to the negative or backward force or the total resistance overcome; but the resistance now consists partly of the frictional resistance, and partly the resistance of the train to having its velocity increased. The work done by the engine over and above that expended in overcoming friction is stored up in the moving mass of the train as energy of motion, or kinetic energy (see p. 14).

UNITS OF FORCE.

Force.	Mass.	Acceleration.
Poundal.	One pound.	One foot per second per second.
Dyne.	One gram.	One centimetre per second per second.

1 poundal = 13,825 dynes.
1 pound = 445,000 dynes.

Weight (W).—The weight of a body is the force that gravity exerts on that body. It depends (1) on the mass of the body; (2) on the acceleration of gravity (*g*), which varies inversely as the square of the distance from the centre of the earth, hence the weight of a body depends upon its position as regards the centre of the earth. The distance, however, of all inhabited places on the earth from the centre is so nearly constant, that for all practical purposes we assume that the acceleration of gravity is constant (the extreme variation is about one-third of one per cent.). Consequently for practical purposes we compare masses by their weights.

$$\text{Weight} = \text{mass} \times \text{acceleration of gravity}$$
$$W = Mg$$

We nave shown above that—

$$\text{Force} = \text{mass} \times \text{acceleration}^{1}$$

[1] Expressing this in absolute units, we have—
Weight or force (poundals) = mass (pounds) × acceleration (feet per second per second)
Then—
Force of gravity on a mass of one pound = 1 × 32·2 = 32·2 poundals
But, as poundals are exceedingly inconvenient units to use for practical

hence we speak of forces as being equal to the weight of so many pounds; but for convenience of expression we shall speak of forces of so many pounds, or of so many tons, as the case may be.

VALUES OF g.[1]

	In foot-pounds, secs.	In centimetre-grammes, secs.
The equator ...	32·091	978·10
London ...	32·191	981·17
The pole ...	32·255	983·11

Work.—When a body is moved so as to overcome a resistance, we know that it must have been acted upon by a force acting in the direction of the displacement. The force is then said to perform work, and the measure of the work done is the product of the force and the displacement. The absolute *unit of work* is unit force (one poundal) acting through unit displacement (foot), *or one foot-poundal.* Such a unit of work is, however, never used by engineers; the unit nearly always used in England is the "foot-pound," *i.e.* one pound weight lifted one foot high.

$$\text{Work} = \text{force} \times \text{displacement} = FS$$

The dimensions of the unit of work are therefore $\frac{MS^2}{t^2}$.

---

purposes, we shall adopt the engineer's unit of one pound weight, *i.e.* a unit 32·2 times as great; then, in order that the fundamental equation may hold for this unit, viz.—

$$\text{Weight or force (pounds)} = \text{mass} \times \text{acceleration}$$

we must divide our weight or force expressed in poundals by 32·2, and we get—

$$\text{Weight or force (pounds)} = \frac{\text{weight or force (poundals)}}{32{\cdot}2} = \frac{\text{mass} \times \text{acceleration}}{32{\cdot}2}$$

or—

$$\text{weight or force (pounds)} = \frac{\text{mass in pounds}}{32{\cdot}2} \times \text{acceleration in ft.-sec. per sec.}$$

Thus we must take our new unit of mass as 32·2 times as great as the absolute unit of mass.

Readers who do not see the point in the above had better leave it alone—at any rate, for the present, as it will not affect any question we shall have to deal with. As a matter of fact, engineers always do (probably unconsciously) make the assumption, but do not explicitly state it.

[1] Hicks's "Elementary Dynamics," p. 45.

Frequently we shall have to deal with a variable force acting through a given displacement; the work done is then the average[1] force multiplied by the displacement. Methods of finding such averages will be discussed later on. In certain cases it will be convenient to remember that the work done in lifting a body is the weight of the body multiplied by the height through which the centre of gravity of the body is lifted.

UNITS OF WORK.

Force.	Displacement.	Unit of work.
Pound.	Foot.	Foot-pound.
Kilogram.	Metre.	Kilogrammetre.
Dyne.	Centimetre.	Erg.

1 foot-pound = 32·2 foot-poundals.
" = 13,560,000 ergs.

Power.—Power is the rate of doing work. *Unit power* is unit work done in unit time, or one foot-pound per second.

$$\text{Power} = \frac{\text{total work done}}{\text{time taken to do it}} = \frac{Fs}{t}$$

The dimensions of the unit of power are therefore $\frac{Ms^2}{t^3}$.

The unit of power commonly used by engineers is an arbitrary unit established by James Watt, viz. a horse-power, which is 33,000 foot-pounds of work done per minute.

$$\text{Horse-power} = \frac{\text{foot-pounds of work done in a given time}}{\text{time (in minutes) occupied in doing the work} \times 33{,}000}$$

1 horse-power = 33,000 foot-pounds per minute
= $7{\cdot}46 \times 10^9$ ergs per second.
1 French horse-power = 32,500 foot-pounds per minute
= $7{\cdot}36 \times 10^9$ ergs per second.
1 horse-power = 746 watts
1 watt = 10^7 ergs per second.

Couples.—When forces act upon a body in such a manner as to tend to give it a spin or a rotation about an axis without any tendency to shift its c. of g., the body is said to be acted

[1] Space-average.

upon by a couple. Thus, in the figure the force F tends to turn the body round about the point O. If, however, this were the only force acting on the body, it would have a motion of translation in the direction of the force as well as a spin round the axis; in order to prevent this motion of translation, another force, F_1, equal and parallel but opposite in direction to F, must be applied to the body in the same plane. Thus, a couple is said to consist of two parallel forces of equal magnitude acting in opposite directions, but not in the same straight line.

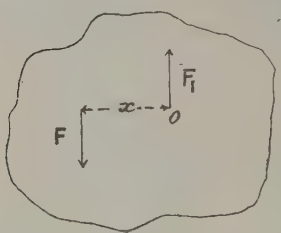


FIG. 4.

The perpendicular distance x between the forces is termed the *arm* of the couple. The tendency of a couple is to turn the body to which it is applied in the plane of the couple. When it tends to turn it in the direction of the hands of a watch, it is termed a clockwise, or positive (+) couple, and in the contrary direction, a contra-clockwise, or negative (−) couple.

It is readily proved[1] that not only may a couple be shifted anywhere in its own plane, but its arm may be altered (as long as its moment is kept the same) without affecting the equilibrium of the body.

Moments.—The moment of a couple is the product of one of the forces and the length of the arm. It is usual to speak of the moment of a force about a given point—that is, the product of the force and the perpendicular distance from its line of action to the point in question.

As in the case of couples, moments are spoken of as clockwise and contra-clockwise.

If a rigid body be in equilibrium under any given system of moments, the algebraic sum of all the moments in any given plane must be zero, or the clockwise moments must be equal to the contra-clockwise moments in any given plane.

$$\text{Moment} = \text{force} \times \text{arm}$$
$$= Fx$$

The dimensions of a moment are therefore $\frac{ML^2}{T^2}$.

[1] See Hicks's "Elementary Mechanics."

Centre of Gravity (c. of g.).—The gravitation forces acting on the several particles of a body may be considered to act parallel to one another.

If a point be so chosen in a body that the sum of the moments of all the gravitation forces acting on the several particles about the one side of any straight line passing through that point be equal to the sum of the moments on the other side of the line, that point is termed the centre of gravity of the body.

Thus, the resultant of all the gravitation forces acting on a body passes through its centre of gravity, however the body may be tilted about.

Centroid.—The corresponding point in a geometrical surface which has no weight is frequently termed the centroid; such cases are fully dealt with in Chapter III.

Energy.—Capacity for doing work is termed energy.

Conservation of Energy.—Experience shows us that energy cannot be created or destroyed; it may be dissipated, or it may be transformed from any one form to any other, hence the whole of the work supplied to any machine must be equal to the work got out of the machine, together with the work converted into heat,[1] either by the friction or the impact of the parts one on the other.

Mechanical Equivalent of Heat.—It was experimentally shown by Joule that in the conversion of mechanical into heat energy,[2] 772 foot-lbs. of work have to be expended in order to generate one thermal unit.

Efficiency of a Machine.—The efficiency of a machine is the ratio of the useful work got out of the machine to the gross work supplied to the machine.

$$\text{Efficiency} = \frac{\text{work got out of the machine}}{\text{work supplied to the machine}}$$

This ratio is necessarily less than unity.

The counter-efficiency is the reciprocal of the efficiency, and is always greater than unity.

$$\text{Counter-efficiency} = \frac{\text{work supplied to the machine}}{\text{work got out of the machine}}$$

[1] To be strictly accurate, we should also say light, sound, electricity, etc.

[2] By far the most accurate determination is that recently made by Professor Osborne Reynolds and Mr. W. H. Moorby, who obtained the value 776·94 (see *Phil. Trans.*, vol. 190, pp. 301–422) from 32° F. to 212° F., which is equivalent to about 773 at 39° F. and 778 at 60° F.

Kinetic Energy.—From the principle of the conservation of energy, we know that when a body falls freely by gravity, the work done on the falling body must be equal to the energy of motion stored in the body (neglecting friction).

The work done by gravity on a weight of W pounds in falling through a height h ft. = Wh foot-lbs. But we have shown above that $h = \frac{v^2}{2g}$, where v is the velocity after falling through a height h; whence—

$$Wh = \frac{Wv^2}{2g}, \text{ or } \frac{Mv^2}{2}$$

This quantity, $\frac{Wv^2}{2g}$, is known as the kinetic energy of the body, or the energy due to its motion.

Inertia.—Since energy has to be expended when the velocity of a body is increased, a body may be said to offer a resistance to having its velocity increased, this resistance is known as the inertia of the body. Inertia is sometimes defined as the "deadness of matter."

Moment of Inertia (I).—We may define inertia as the capacity of a body to possess momentum, and momentum as the product of mass and velocity (Mv). If we have a very small body of mass M rotating about an axis at a radius r, with an angular velocity ω, the linear velocity of the body will be $v = \omega r$, and the momentum will be Mv. But if the body be shifted further from the axis of rotation, and r be thereby increased, the momentum will also be increased in the same ratio. Hence, when we are dealing with a rotating body, we have not only to deal with its mass, but with the *arrangement* of the body about the axis of rotation, *i.e.* with its moment about the axis.

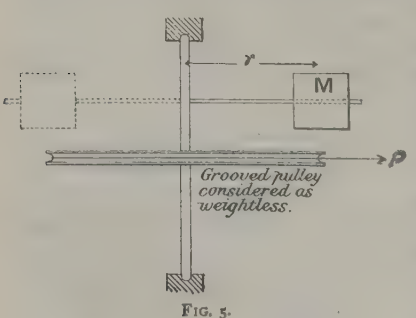


FIG. 5.

Let the body be acted upon by a twisting moment, Pr = T,

then, as the force P acts at the same radius as that of the body, it may be regarded as acting on the body itself. The force P acting at a radius r will produce the same effect as a force nP acting at a radius $\frac{r}{n}$. The force P acting on the mass M gives it a linear acceleration f_a, where $P = Mf_a$, or $f_a = \frac{P}{M}$. The angular velocity ω is $\frac{1}{r}$ times the linear velocity, hence the angular acceleration is $\frac{1}{r}$ times the linear acceleration. Let A = the angular acceleration; then—

$$A = \frac{f_a}{r} = \frac{P}{Mr} = \frac{Pr}{Mr^2} = \frac{T}{Mr^2}$$

$$\text{or angular acceleration} = \frac{\text{twisting moment}^1}{\text{mass} \times (\text{radius})^2}$$

In the case we have just dealt with, the mass M is supposed to be exceedingly small, and every part of it at a distance r from the axis. When the body is great, it may be considered to be made up of a large number of small masses, M_1, M_2, etc., at radii r_1, r_2, etc., respectively; then the above expression becomes—

$$A = \frac{T}{(M_1r_1^2 + M_2r_2^2 + M_3r_3^2 +, \text{etc.})}$$

The quantity in the denominator is termed the "moment of inertia" of the body.

We stated above that the capacity of a body to possess momentum is termed the "inertia of the body." Now, in a case in which the capacity of the body to possess angular momentum depends upon the moment of the several portions of the body about a given axis, we see why the capacity of a rotating body to possess momentum should be termed the "moment of inertia."

Let M = mass of the whole body, then $M = M_1 + M_2 + M_3$, etc.; then the moment of inertia of the body, $I, = M\kappa^2 = (M_1r_1^2 + M_2r_2^2, \text{etc.})$.

Radius of Gyration (κ).—The κ in the paragraph above is known as the radius of gyration of the body. Thus, if we could condense the whole body into a single particle at a distance κ from the axis of rotation, the body would still have

[1] The reader is advised to turn back to the paragraph on "couples," so that he may not lose sight of the fact that a couple involves *two* forces.

the same capacity for possessing energy, due to rotation about that axis.

Representation of Displacements, Velocities,[1] Accelerations, Forces by Straight Lines. — Any {displacement / velocity / acceleration / force} is fully represented when we state its magnitude and its direction, and, in the case of force, its point of application.

Hence a straight line may be used to represent any {displacement / velocity / acceleration / force}, the length of which represents its magnitude, and the direction of the line the direction in which the force, etc., acts.

Two or more {displacements / velocities / accelerations / forces}, meeting at a point, may be replaced by one force, etc., passing through the same point, which is termed the resultant force, etc.

If two {displacements / velocities / accelerations / forces}, not in the same straight line, meeting at a point *a*, be represented by two straight lines, *ab*, *ac*, and if two other straight lines, *dc*, *bd*, be drawn parallel to them from their extremities to form a parallelogram, *abdc*, the diagonal of the parallelogram *ad* which passes through that point will represent the resultant {displacement / velocity / acceleration / force} in magnitude and direction.

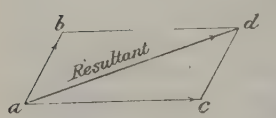


FIG. 6.

Hence, if a force equal and opposite to *ad* act on the point in the same plane, the point will be in equilibrium.

It is evident from the figure that *bd* is equal in every

[1] Including angular velocities or spins.

respect to *ac*; then the three forces are represented by the three sides of the triangle *ab*, *bd*, *ad*. Hence we may say that if three forces act upon a point in such a manner that they are equal and parallel to the sides of a triangle, the point is in equilibrium under the action of those forces. This is known as the theorem of the "triangle of forces."

Many special applications of this method will be dealt with in future chapters.

The proof of the above statements will be found in all elementary books on Mechanics.

Hodograph.—The motion of a body moving in a curved path may be very conveniently analyzed by means of a curve called a "hodograph." In Fig. 7, suppose a point moving along the path P, P_1, P_2, with varying velocity. If a line, *op*, known as a "radius vector," be drawn so that its length represents on any given scale the speed of the point at P, and the direction of the radius vector the direction in which P is moving, the line *op* completely represents the velocity of the point P. If other radii are drawn in the same manner, the curve traced out by their extremities is known as the "hodograph" of the point P. The change of velocity of the point P in passing from P to P_1 is represented on the hodograph by the distance pp_1, consisting of a change in the length of the line, viz. q_1p_1 representing the change in speed of the point P, and pq_1 the change of velocity due to change of direction, if a radius vector be drawn each second; then pp_1 will represent the average change of velocity per second, or in the limit the rate of change of velocity of the point P, or, in other words, the acceleration (see p. 5) of the point P; thus the velocity of *p* represents the acceleration of the point P.

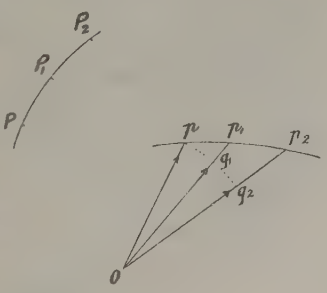


FIG. 7.

If the speed of the point P remained constant, then the length of the line *op* would also be constant, and the hodograph would become the arc of a circle, and the only change in the velocity would be the change in direction pq_1.

Centrifugal Force.—If a heavy body be attached to the end of a piece of string, and the body be caused to move round

in a circular path, the string will be put into tension, the amount of which will depend upon (1) the mass of the body, (2) the length of the string, and (3) the velocity with which the body moves. The tension in the string is equal to the centrifugal force. We will now show how the exact value of this force may be calculated in any given instance.[1]

Let the speed with which the body describes the circle be constant; then the radius vector of the hodograph will be of constant length, and the hodograph itself will be a circle. Let the body describe the outer of the two circles shown in the figure, with a velocity v, and let its velocity at A be represented by the radius OP, the inner circle being the hodograph of A. Now let A move through an extremely small space to A_1, and the corresponding radius vector to OP_1; then the line PP_1 represents the change in velocity of A while it was moving to A_1. (The reader should never lose sight of the fact that change of velocity involves change of direction as well as change of speed, and as the speed is constant in this case, the change of velocity is wholly a change of direction.)

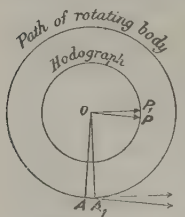


FIG. 8.

As the distance AA_1 becomes smaller, PP_1 becomes more nearly perpendicular to OP, and in the limit it does become perpendicular, and parallel to OA; thus the change of velocity is radial and towards the centre.

We have shown on p. 17 that the velocity of P represents the acceleration of the point A; then, as both circles are described in the same time—

$$\frac{\text{velocity of P}}{\text{velocity of A}} = \frac{\text{OP}}{\text{OA}}$$

But OP was made equal to the velocity of A, viz. v, and OA is the radius of the circle described by the body. Let OA = R; then—

$$\frac{\text{velocity of P}}{v} = \frac{v}{\text{R}}$$

$$\text{or velocity of P} = \frac{v^2}{\text{R}}$$

[1] For another method of treatment, see Barker's "Graphic Methods of Engine Design."

$$\text{and acceleration of A} = \frac{v^2}{R}$$

$$\text{and since force} = \text{mass} \times \text{acceleration}$$

$$\text{we have centrifugal force } C = \frac{Mv^2}{R}$$

$$\text{or in gravitational units, } C = \frac{Wv^2}{gR}$$

This force acts radially outwards from the centre.

Sometimes it is convenient to have the centrifugal force expressed in terms of the angular velocity of the body. We have—

$$v = \omega R$$

$$\text{hence } C = M\omega^2 R$$

$$\text{or } C = \frac{W\omega^2 R}{g}$$

Change of Units.—It frequently happens that we wish to change the units in a given expression to some other units more convenient for our immediate purpose; such an alteration in units is very simple, provided we set about it in systematic fashion. The expression must first be reduced to its fundamental units; then each unit must be multiplied by the required constant to convert it into the new unit. For example, suppose we wish to convert foot-pounds of work to ergs, then—

$$\text{The dimensions of work are } \frac{ms^2}{t^2}$$

$$\text{work in ft.-poundals} = \frac{\text{pounds} \times (\text{feet})^2}{(\text{seconds})^2}$$

$$\text{work in ergs} = \frac{\text{grams} \times (\text{centimetres})^2}{(\text{seconds})^2}$$

$$1 \text{ pound} = 453{\cdot}6 \text{ grams}$$

$$1 \text{ foot} = 30{\cdot}48 \text{ centimetres}$$

Hence—

$$1 \text{ foot-poundal} = 453{\cdot}6 \times 30{\cdot}48^2 = 421{,}390 \text{ ergs}$$

$$\text{and } 1 \text{ foot-pound} = 32{\cdot}2 \text{ foot-poundals}$$

$$= 32{\cdot}2 \times 421{,}390 = 13{,}560{,}000 \text{ ergs}$$

CHAPTER II.

MENSURATION.

MENSURATION consists of the measurement of lengths, areas, and volumes, and the expression of such measurements in terms of a simple unit of length.

Length.—If a point be shifted through any given distance, it traces out a line in space, and the length of the line is the distance the point has been shifted. A simple statement in units of length of this *one* shift completely expresses its only dimension, *length*; hence a line is said to have but *one* dimension, and when we speak of a line of length l, we mean a line containing l length units.

Area.—If a straight line be given a side shift in any given plane, the line sweeps out a surface in space. The area of the surface swept out is dependent upon *two* distinct shifts of the generating point: (1) on the length of the original shift of the point, *i.e.* on the length of the generating line (l); (2) on the length of the side shift of the generating line (d).

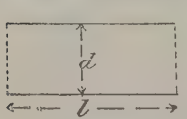


FIG. 9.

Thus a statement of the area of a given surface must involve *two* length quantities, l and d, both expressed in the same units of length. Hence a surface is said to have *two* dimensions, and the area of a surface ld must always be expressed as the product of two lengths, each containing so many length units, viz.—

$$\text{Area} = \text{length units} \times \text{length units} = (\text{length units})^2$$

Volume.—If a plane surface be given a side shift to bring it into another plane, the surface sweeps out a volume in space.

The volume of the space swept out is dependent upon *three* distinct shifts of the generating point: (1) on the length of the original shift of the generating point, *i.e.* on the length of the generating line l; (2) on the length of the side shift of the generating line d; (3) on the side shift of the generating surface t. Thus the statement of the volume of a given body or space must involve *three* length quantities, l, d, t, all expressed in the same units of length.

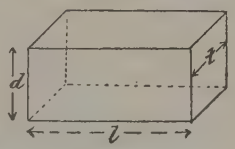


FIG. 10.

Hence a volume is said to have *three* dimensions, and the volume of a body must always be expressed as the product of *three* lengths, each containing so many length units, viz.—

$$\text{Volume} = \text{length units} \times \text{length units} \times \text{length units}$$
$$= (\text{length units})^3$$

Lengths.

Straight line.

Circumference of circle.

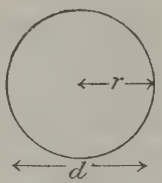


FIG. 11.

Length of circumference $= \pi d$
$= 3{\cdot}1416d$
or $= 2\pi r$
$= 6{\cdot}2832r$

The last two decimals above may usually be neglected; the error will be less than $\frac{1}{8}$ in. on a 10-ft. circle.

Arc of circle.

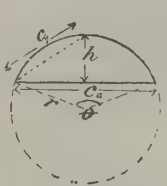


FIG. 12.

Length of arc $= \dfrac{\pi d\theta}{360}$

or $= \dfrac{2\pi r\theta}{360} = \dfrac{r\theta}{57{\cdot}3}$

For an arc less than a semicircle—

Length $= \dfrac{8C_1 - C_a}{3}$ approximately

Arc of ellipse.

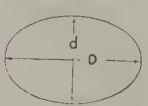


FIG. 13.

Length of circumference approx.

$$= \pi d + 2(D - d) - \frac{d(D - d)}{\sqrt{(D + d)(D + 2d)}}$$

The length of lines can be measured to within $\frac{1}{100}$ in. with a scale divided into either tenths or twentieths of an inch. With special appliances lengths can be measured to within $\frac{1}{1000000}$ in. if necessary.

The mathematical process by which the value of π is determined is too long for insertion here. One method consists of calculating the perimeter of a many-sided polygon described about a circle, also of one inscribed in a circle. The perimeter of the outer polygon is greater, and that of the inner less, than the perimeter of the circle. The greater the number of sides the smaller is the difference. The value of π has been found to 750 places of decimals, but it is rarely required for practical purposes beyond three or four places. For a simple method of finding the value of π, see "Longmans' School Mensuration," p. 48.

The length of the arc is less than the length of the circumference in the ratio $\frac{\theta}{360}$.

$$\text{Length of arc} = \pi d \times \frac{\theta}{360} = \frac{\pi d \theta}{360}$$

The approximate formula given is extremely near when h is not great compared with C_a; even for a semicircle the error is only about 1 in 80. The proof is given in Lodge's "Mensuration for Senior Students" (Longmans).

No simple expression for the exact value of the length of an elliptic arc can be given, the value opposite is due to Mr. M. Arnold Pears, of New South Wales, see Trautwine's "Pocket-book," 18th edition, p. 189.

Arc of parabola.

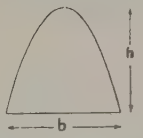


FIG. 14.

$$\text{Length of arc} = 2\sqrt{\frac{b^2}{4} + \frac{4h^2}{3}} \quad \text{(approximately)}$$

Irregular curved line abc.

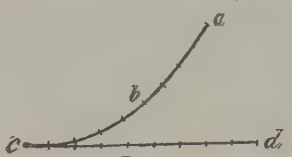


FIG. 15.

Set off tangent *cd*. With pair of dividers start from *a*, making small steps till the point *c* is reached, or nearly so. Count number of steps, and step off same number along tangent.

Areas.

Parallelograms.

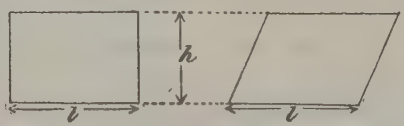


FIG. 16.

$$\text{Area of figure} = lh$$

Triangles.

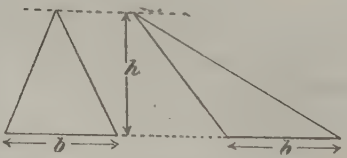


FIG. 17.

$$\text{Area of figure} = \frac{bh}{2}$$

Equilateral triangle.

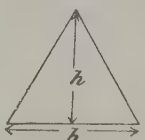


FIG. 18.

$$\text{Area of figure} = \frac{\sqrt{3}}{4}b^2 = 0{\cdot}433b^2$$

No simple expression can be given for the length of a parabolic arc—a common approximation is that given on the opposite page. The error is negligible when h is small compared with b, but when h is equal to b the error amounts to about $8\frac{1}{2}$ per cent.

The stepping should be commenced at the end remote from the tangent; then if the last step does not exactly coincide with c, the backward stepping can be commenced from the last point without causing any appreciable error. The greater the accuracy required, the greater must be the number of steps.

Areas.

See Euc. I. 35.

See Euc. I. 41.

$$h = \sqrt{b^2 - \left(\frac{b}{2}\right)^2} = \sqrt{\frac{3b^2}{4}} = \frac{\sqrt{3}b^2}{2}$$

$$\text{area} = \frac{bh}{2} = \frac{\sqrt{3}}{4}b^2 = 0{\cdot}433b^2$$

Triangle.

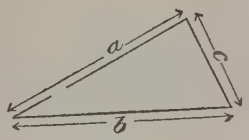


FIG. 19.

Let $s = \frac{a+b+c}{2}$

Area of figure $= \sqrt{s(s-a)(s-b)(s-c)}$

Quadrilateral.

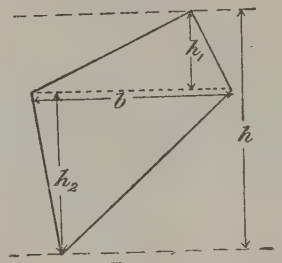


FIG. 20.

Area of figure $= \frac{bh}{2}$

Trapezium.

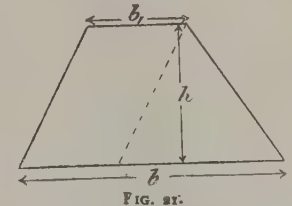


FIG. 21.

Area of figure $= \left(\frac{b+b_1}{2}\right)h$

Irregular straight-lined figure.

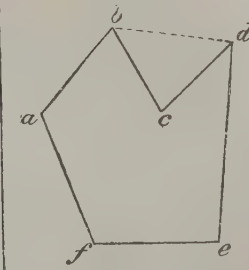


FIG. 22.

Area of figure = area *abdef* − area *bcd*
or area of triangles (*acb* + *acf* + *cfe* + *ced*)

The proof is somewhat lengthy, but perfectly simple (see "Longmans' Mensuration," p. 18).

Area of upper triangle $= \frac{bh_1}{2}$

" lower triangle $= \frac{bh_2}{2}$

" both triangles $= h\left(\frac{h_1 + h_2}{2}\right) = \frac{bh}{2}$

Area of parallelogram $= b_1 h$

Area of triangle $= \frac{(b - b_1)h}{2}$

Area of whole figure $= \frac{(b - b_1 + 2b_1)h}{2} = \frac{(b + b_1)h}{2}$

Simple case of addition and subtraction of areas.

Circle.

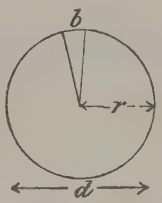


FIG. 23.

Area of figure $= \pi r^2 = 3{\cdot}1416r^2$

or $\frac{\pi d^2}{4} = 0{\cdot}7854d^2$

Sector of circle.

FIG. 24.

Area of figure $= \frac{\pi r^2 \theta}{360}$

Segment of circle.

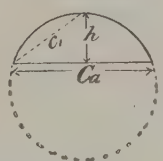


FIG. 25.

Area of figure $= \frac{2}{3}C_a h$ when h is small

$= \frac{h}{15}(6C_a + 8C_1)$ nearly

Hollow circle.

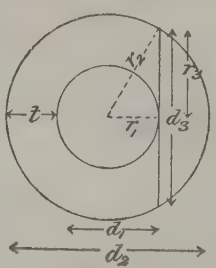


FIG. 26.

Area of figure = area of outer circle − area of inner circle

$= \pi r_2^2 - \pi r_1^2$

$= \pi(r_2^2 - r_1^2)$

$= \pi r_3^2$

or $= 0{\cdot}785 d_3^2$

or $= \frac{\pi(d_1 + d_2)}{2} t$, *i.e.* mean circumf. × thickness

The circle may be conceived to be made up of a great number of tiny triangles, such as the one shown, the base of each little triangle being b units, then the area of each triangle is $\frac{br}{2}$; but the sum of all the bases equals the circumference, or $\Sigma b = 2\pi r$, hence the area of all the triangles put together, *i.e.* the area of the circle, $= \frac{2\pi r \cdot r}{2} = \pi r^2$.

The area of the sector is less than the area of the circle in the ratio $\frac{\theta}{360}$, hence the area of the sector $= \frac{\pi r^2 \theta}{360}$; if β be the angle expressed in circular measure, then the above ratio becomes $\frac{\beta}{2\pi}$.

$$\text{The area} = \frac{r^2\beta}{2}$$

When h is less than $\frac{C_a}{4}$, the arc of the circle very nearly coincides with a parabolic arc (see p. 31). For proof of second formula, see Lodge's "Mensuration for Senior Students" (Longmans).

Simple Case of Subtraction of Areas.—The substitution of r_3^2 for $r_2^2 - r_1^2$ follows from the properties of the right-angled triangle (Euc. I. 47).

The mean circumference × thickness is a very convenient form of expression; it is arrived at thus—

$$\text{Mean circumference} = \frac{\pi(d_2 + d_1)}{2}$$

$$\text{thickness} = \frac{d_2 - d_1}{2}$$

$$\text{product} = \frac{\pi(d_2 + d_1)}{2} \times \frac{d_2 - d_1}{2} = \frac{\pi}{4}(d_2^2 - d_1^2)$$

Ellipse.

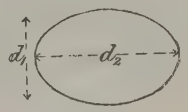


FIG. 27.

Area of figure $= \pi r_1 r_2$
or $= \frac{\pi}{4} d_1 d_2$

Parabolic segments.

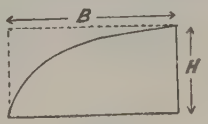


FIG. 28.

Area of figure $= \frac{2}{3}$BH
i.e. $\frac{2}{3}$(area of circumscribing rectangle)

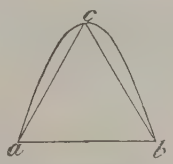


FIG. 29.

Area of figure $= \frac{4}{3}$ area of $\triangle$ *abc*

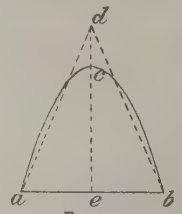


FIG. 30.

Make $dc = \frac{1}{3}ce$
area of figure = area of $\triangle$ *abd*

An ellipse may be regarded as a flattened or an elongated circle; hence the area of an ellipse is $\left\{\begin{matrix}\text{less}\\\text{greater}\end{matrix}\right\}$ than the area of a circle whose diameter is the $\left\{\begin{matrix}\text{major}\\\text{minor}\end{matrix}\right\}$ axis of an ellipse $\left\{\begin{matrix}d_2\\d_1\end{matrix}\right\}$, in the ratio $\left\{\begin{matrix}d_1 \text{ to } d_2.\\d_2 \text{ to } d_1\end{matrix}\right.$

$$\text{Area} = \frac{\pi d_2^2}{4} \times \frac{d_1}{d_2} = \frac{\pi}{4} d_1 d_2, \text{ or } \frac{\pi d_1^2}{4} \times \frac{d_2}{d_1} = \frac{\pi d_1 d_2}{4}$$

From the properties of the parabola, we have—

$$\frac{h^2}{H^2} = \frac{b}{B}$$

$$h = \sqrt{\frac{bH^2}{B}} = \frac{b^{\frac{1}{2}}H}{B^{\frac{1}{2}}}$$

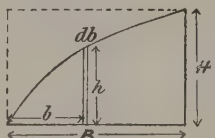


FIG. 28a.

$$\text{area of strip} = h \,.\, db = \left(\frac{Hb^{\frac{1}{2}}}{B^{\frac{1}{2}}}\right) db$$

$$\text{area of whole figure} = \frac{H}{B^{\frac{1}{2}}} \int_{b=0}^{b=B} b^{\frac{1}{2}} \,.\, db = \frac{H}{B^{\frac{1}{2}}} \times \frac{B^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \tfrac{2}{3}HB$$

The area *acbg* has been shown to be $\frac{2}{3}$HB. Take from each the area of the △ *abg*, then the remainder *abc* = $\frac{1}{3}$ the △ *abe*; but, from the properties of the parabola, we have *ed* = $\frac{1}{2}$*eb*, hence the area *abc* = $\frac{2}{3}$ area of the circumscribing △ *abd*.

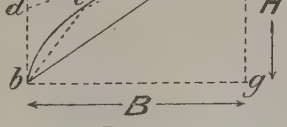


FIG. 29a.

From the properties of the parabola, we also have the height of the △ *abd* = 2(height of the △ *abc*); hence the area of the △ *abd* = 2(area of △ *abc*), and the area of the parabolic segment = 2 × $\frac{2}{3}$ area △ *abc* = $\frac{4}{3}$ area △ *abc*.

By increasing the height of the △ *abc* to $\frac{4}{3}$ its original height, we increase its area in the same ratio, and consequently make it equal to the area of the parabolic segment.

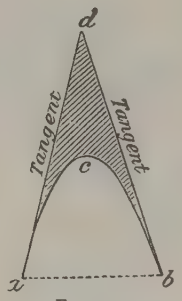


FIG. 31.

Area of shaded figure = $\frac{1}{3}$ area of $\triangle$ *abd*

Surfaces bounded by an irregular curve.

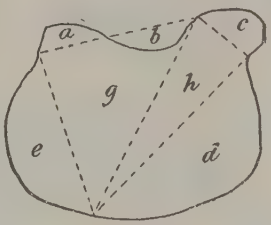


FIG. 32.

Area of figure = areas of parabolic segments $(a - b + c + d + e)$ + areas of triangles $(g + h)$.

Mean ordinate method.

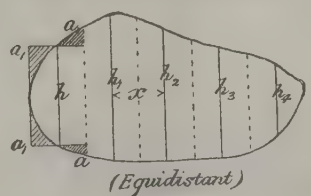


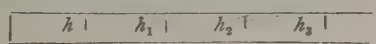
FIG. 33.

Area of figure = $(h + h_1 + h_2 + h_3 +$, etc.$)x$

The area $abc = \frac{2}{3}$ area of triangle abd, hence the remainder $= \frac{1}{3}$ of triangle abd.

Simply a case of addition and subtraction of areas. It is a somewhat clumsy and tedious method, and is not recommended for general work. One of the following methods is considered to be better.

This is a fairly accurate method if a large number of ordinates are taken. The value of $h + h_1 + h_2 + h_3$, etc., is most



and so on.

FIG. 33*a*.

easily found by marking them off continuously on a strip of paper.

The value of x must be accurately found; thus, If n be the number of ordinates, then $x = \frac{l}{n}$.

The method assumes that the areas a, a cut off are equal to the areas a_1, a_1 put on.

Simpson's Method.

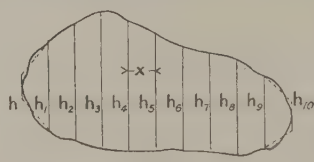


FIG. 34.

$$\text{Area of figure} = \frac{x}{3}(h + 4h_1 + 2h_2 + 4h_3 + 2h_4 + 4h_5 + 2h_6 + 4h_7 + 2h_8 + 4h_9 + h_{10})$$

The end ordinates should be obtained by drawing the mean lines (shown broken). If they are taken as zero the expression gives too low a result.

Any *odd* number of ordinates may be taken; the greater the number the greater will be the accuracy.

Curved surface of a spherical indentation.

FIG. 34*b*.

$$\text{Curved surface} = 2\pi R\left(R - \sqrt{R^2 - \frac{d^2}{4}}\right)$$

This is by far the most accurate and useful of all methods of measuring such areas. The proof is as follows :—

The curve *gfedc* is assumed to be a parabolic arc.

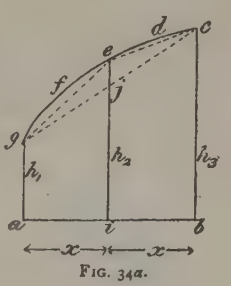


FIG. 34a.

Area *aieg* $= x\left(\frac{h_1 + h_2}{2}\right)$ . . . . (i.)

„ *ibce* $= x\left(\frac{h_2 + h_3}{2}\right)$ . . . . (ii.)

„ *abceg* $= \frac{x}{2}(h_1 + 2h_2 + h_3)$. (i.+ii.)

„ *abcjg* $= 2x\left(\frac{h_1 + h_3}{2}\right) = x(h_1 + h_3)$ (iii.)

Area of $\triangle$ *gce* = (i.) + (ii.) − (iii.)

$$= \frac{x}{2}(h_1 + 2h_2 + h_3) - x(h_1 + h_3)$$

$$= \frac{x}{2}(2h_2 - h_1 - h_3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

Area of parabolic segment *gcdef* $= \frac{4}{3}$(iv.) $= \frac{2x}{3}(2h_2 - h_1 - h_3)$. (v.)

Whole figure = (iii.) + (v.) $= x(h_1 + h_3) + \frac{2x}{3}(2h_2 - h_1 - h_3)$

$$= \frac{x}{3}(h_1 + 4h_2 + h_3)$$

If two more slices were added to the figure, the added area would be as above $= \frac{x}{3}(h_3 + 4h_4 + h_5)$, and when the two are added they become $= \frac{x}{3}(h_1 + 4h_2 + 2h_3 + 4h_4 + h_5)$.

The curved surface of the slice $= 2\pi r_1 s$

By similar triangles we have

$$\frac{S}{\delta y} = \frac{R}{r_1}$$

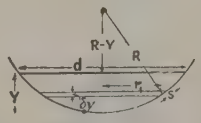


FIG. 34c.

Substituting the value of S we have

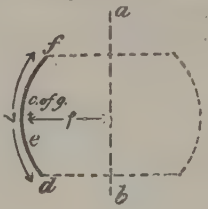
Curved surface of slice $= 2\pi R\delta y$

„ „ indentation $= 2\pi RY$

Expressing Y in terms of R and d we have

$$2\pi RY = 2\pi R\left(R - \sqrt{R^2 - \frac{d^2}{4}}\right)$$

Surfaces of revolution.



Pappus' or Guldinus' Method.—

Area of surface swept out by the revolution of the line *def* about the axis *ab* $= L \times 2\pi\rho$

Length of line $= L$

Radius of c. of g. of line *def* considered as a fine wire $= \rho$

FIG. 35.

This method also holds for any part of a revolution as well as for a complete revolution. The area of such figures as circles, hollow circles, sectors, parallelograms ($\rho = \infty$), can also be found by this method.

Surface of sphere.

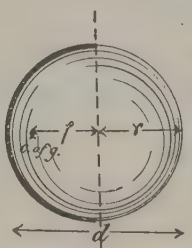


FIG. 36.

Area of surface of sphere $= 4\pi r^2$
$= \pi d^2$

The surface of a sphere is the same as the curved surface of a cylinder of same diameter and length $= d$.

Surface of cone.

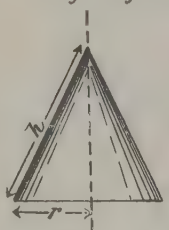


FIG. 37.

Area of curved surface of cone $= \pi r h$

The area of the surface traced out by a narrow strip of length $\begin{cases} l_0 \text{ and radius } \rho_0 = 2\pi l_0\rho_0 \\ l_1 \quad \text{,,} \quad \rho_1 = 2\pi l_1\rho_1 \end{cases}$, and so on.

Area of whole surface
$= 2\pi(l_0\rho_0 + l_1\rho_1 +, \text{ etc.})$
$= 2\pi$(each elemental length of wire × its distance from axis of revolution)
$= 2\pi$(total length of revolving wire × distance of c. of g. from axis of revolution) (see p. 58)
= (total length of revolving wire × length of path described by its centre of gravity)
$= L2\pi\rho$

FIG. 35*a*.

N.B.—The revolving wire must lie wholly on one side of the axis of revolution and in the same plane.

The distance of the c. of g. of any circular *arc*, or wire bent to a circular arc, from the centre of the circle is $y = \frac{rc}{a} = \rho$, where r = radius of circle, c chord of arc, a length of arc (see p. 64).

In the spherical surface $a = L = \pi r$, $c = 2r$, $\rho = \frac{2r^2}{\pi r} = \frac{2r}{\pi}$

$$\text{Surface of sphere} = \pi r \cdot 2\pi \cdot \frac{2r}{\pi} = 4\pi r^2$$

Length of revolving wire $= L = h$

radius of c. of g. ,, ,, $= \rho = \frac{r}{2}$

$$\text{surface of cone} = \frac{h2\pi r}{2} = \pi rh$$

Hyperbola.

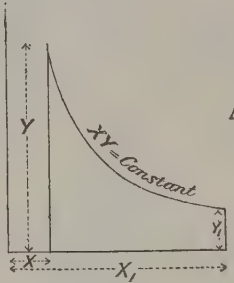


FIG. 38.

$$\text{Area of figure} = XY \log_e r$$

$$r = \frac{Y}{Y_1} \text{ or } \frac{X_1}{X}$$

$$\log_e = 2{\cdot}31 \times \text{ordinary log}$$

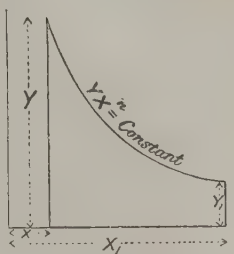


FIG. 39.

$$\text{Area of figure} = \frac{XY - X_1Y_1}{n - 1}$$

In the hyperbola we have—

$$XY = X_1Y_1 = xy$$

$$\text{hence } y = \frac{XY}{x}$$

$$\text{area of strip} = y \,.\, dx = XY\frac{dx}{x}$$

$$\left.\begin{matrix}\text{Area of}\\ \text{whole}\\ \text{figure}\end{matrix}\right\} = XY\int_{x=X}^{x=X_1}\frac{dx}{x}$$

$$= XY\,(\log_e X_1 - \log_e X)$$

$$= XY \log_e \frac{X_1}{X}$$

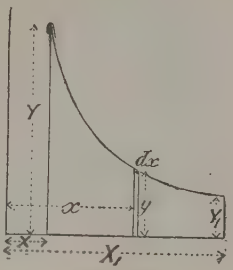


FIG. 38*a*.

Using the figure above, in this case we have—

$$YX^n = Y_1X_1^n = yx^n$$

$$\text{hence } y = \frac{YX^n}{x^n}$$

$$\text{area of strip} = y \,.\, dx = \frac{YX^n}{x^n}dx = YX^n x^{-n}dx$$

$$\text{area of whole figure} = YX^n\int_{x=X}^{x=X_1} x^{-n}dx = YX^n\left(\frac{X_1^{1-n} - X^{1-n}}{1-n}\right)$$

$$= \frac{YX^nX_1^{1-n} - YX}{1-n}$$

But $Y_1X_1^n = YX^n$

Multiply both sides by X_1^{1-n}

$$\text{then } Y_1X_1 = YX^nX_1^{1-n}$$

Substituting, we have—

$$\text{Area of whole figure} = \frac{Y_1X_1 - YX}{1-n}$$

$$\text{or} = \frac{YX - Y_1X_1}{n-1}$$

Irregular areas.

Irregular areas of every description are most easily and accurately measured by a planimeter, such as Amsler's or Goodman's.

A very convenient method is to cut out a piece of thin cardboard or sheet metal to the exact dimensions of the area; weigh it, and compare with a known area (such as a circle or square) cut from the same cardboard or metal. A convenient method of weighing is shown on the opposite page, and gives very accurate results if reasonable care be taken.

Volumes.

Prisms.

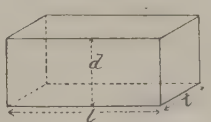


FIG. 41.

Let A = area of the end of prism;
l = length of prism.

Volume = lA

Parallelopiped.

Volume = ldt

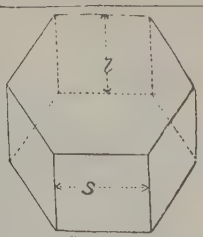


FIG. 42.

Hexagonal prism.

Volume = $2{\cdot}598s^2l$

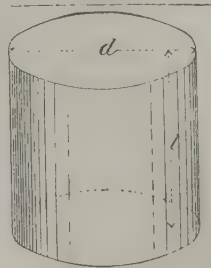


FIG. 43.

Cylinder.

$$\text{Volume} = \frac{\pi d^2 l}{4} = 0{\cdot}785d^2l$$

Suspend a knitting-needle or a straight piece of wire or wood by a piece of cotton, and accurately balance by shifting the cotton. Then suspend the two pieces of cardboard by pieces of cotton or silk; shift them till they balance; then measure the distances x and y.

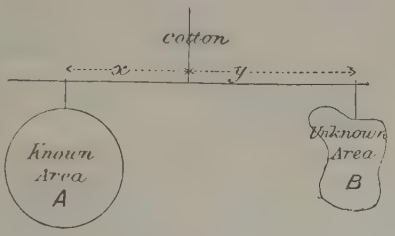


FIG. 40.

Then $Ax = By$

or $B = \frac{Ax}{y}$

The area of A should not differ very greatly from the area of B, or one arm becomes very short, and error is more likely to occur.

Area of end $= td$

volume $= ltd$

Area of hexagon = area of six equilateral triangles

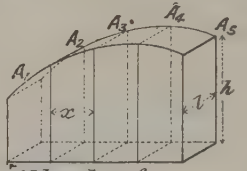
$= 6 \times 0{\cdot}433S^2$ (see Fig. 18)

volume $= 2{\cdot}598S^2l$

or say $2{\cdot}6S^2l$

Area of circular end $= \frac{\pi}{4}d^2$

volume $= \frac{\pi d^2 l}{4}$



Odd number of Equidistant slices

Prismoid.

Simpson's Method.—

$$\text{Volume} = \frac{x}{3}(A_1 + 4A_2 + 2A_3 + 4A_4 + A_5)$$

and so on for any odd number of sections.

FIG. 44.

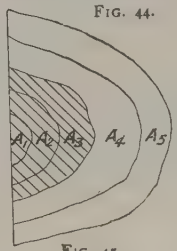


FIG. 45.

Contoured volume.

N.B.—Each area is to be taken as including those within it, *not* the area between the two contours. A_3 is shaded over to make this clear.

Solids of revolution.

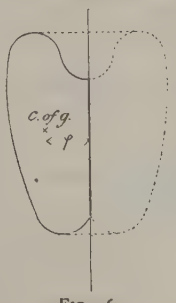


FIG. 46.

Method of Pappus or Guldinus.—

Let A = area of full-lined surface;
ρ = radius of c. of g. of *surface.*

Volume of solid of revolution = $2\pi\rho A$

N.B.—The surface must lie wholly on one side of the axis of revolution, and in the same plane.

This method is applicable to a great number of problems, spheres, cones, rings, etc.

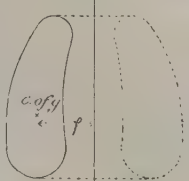


FIG. 47.

Area of end (or side) $= \frac{x}{3}(h_1 + 4h_2 + 2h_3 +, \text{etc.})$ (see p. 34),

where h_1, h_2, etc., are the heights of the sections.

$$\text{Volume} = \frac{x}{3}(h_1 l + 4h_2 l + 2h_3 l +, \text{etc.})$$

$$= \frac{x}{3}(A_1 + 4A_2 + 2A_3 +, \text{etc.})$$

The above proof assumes that the sections are parallelograms, *i.e.* the solid is flat-topped along its length. We shall later on show that the formula is accurate for many solids having surfaces curved in all directions, such as a sphere, ellipsoid, paraboloid, hyperboloid.

If the number of sections be *even*, calculate the volume of the greater portion by this method, and treat the volume of the remainder as a paraboloid of revolution or as a prism.

Let the area be revolved around the axis; then—

The volume swept out by an elemental area a_0, when revolving round the axis at a distance ρ_0 $\Big\}$ $= a_0 \times 2\pi\rho_0$

Ditto ditto a_1 and ρ_1 $= a_1 \times 2\pi\rho_1$

and so on.

Whole volume swept out by all the elemental areas, a_0, a_1, etc., when revolving round the axis at their respective distances, ρ_0, ρ_1, etc. $\Big\}$ $= 2\pi(a_0\rho_0 + a_1\rho_1 +, \text{etc.})$

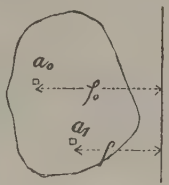


FIG. 46a.

$= 2\pi$(each elemental area, a_0, a_1, etc. × their respective distances, ρ_0, ρ_1, from the axis of revolution)

$= 2\pi$(sum of elemental areas, or whole area × distance of c. of g. of whole area from the axis of revolution) (see p. 58)

$= A \times 2\pi\rho = 2\pi\rho A$

But $2\pi\rho$ is the distance the c. of g. has moved through, or the length of the path of the c. of g.; hence—

Whole volume = area of generating surface × the length of the path of the c. of g. of the area

This proof holds for any part of a revolution, and for any value of ρ; when ρ becomes infinite, the path becomes a straight line, in such a case as a prism.

Sphere.

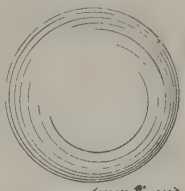


FIG. 48.

Volume of sphere $= \frac{\pi d^3}{6}$, or $\frac{4}{3}\pi r^3$

Volume of sphere $= \frac{2}{3}$ volume of circumscribing cylinder

Hollow sphere.

Volume of hollow sphere $\Big\} = \Big\{$ volume of outer sphere $-$ volume of inner sphere

$$= \frac{\pi d_e^3}{6} - \frac{\pi d_i^3}{6}$$

$$= \frac{\pi}{6}(d_e^3 - d_i^3)$$

Slice of sphere.

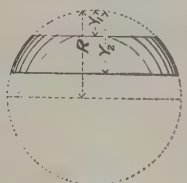


FIG. 49.

Volume of slice $= \frac{\pi}{3}\{3R(Y_2^2 - Y_1^2) - Y_2^3 + Y_1^3\}$

N.B.—The slice must be taken wholly on one side of the diameter; if the slice includes the diameter, it must be treated as two of the following slices.

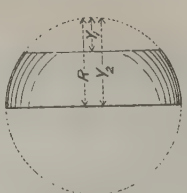


FIG. 50.

Special case in which $Y_2 = R$.

Volume of slice $= \frac{\pi}{3}(2R^3 - 3RY_1^2 + Y_1^3)$

Sphere.—The revolving area is a semicircle of area $\frac{\pi r^2}{2}$

The distance of the c. of g. from the diameter $\Big\} = \rho = \frac{4r}{3\pi}$ (see p. 66)

$$\text{volume swept out} = 2\pi \times \frac{4r}{3\pi} \times \frac{\pi r^2}{2} = \tfrac{4}{3}\pi r^3 = \frac{\pi d^3}{6}$$

or by Simpson's rule—

$$\text{Volume of sphere} = \frac{r}{3}(0 + 4\pi r^2 + 0) = \tfrac{4}{3}\pi r^3$$

Volume of elemental slice $= \pi c^2 dy$

$$= \pi\{R^2 - (R^2 + y^2 - 2Ry)\}dy$$

$$= \pi(2Ry - y^2)dy$$

Volume of whole slice $\Big\} = \pi \int_{y=Y_1}^{y=Y_2} (2Ry - y^2)dy$

$$= \pi \left[\frac{2Ry^2}{2} - \frac{y^3}{3}\right]_{y=Y_1}^{y=Y_2}$$

$$= \pi\left\{\frac{2R(Y_2^2 - Y_1^2)}{2} - \frac{Y_2^3 - Y_1^3}{3}\right\}$$

$$= \frac{\pi}{3}\{3R(Y_2^2 - Y_1^2) - Y_2^3 + Y_1^3\}$$

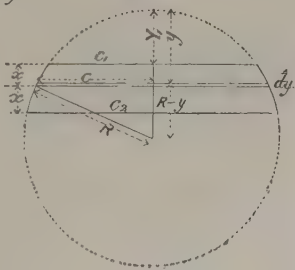


FIG. 49a.

The same result can be obtained by Simpson's method—

$$\text{Volume} = \frac{x}{3}(\pi C_2^2 + 4\pi C^2 + \pi C_1^2)$$

For x substitute $\frac{Y_2 - Y_1}{2}$

c^2 ,, $(2Ry - y^2)$ with the proper suffixes.

The algebraic work is long, but the results by the two methods will be found to be identical.

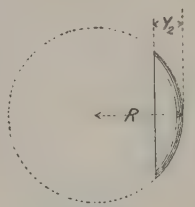


FIG. 51.

Special case in which $Y_1 = 0$.

$$\text{Volume of slice} = \frac{\pi}{3}(3RY_2^2 - Y_2^3)$$

When $Y_2 = R$, and $Y_1 = 0$, the slice becomes a hemisphere, and the—

$$\text{Volume of hemisphere} = \frac{\pi}{3}(2R^3) = \tfrac{2}{3}\pi R^3$$

which is one-half the volume of the sphere found by the other method.

Paraboloid.

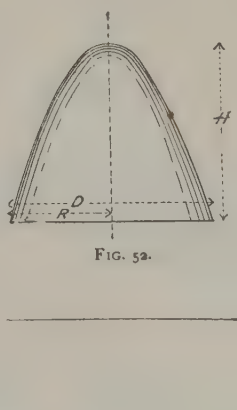


FIG. 52.

$$\text{Volume of paraboloid} = \frac{\pi}{2}R^2H, \text{ or } \frac{\pi}{8}D^2H$$
$$= 1{\cdot}57R^2H, \text{ or } 0{\cdot}39D^2H$$
$$= \tfrac{1}{2} \text{ volume circumscribing cylinder}$$

Cone.

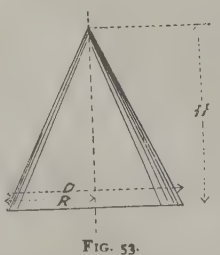


FIG. 53.

$$\text{Volume of cone} = \frac{\pi}{3}R^2H = \frac{\pi}{12}D^2H$$
$$= \tfrac{1}{3} \text{ volume circumscribing cylinder}$$

(*Continued from page* 45.)

For the hemisphere it comes out very easily, thus—

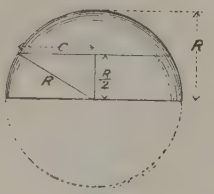


FIG. 51*a*.

$$x = \frac{R}{2} \qquad c^2 = R^2 - \frac{R^2}{4}$$

$$\text{Volume} = \frac{R}{6}\{0 + \pi(4R^2 - R^2) + \pi R^2\}$$
$$= \tfrac{2}{3}\pi R^3$$

From the properties of the parabola, we have—

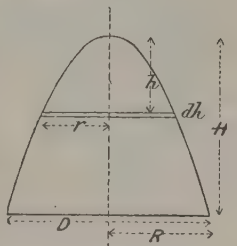


FIG. 52*a*.

$$\frac{r^2}{R^2} = \frac{h}{H}$$

$$r^2 = \frac{R^2 h}{H}$$

$$\text{Volume of slice} = \pi r^2 dh = \frac{\pi R^2 h}{H} dh$$

$$\text{volume of solid} = \frac{\pi R^2}{H} \int_0^H h \, . \, dh$$

$$= \frac{\pi R^2 H^2}{2H} = \frac{\pi R^2 H}{2}$$

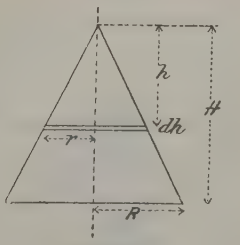


FIG. 53*a*.

$$\frac{r}{R} = \frac{h}{H}$$

$$r = \frac{Rh}{H}$$

$$\text{Volume of slice} = \pi r^2 dh = \frac{\pi R^2 h^2}{H^2} dh$$

$$\text{volume of cone} = \frac{\pi R^2}{H^2} \int_0^H h^2 dh$$

$$= \frac{\pi R^2}{H^2} \times \frac{H^3}{3} = \frac{\pi R^2 H}{3}$$

Pyramid.

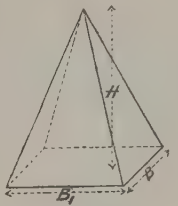


FIG. 54.

$$\text{Volume of pyramid} = \frac{B_1 B H}{3}$$
$$= \frac{H^3}{3}, \text{ when } B_1 = B = H$$
$$= \tfrac{1}{3} \text{ volume circumscribing solid}$$

Slightly tapered body.

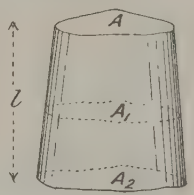


FIG. 55.

Mean Areas Method.—

$$\text{Volume of body} = \left(\frac{A + A_1 + A_2}{3}\right) l \text{ (approx.)}$$
$$= (\text{mean area}) l$$

Ring.

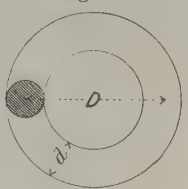


FIG. 56.

$$\text{Volume of ring} = \frac{\pi d^2}{4} \times \pi D = 2{\cdot}47 d^2 D$$

WEIGHT OF MATERIALS.

Aluminium	0·093	lb. per cubic inch.
Brass and bronze	0·30	,, ,,
Copper	0·32	,, ,,
Iron—cast	0·26	,, ,,
,, wrought	0·278	,, ,,
Steel	0·283	,, ,,
Lead	0·412	,, ,,
Brickwork	100 to 140	lbs. per cubic foot.
Stone	150 to 180	,, ,,

This may be proved in precisely the same manner as the cone, or thus by Simpson's method—

$$\text{Volume} = \frac{H}{2 \times 3}\left\{O + 4\left(\frac{B}{2} \times \frac{B_1}{2}\right) + B \times B_1\right\}$$

$$= \frac{H}{6}(2BB_1) = \frac{BB_1H}{3}$$

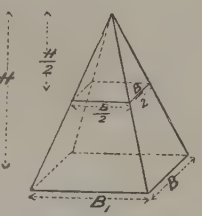


Fig. 54*a*.

This method is only approximately true when the taper is very slight. For such a body as a pyramid it would be seriously in error; the volume obtained by this method would be $\frac{5}{12}H^3$ instead of $\frac{4}{12}H^3$.

The diameter D is measured from centre to centre of the sections of the ring, *i.e.* their centres of gravity—

Volume = area of surface of revolution × length of path of c. of g. of section

$$= \frac{\pi d^2}{4} \times \pi D = \frac{\pi^2 d^2 D}{4}$$

Weight of Materials.

Concrete	130 to 150 lbs.	per	cubic foot.
Pine and larch	30 to 40	,,	,,
Pitch pine and oak	40 to 60	,,	,,
Teak	45 to 55	,,	,,
Greenheart	65 to 75	,,	,,

CHAPTER III.

MOMENTS.

THAT branch of applied mechanics which deals with moments is of the utmost importance to the engineer, and yet perhaps it gives the beginner more trouble than any other part of the subject. The following simple illustrations may possibly help to make the matter clear. We have already (see p. 12) explained the meaning of the terms "clockwise" and "contra-clockwise" moments.

In the figures that follow, the two pulleys of radii R and R_1 are attached to the same shaft, so that they rotate together. We shall assume that there is no friction on the axle.

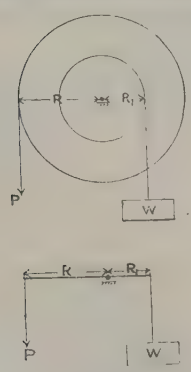


FIG. 57.

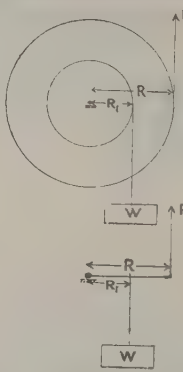


FIG. 58.

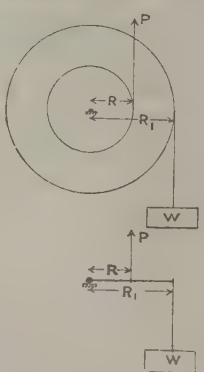


FIG. 59.

Let a cord be wound round each pulley in such a manner that when a force P is applied to one cord, the weight W will be lifted by the other.

Now let the cord be pulled through a sufficient distance to cause the pulleys to make one complete revolution; we shall then have—

The work done by pulling the cord $= P \times 2\pi R$
„ „ in lifting the weight $= W \times 2\pi R_1$

These must be equal, as it is assumed that no work is wasted in friction; hence—

$$P 2\pi R = W 2\pi R_1$$
$$\text{or } PR = WR_1$$

or the contra-clockwise moment = the clockwise moment

It is clear that this relation will hold for any portion of a revolution, however small; also for any size of pulleys.

The levers shown in the same figures may be regarded as small portions of the pulleys; hence the same relations hold in their case.

It may be stated as a general principle that if a rigid body be in equilibrium under any given system of moments, the algebraic sum of all the moments in any given plane must be zero, or the clockwise moments must be equal to the contra-clockwise moments.

First Moments.—The product of a $\left\{\begin{matrix}\text{force } (f)\\ \text{mass } (m)\\ \text{area } (a)\\ \text{volume } (v)\end{matrix}\right\}$ by the length of its arm l, viz. $\left\{\begin{matrix}fl\\ ml\\ al\\ vl\end{matrix}\right\}$, is termed the *first moment* of the $\left\{\begin{matrix}\text{force}\\ \text{mass}\\ \text{area}\\ \text{volume}\end{matrix}\right\}$, or sometimes simply the *moment.*

A statement of the first moment of a $\left\{\begin{matrix}\text{force}\\ \text{mass}\\ \text{area}\\ \text{volume}\end{matrix}\right\}$ must consist of the product of $\left\{\begin{matrix}\text{force units} \times \text{length units.}\\ \text{mass units} \times \text{length units.}\\ \text{area units} \times \text{length units.}\\ \text{volume units} \times \text{length units.}\end{matrix}\right\}$

In speaking of moments, we shall always put the units of force, etc., first, and the length units afterwards. For example, we shall speak of a moment as so many pounds-feet or tons-inches, to avoid confusion with work units.

Second Moments.—The product of a $\left\{\begin{matrix}\text{force } (f)\\ \text{mass } (m)\\ \text{area } (a)\\ \text{volume } (v)\end{matrix}\right\}$ by the square or second power of the length (l) of its arm, viz. $\left\{\begin{matrix}fl^2\\ ml^2\\ al^2\\ vl^2\end{matrix}\right\}$, is termed the *second* moment of the $\left\{\begin{matrix}\text{force}\\ \text{mass}\\ \text{area}\\ \text{volume}\end{matrix}\right\}$. The *second moment* of a volume or an area is sometimes termed the "moment of inertia" (see p. 78) of the volume or area. Strictly, this term should only be used when dealing with questions involving the inertia of bodies; but in other cases, where the second moment has nothing whatever to do with inertia, the term "second moment" is preferable.

A statement of the second moment of a $\left\{\begin{matrix}\text{force}\\ \text{mass}\\ \text{area}\\ \text{volume}\end{matrix}\right\}$ must consist of the product of $\left\{\begin{matrix}\text{force units} \times (\text{length units})^2.\\ \text{mass units} \times (\text{length units})^2.\\ \text{area units} \times (\text{length units})^2.\\ \text{volume units} \times (\text{length units})^2.\end{matrix}\right\}$

First Moments.

Levers.

Neglecting weight of lever.

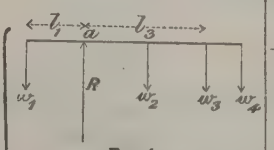


FIG. 60.

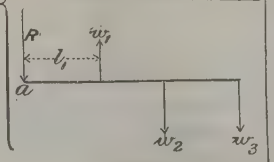
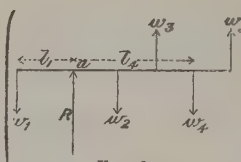
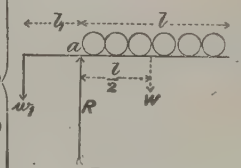
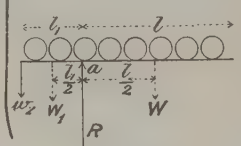


FIG. 61.

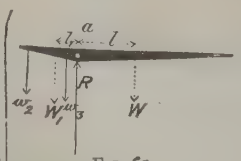
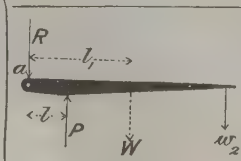
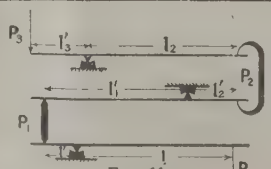
Clockwise moments about the point a.	Contra-clockwise moments about the point a.
$w_2l_2 + w_3l_3 + w_4l_4$	$= w_1l_1$
$w_2l_2 + w_3l_3$	$= w_1l_1$

Reaction R at fulcrum *a*, *i.e.* the resultant of all the forces acting on lever.	REMARKS.
$w_1+w_2+w_3+w_4$	To save confusion in the diagrams, the l has in some cases been omitted. In every case the suffix of l indicates the distance of the weight w bearing the same suffix from the fulcrum.
$w_1-w_2-w_3$	

Neglecting weight of lever.

	Clockwise moments about the point *a*.	Contra-clockwise moments about the point *a*.
 FIG. 62.	$w_2 l_2 + w_4 l_4$	$= w_1 l_1 + w_3 l_3 + w_5 l_5$
 FIG. 63.	$W\frac{l}{2}$ or $\frac{wl^2}{2}$ If w = distributed load per unit length, $wl = W$	$= w_1 l_1$
 FIG. 64.	$W\frac{l}{2}$ or $\frac{wl^2}{2}$	$= W_1\frac{l_1}{2} + w_1 l_1$ or $\frac{wl_1^2}{2} + w_1 l_1$

Allowing for weight of lever.

 FIG. 65.	Wl	$= W_1 l_1 + w_2 l_2 + w_3 l_3$
	W = weight of long arm of lever	l = distance of c. of g. of long arm from *a*
	W_1 = weight of short arm of lever	l_1 = distance of c. of g. of short arm from *a*
 FIG. 66.	$w_2 l_2 + Wl_1$	$= Pl$
	W = weight of whole lever	l_1 = distance of c. of g. of lever from *a*
 FIG. 66*a*.	$Pl + P_1 l_1' + P_2 l_2$ or $\frac{P l l_1' l_2}{l' l_2' l_3'}$	$= P_1 l' + P_2 l_2' + P_3 l_3'$ $= P_3$

Reaction R at fulcrum *a*, *i.e.* the resultant of all the forces acting on lever.	Remarks.
$w_1+w_2-w_3+w_4-w_5$	
w_1+W	N.B.—The unit length for *w* must be of the same kind as the length units of the lever.
w_1+W_1+W	
$w_2+W_1+w_3+W$	This is the arrangement of the lever of the Buckton testing machine. Instead of using a huge balance weight on the short arm, the travelling weight w_2 has a contra-clockwise moment when the lever is balanced, and the load on the specimen, viz. w_3, is zero. As w_2 moves along its moment is decreased, and consequently the load w_3 is increased. When w_2 passes over the fulcrum, its moment is clockwise; then we have $Wl \times w_2l_2 = W_1l_1 + w_3l_3$.
$W+w_2-P$	This is the arrangement of an ordinary lever safety-valve, where P is the pressure on the valve.
	The weight of the levers may be taken into account by the method already shown.

Bell-crank levers. Neglecting weight of lever.

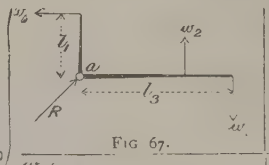


FIG 67.

Clockwise moments about the point *a*.	Contra-clockwise moments about the point *a*.
$w_3 l_3$	$= w_1 l_1 + w_2 l_2$

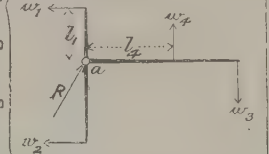


FIG. 68.

$w_2 l_2 + w_3 l_3$	$= w_1 l_1 + w_4 l_4$

Allowing for weight of lever.

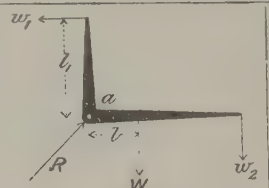
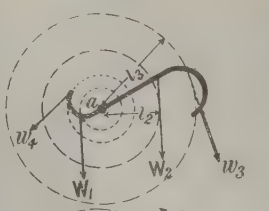


FIG. 69.

$Wl + w_2 l_2$	$= w_1 l_1$
W = weight of horizontal arm	l = distance of c. of g. from *a*

FIG. 70.



$W_2 l_2 + w_3 l_3$	$= W_1 l_1 + w_4 l_4$
W_2 = weight of long curved arm of lever	l_2 = distance of c. of g. of long arm from *a*
W_1 = ditto short arm	l_1 = ditto short arm
	l_4 = perpendicular distance of the line of w_4 from *a*

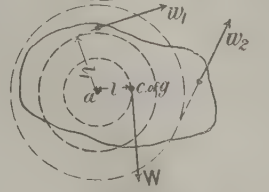


FIG. 71

$Wl + w_1 l_1$	$= w_2 l_2$
W = weight of body	l = perpendicular distance of force W from *a*

Reaction R.	Remarks.
In all these cases it must be found by the parallelogram of forces.	It should be noticed that the direction of the resultant R varies with the *position* of the weights; hence, if a bell-crank lever be fitted with a knife-edge, and the weights travel along, as in some types of testing-machines, the resultant passes through the knife-edge, but not always normal to the seating, thus causing it to chimble away, or to damage its fine edge.
	The shape of the lever makes no difference whatever to the leverage. Consider each force as acting through a cord wrapped round the pulleys as shown, then it will be seen that the moment of each force is the product of the force and the radius of the pulley from which the cord proceeds, *i.e.* the perpendicular distance of the line of action of the force from the fulcrum.

Fig. 68*a*.

Centres of Gravity, and Centroids.—We have already given the following definition of the centre of gravity (see p. 13). If a point be so chosen in a body that the sum of the moments of all the gravitational forces acting on the several particles about the one side of any straight line passing through that point, be equal to the sum of the moments on the other side of the line, that point is termed the centre of gravity; or if the moments on the one side of the line be termed positive (+), and the moments on the other side of the line be termed negative (−), the sum of the moments will be zero.

From this definition it will be seen that, as the particles of any body are acted upon by a system of parallel forces, viz. gravity acting upon each, the algebraic sum of the moments of these forces about a line must be zero when that line passes through the c. of g. of the body.

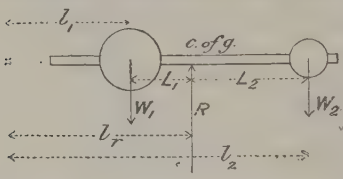


FIG. 72.

Let the weights W_1, W_2, be attached, as shown, to a balanced rod—we need not consider the rod itself, as it is balanced—then, by our definition of the c. of g., we have $W_1L_1 = W_2L_2$.

In finding the position of the c. of g., it will be more convenient to take moments about another point, say x, distant l_1 and l_2 from W_1 and W_2 respectively, and distant l_r (at present unknown) from the c. of g.

$$W_1l_1 + W_2L_2 = Rl_r$$
$$= (W_1 + W_2)l_r$$
$$l_r = \frac{W_1l_1 + W_2l_2}{W_1 + W_2}$$

If we are dealing with a thin sheet of uniform thickness and weighing K pounds per unit of area, the weight of any given portion will be Ka pounds. Then we may put $W_1 = Ka_1$ and $W_2 = Ka_2$;

$$\text{and } l = \frac{K(a_1l_1 + a_2l_2)}{K(a_1 + a_2)} = \frac{a_1l_1 + a_2l_2}{A}$$

or, expressed in words—

distance of c. of g. from the point x

$$= \frac{\text{the sum of the moments of all elemental surfaces about } x}{\text{area of surface}}$$

$$\text{or} = \frac{\text{the moment of surface about } x}{\text{area of surface}}$$

where $A = a_1 + a_2 =$ whole area.

In an actual case there will, of course, be a great number of elemental areas, a, a_1, a_2, a_3, etc., with their corresponding arms, l, l_1, l_2, l_3, etc. Only two have been taken above, they being sufficient to show the principle involved.

When dealing with a body at rest, we may consider its whole mass as being concentrated at its centre of gravity.

When speaking of the c. of g. of a thin weightless lamina or a geometrical surface, it is better to use the term "centroid" instead of centre of gravity.

Position of Centre of Gravity, or Centroid.

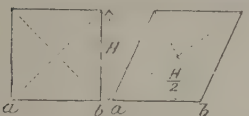


Fig. 73.

Parallelograms.

Intersection of diagonals.

Height above base $ab = \frac{H}{2}$

In a symmetrical figure it is evident that the c. of g. lies on the axis of symmetry. A parallelogram has two axes of symmetry, viz. the diagonals; hence the c. of g. lies on each, and therefore at their intersection, and as they bisect one another, the intersection is at a height $\frac{H}{2}$ from the base.

Triangle.

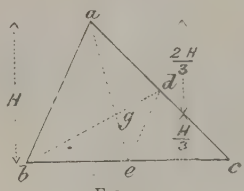


FIG. 74.

Intersection of *ae* and *bd*, where *d* and *e* are the middle points of *ac* and *bc* respectively.

$$\text{Height above base } bc = \frac{H}{3}$$

$$\text{height above apex } a = \frac{2H}{3}$$

Triangle.

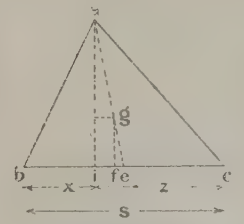


FIG. 74a.

Distance of c. of g. from *b*

$$= bf = \frac{x + S}{3}$$

$$\text{Ditto from } c = cf = \frac{z + S}{3}$$

Trapezium.

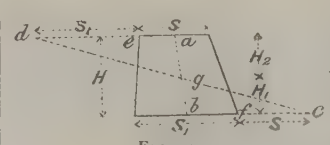


FIG. 75.

Intersection of *ab* and *cd*, where *a* and *b* are the middle points of S and S_1, and $ed = S_1$, $fc = S$.

$$\left.\begin{array}{l}\text{Height above}\\ \text{base } H_1\end{array}\right\} = \frac{H(2S + S_1)}{3(S + S_1)}$$

$$\left.\begin{array}{l}\text{depth below}\\ \text{top } H_2\end{array}\right\} = \frac{H(S + 2S_1)}{3(S + S_1)}$$

Conceive the triangle divided up into a great number of very narrow strips parallel to one of the sides, viz. *bc*. It is evident that the c. of g. of each strip will be at the middle points of each, and therefore will lie on a line drawn from the opposite angle point *a* to the middle point of the side *e*, *i.e.* on *ae*; likewise it will lie on *bd*; therefore the c. of g. is at the intersection of *ae* and *bd*, viz. *g*.

Join *de*. Then by construction $ad = dc = \frac{ac}{2}$, and $be = ec = \frac{bc}{2}$; hence the triangles *acb* and *dce* are similar, and therefore $de = \frac{ab}{2}$. The triangles *agb* and *dge* are also similar, hence $eg = \frac{ag}{2} = \frac{ae}{3}$.

Since *g* is situated at $\frac{1}{3}$ height from the base, we have

$$if = \frac{2}{3} \quad i.e. = \frac{2}{3}\left(\frac{S}{2} - x\right)$$

$$if + x = bf = \frac{x + S}{3}$$

Draw the dotted line parallel to the sloping side of the trapezium in Fig. 75*a*.

Height of c. of g. of figure from base

$$= H_1 = \frac{\text{area of parallg.} \times \text{ht. of its c. of g.} + \text{area of} \triangle \times \text{ht. of its c. of g.}}{\text{area of whole figure}}$$

$$H_1 = \frac{\frac{SH \times H}{2} + (S_1 - S)\frac{H}{2} \times \frac{H}{3}}{\left(\frac{S + S_1}{2}\right)H} = \frac{H(2S + S_1)}{3(S + S_1)}$$

$$H_2 = \frac{\frac{SH \times H}{2} + (S_1 - S)\frac{H}{2} \times \frac{2H}{3}}{\left(\frac{S + S_1}{2}\right)H} = \frac{H(S + 2S_1)}{3(S + S_1)}$$

$$\frac{H_1}{H_2} = \frac{2S + S_1}{2S_1 + S} = \frac{S + \frac{S_1}{2}}{S_1 + \frac{S}{2}}$$

or $\frac{bg}{ga} = \frac{bc}{ad}$

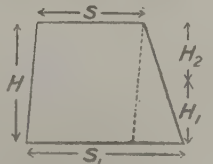


FIG. 75*a*.

Position of Centre of Gravity, or Centroid.

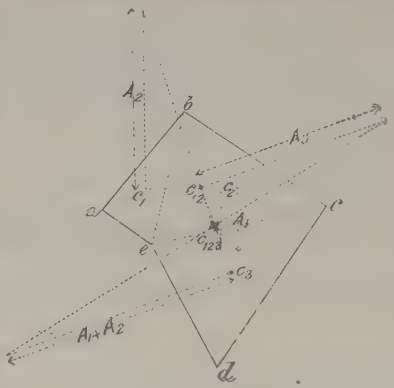


Fig. 76.

Area *abe* = A_1
bce = A_2
cde = A_3
c. of g. of area *abe* C_1
" " *bce* C_2
" " *cde* C_3
" " whole figure $C_{1.2.3.}$

Trapezium and triangles.

Intersection of line joining c. of g. of triangle and c. of g. of trapezium, viz. *ab* and *cd*, where *ac* = area of trapezium, and *db* area of triangle. *ac* is parallel to *bd*.

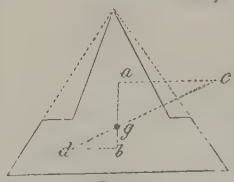


Fig. 77.

Lamina with hole.

Let A = area *abcde*;
H = height of its c. of g. from *ed*;
H_1 = height of its c. of g. from *df* drawn at right angles to *ed*;
a = area of hole *gji*;
h = height of its c. of g. from *ed*;
h_1 = height of its c. of g. from *df*.

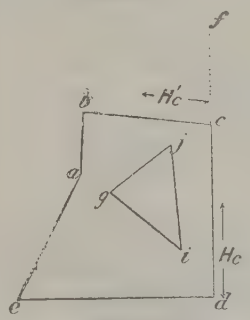


Fig. 78.

Then—

Hc = height of c. of g. of whole figure from *ed*
$H'c$ = height of c. of g. of whole figure from *df*

$$Hc = \frac{AH - ah}{A - a}$$

$$H'c = \frac{AH_1 - ah}{A - a}$$

The principle of these graphic methods is as follows:—

Let the centres of gravity of two areas, A_1 and A_2, be situated at points C_1 and C_2 respectively, and let the common centre of gravity be situated at c, distant x_1 from C_1, and x_2 from C_2; then we shall have $A_1x_1 = A_2x_2$. From C_2 set off a line c_2b_2, whose length represents on some given scale the area A_1, and from C_1 a line c_1b_1 parallel to it, whose length represents on the same scale the area A_2. Join b_1, b_2. Then the intersection of b_1b_2 and C_1C_2 is the common centre of gravity c. The two triangles are similar, therefore—

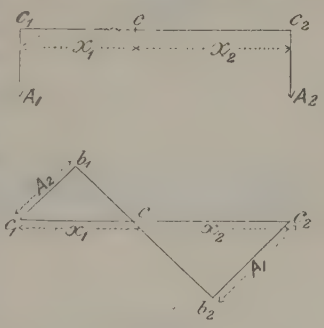


FIG. 76a.

$$\frac{A_1}{A_2} = \frac{x_2}{x_1}, \text{ or } A_1x_1 = A_2x_2$$

N.B.—The lines C_1b_1 and C_2b_2 are set off on *opposite* sides of C_1, C_2, and at opposite ends to their respective areas, at any convenient angle; but it is undesirable to have a very acute angle at c, otherwise the point will not be well defined. When one of the areas, say A_2, is negative, *i.e.* is the area of a hole or a part cut out of a lamina, then the lines c_1b_1 and c_2b_2 must be set off on the same side of the line, thus—

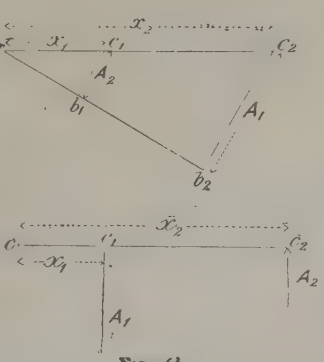


FIG. 76b.

Then—

$$\frac{A_1}{A_2} = \frac{x_2}{x_1}, \text{ or } A_1x_1 = A_2x_2$$

Position of Centre of Gravity, or Centroid.

Graphical method.

Lamina with hole.

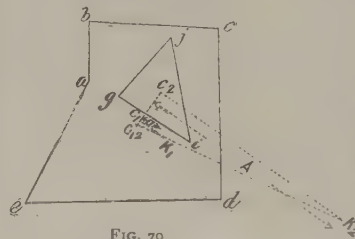


Fig. 70

If c_1 be the c. of g. of *abcde*;
c_2 „ „ *gji*.

Join c_1, c_2, and produce; set off c_2K_2 and c_1K_1 parallel to one another and equal to A and *a* respectively; through the end points K_2, K_1 draw a line to meet the line through c_1, c_2 in $c_{1,2}$, which is the c. of g. of the whole figure.

Note.—The lines c_2K_2, c_1K_1 need not be at right angles to the line c_1c_2, but the line K_2K_1 should not cut it at a very acute angle.

Portion of a regular polygon or an arc of a circle, considered as a thin wire.

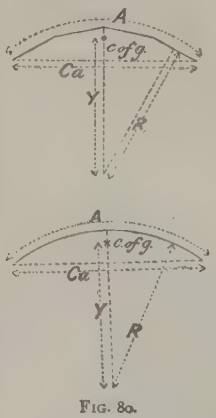


Fig. 80.

Let A = length of the sides of the polygon, or the length of the arc in the case of a circle;
R = radius of a circle inscribed in the polygon, or the radius of the circle itself;
C_a = chord of the arc of the polygon or circle;
Y = distance of the c. of g. from the centre of the circle.

Then $A : R :: C_a : Y$

$$\text{or } \frac{A}{R} = \frac{C_a}{Y}$$

$$\text{or } Y = \frac{R C_a}{A}$$

N.B.—The same expression holds for an arc greater than a semicircle.

See p. 63.

Regard each side of the polygon as a piece of wire of length l; the c. of g. of each side will be at the middle point, and distant y_1, y_2, y_3, etc., from the diameter of the inscribed circle; and let the projected length of each side on the diameter be c_1, c_2, c_3, etc.

The triangles *def* and O*ba* are similar;

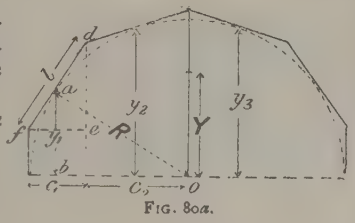


FIG. 80*a*.

therefore $\frac{fd}{fe} = \frac{Oa}{ab}$, or $\frac{l}{c_1} = \frac{R}{y_1}$

$$\text{and } y_1 = \frac{Rc_1}{l}$$

$$\text{likewise } y_2 = \frac{Rc_2}{l}, \text{ and so on}$$

Let Y = distance of c. of g. of portion of polygon from the centre O;

w = weight of each side of the polygon.

Then—

$$Y = \frac{wy_1 + wy_2 +, \text{ etc.}}{wn}$$

where n = number of sides. The w cancels top and bottom.

Substituting the values of y_1, y_2, etc., found above, we have—

$$Y = \frac{R}{nl}(c_1 + c_2 +, \text{ etc.})$$

(*Proof concluded on p.* 67.)

Position of Centre of Gravity or Centroid.

Semicircular arc or wire.

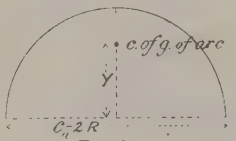


Fig. 81.

$$Y = \frac{2R}{\pi} = \frac{D}{\pi}$$

Circular sector considered as a thin sheet.

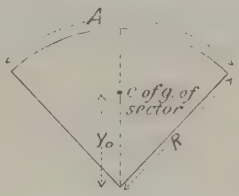


Fig. 82.

$$Y_0 = \frac{2RC_c}{3A}$$

Semicircular lamina or sheet.

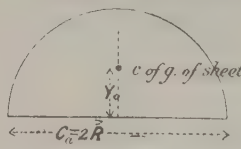


Fig. 83.

$$Y_0 = \frac{4R}{3\pi} = \frac{2D}{3\pi}$$

Parabolic segment.

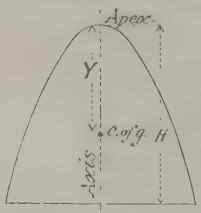


Fig. 84.

$$Y = \tfrac{3}{5}H$$

where Y = distance of c. of g. from apex.

The figure being symmetrical, the c. of g. lies on the axis.

but $c_1 + c_2 +$, etc. = the whole chord subtended by the sides of the polygon

$$= C_a$$

and $nl = A$

$$Y = \frac{RC_a}{A}$$

When n becomes infinitely great, the polygon becomes a circle.

The axis of symmetry on which the c. of g. lies is a line drawn from the centre of the circle at right angles to the chord.

In the case of the sector of a polygon or a circle, we have to find the c. of g. of a series of triangles, instead of their bases, which, as we have shown before, is situated at a distance equal to two-thirds of their height from the apex; hence the c. of g. of a sector is situated at a distance $= \frac{2}{3}Y$ from the centre of the inscribed circle.

From the properties of the parabola, we have—

$$\frac{h}{H} = \frac{b^2}{B^2}$$

$$b = \frac{Bh^{\frac{1}{2}}}{H^{\frac{1}{2}}}$$

Area of strip $= b \,.\, dh = \frac{Bh^{\frac{1}{2}}}{H^{\frac{1}{2}}} dh$

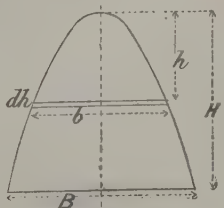


FIG. 84a.

(*Continued on p.* 69.)

POSITION OF CENTRE OF GRAVITY OR CENTROID.

Parabolic segment.

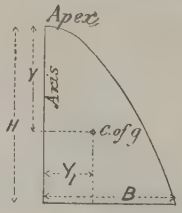


FIG. 85

$Y = \frac{3}{5}H$ (see above)

$Y_1 = \frac{3}{8}B$

where Y_1 = distance of c. of g. from axis.

$$\text{moment of strip about apex} = h \,.\, b \,.\, dh = \frac{\mathrm{B}h^{\frac{3}{2}}}{\mathrm{H}^{\frac{1}{2}}}dh$$

$$\text{moment of whole figure about apex} = \frac{\mathrm{B}}{\mathrm{H}^{\frac{1}{2}}}\int_0^{\mathrm{H}} h^{\frac{3}{2}}dh = \frac{\mathrm{B}}{\mathrm{H}^{\frac{1}{2}}} \times \frac{\mathrm{H}^{\frac{5}{2}}}{\frac{5}{2}}$$

$$= \tfrac{2}{5}\mathrm{BH}^2$$

$$\text{the area of the figure} = \tfrac{2}{3}\mathrm{BH} \text{ (see p. 30)}$$

$$\text{the dist. of the c. of g. from the apex} = \frac{\frac{2}{5}\mathrm{BH}^2}{\frac{2}{3}\mathrm{BH}} = \tfrac{3}{5}\mathrm{H}$$

From the properties of the parabola, we have—

$$\frac{h}{\mathrm{H}} = \frac{b^2}{\mathrm{B}^2}$$

$$\text{or } \frac{\mathrm{H} - h_0}{\mathrm{H}} = \frac{b^2}{\mathrm{B}^2}$$

$$\mathrm{B}^2(\mathrm{H} - h_0) = b^2\mathrm{H}$$

$$\mathrm{B}^2h_0 = \mathrm{B}^2\mathrm{H} - b^2\mathrm{H}$$

$$h_0 = \frac{\mathrm{H}}{\mathrm{B}^2}(\mathrm{B}^2 - b^2)$$

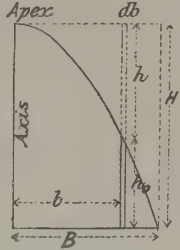


FIG. 85*a*.

$$\text{area of strip} = h_0db$$

$$\text{moment of strip about axis} = b \,.\, h_0db = \frac{\mathrm{H}}{\mathrm{B}^2}(\mathrm{B}^2b - b^3)db$$

$$\left.\begin{matrix}\text{moment of whole area} \\ \text{about axis}\end{matrix}\right\} = \frac{\mathrm{H}}{\mathrm{B}^2}\int_0^{\mathrm{B}}(\mathrm{B}^2b - b^3)db$$

$$\frac{\mathrm{H}}{\mathrm{B}^2}\left[\frac{\mathrm{B}^2b^2}{2} - \frac{b^4}{4}\right]_0^{\mathrm{B}}$$

$$= \frac{\mathrm{H}}{\mathrm{B}^2}\left(\frac{\mathrm{B}^4}{2} - \frac{\mathrm{B}^4}{4}\right) = \frac{\mathrm{HB}^2}{4}$$

$$\text{the area of the figure} = \tfrac{2}{3}\mathrm{BH}$$

$$\left.\begin{matrix}\text{the distance of the c. of g.} \\ \text{from the axis}\end{matrix}\right\} = \frac{\frac{\mathrm{HB}^2}{4}}{\frac{2}{3}\mathrm{BH}} = \tfrac{3}{8}\mathrm{B}$$

Position of Centre of Gravity or Centroid.

Ex-parabolic segment.

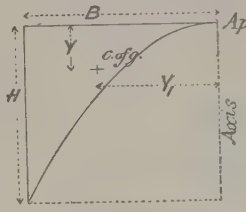
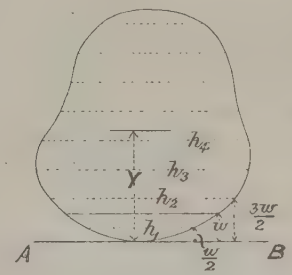


Fig 86.

$$Y_1 = \tfrac{3}{4}B$$
$$Y = \tfrac{3}{10}H$$

where Y_1 = distance of c. of g. from axis;
Y = distance of c. of g. from apex.

Irregular figure.



$$Y = \frac{w(h_1+3h_2+5h_3+7h_4+, \text{etc.})}{2(h_1+h_2+h_3+h_4+, \text{etc.})}$$

where Y = distance of c. of g. from line AB;
w = width of strips;
h = mean height of the strips.

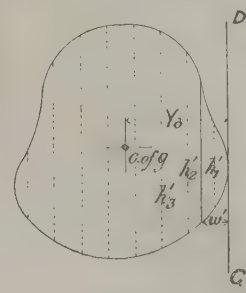


Fig. 87.

$$Y_0 = \frac{w'}{2}\left(\frac{h'_1+3h'_2+5h'_3+7h'_4+, \text{etc.}}{h'_1+h'_2+h'_3+h'_4+, \text{etc.}}\right)$$

where Y_0 = distance of c. of g. from line CD;
w' = width of strips;
h' = mean height of the strips.

By the principle of moments, we have—

$$\text{Distance of c. of g. of figure from axis} = \frac{\left(\text{area of rect.} \times \left\{\begin{matrix}\text{dist. of its c. of g.}\\ \text{from axis}\end{matrix}\right\}\right) - \left(\left\{\begin{matrix}\text{area of para.}\\ \text{segment}\end{matrix}\right\} \times \left\{\begin{matrix}\text{dist. of its c. of g.}\\ \text{from axis}\end{matrix}\right)}{\text{area of figure}}$$

$$Y_1 = \frac{BH \times \frac{B}{2} - \frac{2}{3}BH \times \frac{3}{8}B}{\frac{1}{3}BH} = \frac{3}{4}B$$

Likewise—

$$Y = \frac{BH \times \frac{H}{2} - \frac{2}{3}BH \times \frac{3}{5}H}{\frac{1}{3}BH} = \frac{3}{10}H$$

This is a simple case of moments, in which we have—

$$\left.\begin{matrix}\text{Distance of c. of g.}\\ \text{from line AB}\end{matrix}\right\} = \frac{\text{moment of each strip about AB}}{\text{area of whole figure}}$$

The area of the first strip $= wh_1$	Moment of first strip $= wh_1 \times \frac{w}{2}$
,, ,, second ,, $= wh_2$	,, second ,, $= wh_2 \times \frac{3w}{2}$
,, ,, third ,, $= wh_3$	,, third ,, $= wh_3 \times \frac{5w}{2}$

and so on.

Area of whole figure $= wh_1 + wh_2 + wh_3 +$, etc.

$$\left.\begin{matrix}\text{Distance of c.}\\ \text{of g. from}\\ \text{line AB}\end{matrix}\right\} = Y = \frac{wh_1 \times \frac{w}{2} + wh_2 \times \frac{3w}{2} + wh_3 \times \frac{5w}{2} +, \text{etc.}}{wh_1 + wh_2 + wh_3 +, \text{etc.}}$$

one w cancels out top and bottom, and we have—

$$Y = \frac{w}{2}\left(\frac{h_1 + 3h_2 + 5h_3 + 7h_4 +, \text{etc.}}{h_1 + h_2 + h_3 + h_4 +, \text{etc.}}\right.$$

and similarly with Y_0.

The division of the figure may be done thus: Draw a line, xy, at any angle, and set off equal parts as shown; project the first, third, fifth, etc., on to xz drawn normal to AB.

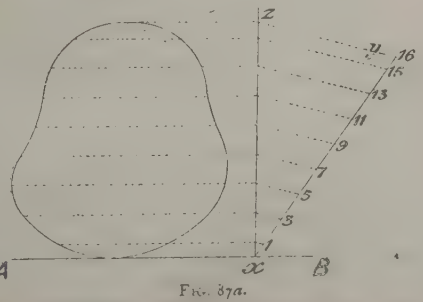


FIG. 87a.

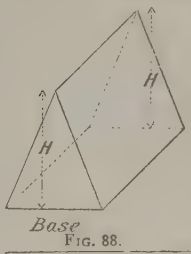


FIG. 88.

POSITION OF CENTRE OF GRAVITY OR CENTROID.

Wedge.

On a plane midway between the ends, and at a height $\frac{H}{3}$ from base.

For frustum of wedge, see Trapezium.

Pyramid or cone.

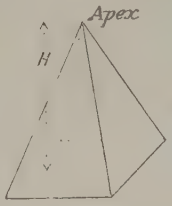


FIG. 89.

On a line drawn from the middle point of the base to the apex, and at a distance $\frac{3}{4}$H from the apex.

Frustum of pyramid or cone.

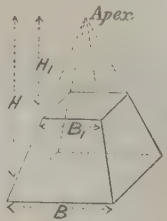


FIG. 90.

On a line drawn from the middle point of the base to the apex, and at a height $\frac{3}{4}H\left(\frac{1-n^4}{1-n^3}\right)$ from the apex, where $n = \frac{H_1}{H}$ or $\frac{B_1}{B}$

A wedge may be considered as a large number of triangular laminæ placed side by side, the c. of g. of each being situated at a height $\frac{H}{3}$ from the base.

Volume of layer $= b^2 \cdot dh$

moment of layer about apex $= b^2 \cdot h \cdot dh$

But $\frac{b}{h} = \frac{B}{H}$

$$b = \frac{h \cdot B}{H}$$

moment of layer about apex $= \frac{B^2 h^3}{H^2} dh$

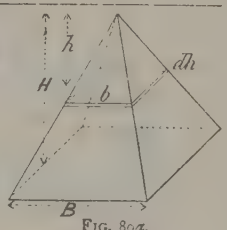


FIG. 89a.

$$\text{moment of the whole pyramid about apex} = \frac{B^2}{H^2}\int_0^H h^3 \cdot dh = \frac{B^2H^4}{4H^2} = \frac{B^2H^2}{4}$$

$$\text{volume of pyramid} = \frac{B^2H}{3}$$

$$\text{distance of c. of g. from apex} = \frac{\frac{B^2H^2}{4}}{\frac{B^2H}{3}} = \tfrac{3}{4}H$$

In the case above, instead of integrating between the limits of H and o for the moment about the apex, we must integrate between the limits H and H_1; thus—

$$\text{Moment of frustum of pyramid about the (imaginary) apex} = \frac{B^2}{H^2}\int_{H_1}^{H} h^3 \cdot dh$$

$$= \frac{B^2}{H^2}\left(\frac{H^4}{4} - \frac{H_1^4}{4}\right) \quad \text{(i.)}$$

$$\text{volume of frustum} = \frac{B^2H}{3} - \frac{B_1^2H_1}{3} \quad \text{(ii.)}$$

substituting the value $\frac{B_1}{B} = \frac{H_1}{H} = n$

$$\text{then the distance of the c. of g. from the apex} = \frac{\text{(i.)}}{\text{(ii.)}} = \tfrac{3}{4}H\left(\frac{1-n^4}{1-n^3}\right)$$

Position of Centre of Gravity or Centroid.

Locomotive or other symmetrical body.

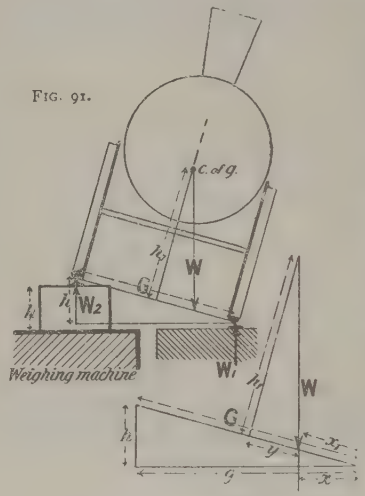


Fig. 91.

Fig. 92.

The height of the c. of g. above the rails can be found graphically, after calculating x, by erecting a perpendicular to cut the centre line.

W, the weight of the engine, is found by weighing it in the ordinary way. W_2 is found by tilting the engine as shown, with one set of wheels resting on blocks on the platform of a weighing machine, and the other set resting on the ground.

Let h_1 be the height of the c. of g. above the rails.

$$h_1 = \frac{G}{h}\left(\frac{1}{2} - \frac{W_2}{W}\right)\sqrt{G^2 - h^2}$$

Irregular surfaces.

Also see Barker's "Graphical Calculus," p. 179, for a graphical integration of irregular surfaces.

By taking moments about the lower rail, we have—

$$Wx = W_2 g$$

$$\text{But } \frac{x}{x_1} = \frac{g}{G}$$

$$\text{whence } \frac{W g x_1}{G} = W_2 g$$

$$x_1 = \frac{W_2 G}{W}$$

$$y = \frac{G}{2} - x_1 = \frac{G}{2} - \frac{W_2 G}{W}$$

By similar triangles—

$$\frac{h_1}{y} = \frac{g}{h} \qquad h_1 = \frac{yg}{h}$$

$$h_1 = \frac{G\left(\frac{1}{2} - \frac{W_2}{W}\right)g}{h}$$

$$h_1 = \frac{G}{h}\left(\frac{1}{2} - \frac{W_2}{W}\right)\sqrt{G^2 - h^2}$$

(These symbols refer to Fig. 92 only.)

The c. of g. is easily found by balancing methods; thus, if the c. of g. of an irregular surface be required, cut out the required figure in thin sheet metal or cardboard, and balance on the edge of a steel straight-edge, thus: The points *a*, *a* and *b*, *b* are marked and afterwards joined: the point where they cut is the c. of g. As a check on the result, it is well to balance about a third line *cc*; the three lines should intersect at one point, and not form a small triangle.

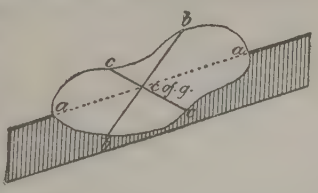


Fig. 93.

The c. of g. of many solids can also be found in a similar manner, or by suspending them by means of a wire, and dropping a perpendicular through the points of suspension.

Second Moments—Moments of Inertia.

A definition of a second moment has been given on p. 52. In every case we shall find the second moment by summing up

or integrating the product of every element of the body or surface by the square of its distance from the axis in question. In some cases we shall find it convenient to make use of the following theorems :—

Let I_0 = the second moment, or moment of inertia, of any surface (treated as a thin lamina) or body about a given axis;

I = the second moment, or moment of inertia, of any surface (treated as a thin lamina) or body about a parallel axis passing through the c. of g.;

M = mass of the body;

A = area of the surface;

R_0 = the perpendicular distance between the two axes.

Then $I_0 = I + MR_0^2$, or $I + AR_0^2$

Let xy be the axis passing through the c. of g.

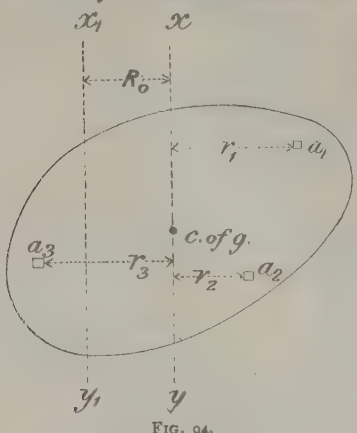


FIG. 94.

Let x_1y_1 be the axis of revolution, parallel to xy and in the plane of the surface or lamina.

Let the elemental areas, a_1, a_2, a_3, etc., be situated at distances r_1, r_2, r_3, etc., from xy. Then we have—

$$\begin{aligned} I_0 &= a_1(R_0 + r_1)^2 + a_2(R_0 + r_2)^2 +, \text{ etc.} \\ &= a_1(R_0^2 + r_1^2 + 2R_0r_1) + a_2(R_0^2 + r_2^2 + 2R_0r_2) +, \text{ etc.} \\ &= a_1r_1^2 + a_2r_2^2 +, \text{ etc.} + R_0^2(a_1 + a_2 +, \text{ etc.}) + 2R_0(a_1r_1 + a_2r_2 +, \text{ etc.}) \end{aligned}$$

But as xy passes through the c. of g. of the section, we have $a_1r_1 + a_2r_2 +$, etc. $= 0$ (see p. 58), for some r's are positive and some negative; hence the latter term vanishes.

The second term, $a_1 + a_2 +$, etc. = the whole area (A); whence it becomes R_0^2A.

In the first term, we have simply the second moment, or moment of inertia, about the axis passing through the c. of g. $= I$; hence we get—

$$I_0 = I + R_0^2A$$

We may, of course, substitute m_1, m_2, etc., for the elemental masses, and M for the mass of the body instead of A.

When a body or surface (treated as a thin lamina) revolves about an axis or pole perpendicular to its plane of revolution, the second moment, or moment of inertia, is termed the second polar moment, or polar moment of inertia.

The second polar moment of any surface is the sum of the second moments about any two rectangular axes in its own plane passing through the axis of revolution, or

$$I_p = I_x + I_y$$

Consider any elemental area a, distant r from the pole.

I_x about $ox = ay^2$
I_y „ $oy = ax^2$
I_p „ pole $= ar^2$
But $r^2 = x^2 + y^2$
and $ar^2 = ax^2 + ay^2$
hence $I_p = I_x + I_y$

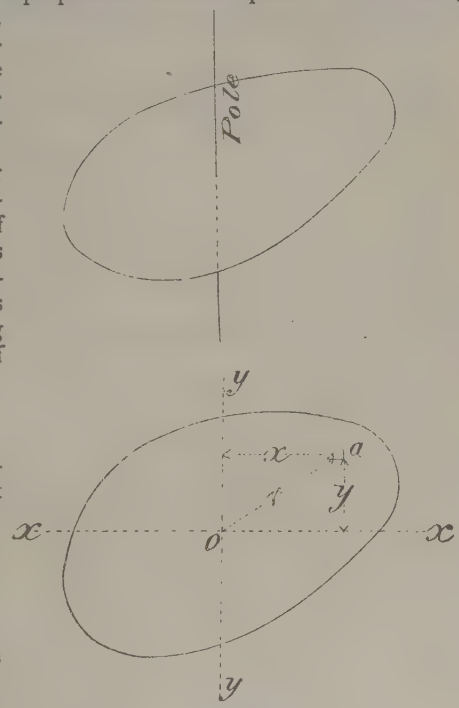


FIG. 95.

In a similar way, it may be proved for every element of the surface.

When finding the position of the c. of g., we had the following relation:—

$$\text{Distance of c. of g. from the axis } xy\ (\kappa) = \frac{\text{first moment of surface about } xy}{\text{area of surface}}$$

$$\text{or} = \frac{\text{first moment of body about } xy}{\text{volume of body}}$$

$$\kappa = \frac{\Sigma ar}{A}$$

Now, when dealing with the second moment, we have a corresponding centre, termed the centre of gyration, at which the whole of a moving body or surface may be considered to be concentrated; the distance of the centre of gyration from the axis of revolution is termed the "radius of gyration." When finding its value, we have the following relation:—

$$\left.\begin{array}{l}\text{Radius of gyration} \\ \text{about the axis } xy\ (\kappa^2)\end{array}\right\}^2 = \frac{\text{second moment of surface about } xy}{\text{area of surface}}$$

$$\text{or} = \frac{\text{second moment of body about } xy}{\text{volume of body}}$$

$$\kappa^2 = \frac{\Sigma a r^2}{A} = \frac{I}{A}, \text{ or } I = A\kappa^2$$

SECOND MOMENT, OR MOMENT OF INERTIA (I).

Parallelogram treated as a thin lamina about its extreme end.

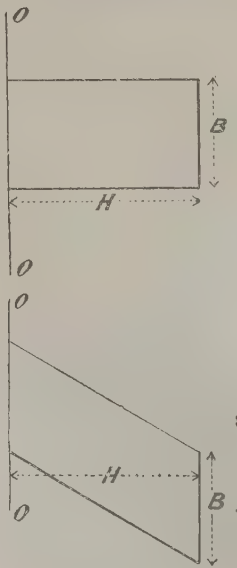


FIG. 96.

$$I_0 = \frac{BH^3}{3}$$

If the figure be a square—

$$B = H = S$$

we have $I_0 = \frac{S^4}{3}$

Radius of gyration κ.

$$\frac{H}{\sqrt{3}}$$

For other cases of moments of inertia or second moments, see Chapter XI., Beams.

Area of elemental strip $= B \cdot dh$
second moment of strip $= B \cdot h^2 \cdot dh$

$$\left.\begin{matrix}\text{second moment}\\ \text{of whole surface}\end{matrix}\right\} = B \int_0^H h^2 \cdot dh = \frac{BH^3}{3}$$

$$\left.\begin{matrix}\text{area of whole}\\ \text{surface}\end{matrix}\right\} = BH$$

$$\left.\begin{matrix}\text{square of radius}\\ \text{of gyration}\end{matrix}\right\} = \frac{BH^3}{3BH} = \frac{H^2}{3}$$

$$\text{radius of gyration} = \frac{H}{\sqrt{3}}$$

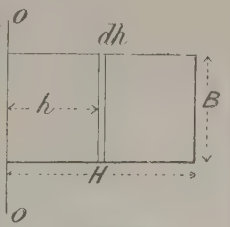


FIG. 96*a*.

It will be seen that the above reasoning holds, however the parallelogram may be distorted sideways, as shown.

Second Moment, or Moment of Inertia (I).

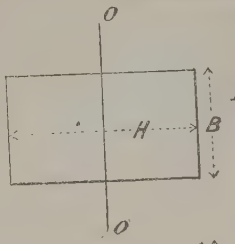
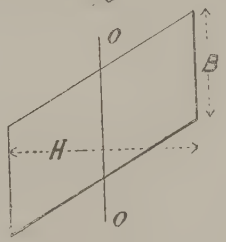
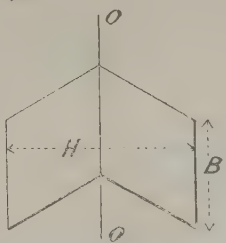
		Radius of gyration κ.
 *Parallelogram treated as a thin lamina about its central axis.*	$I = \frac{BH^3}{12}$ or $\frac{S^4}{12}$	$\frac{H}{\sqrt{12}}$
		
		

FIG. 97.

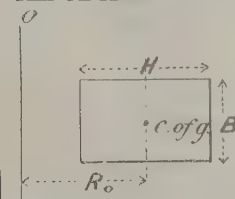
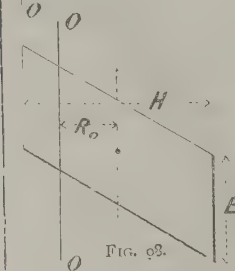
 *Parallelogram treated as a thin lamina about an axis distant R_0 from its c. of g.*	$I_0 = BH\left(\frac{H^2}{12} + R_0^2\right)$	$\sqrt{\frac{H^2}{12} + R_0^2}$
		

FIG. 98.

This is simply a case of two parallelograms such as the above put together axis to axis, each of length $\frac{H}{2}$;

$$\text{Then the second moment of each} = \frac{B\left(\frac{H}{2}\right)^3}{3} = \frac{BH^3}{8 \times 3}$$

$$\left.\begin{array}{r}\text{then the second moment} \\ \text{of the two together}\end{array}\right\} = 2\left(\frac{BH^3}{24}\right) = \frac{BH^3}{12}$$

$$\text{area of whole surface} = BH$$

$$\text{radius of gyration} = \sqrt{\frac{BH^3}{12BH}} = \frac{H}{\sqrt{12}}$$

From the theorem given above (p. 76), we have—

$$I_0 = I + R_0^2 A \qquad I = \frac{BH^3}{12}$$

$$= \frac{BH^3}{12} + R_0^2 BH \qquad A = BH$$

$$= BH\left(\frac{H^2}{12} + R_0^2\right)$$

when $R_0 = 0$, $I_0 = I$

Second Moment, or Moment of Inertia (I).

Hollow parallelogram.

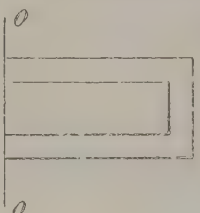


Fig. 99.

Let I_e = second moment of external figure;
I_i = second moment of internal figure;
I_0 = second moment of hollow figure;
$I_0 = I_e - I_i$.

Radius of gyration κ.

Triangle about an axis parallel to the base passing through the apex.

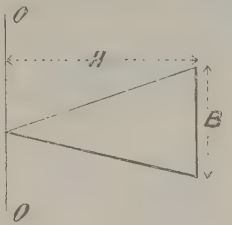


Fig. 100.

$$I_0 = \frac{BH^3}{4}$$

$\kappa = \frac{H}{\sqrt{2}}$

Triangle about an axis parallel to the base passing through the c. of g.

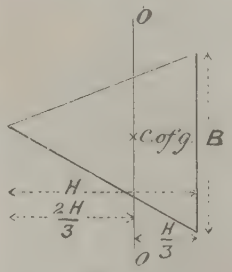


Fig. 101.

$$I = \frac{BH^3}{36}$$

$\kappa = \frac{H}{\sqrt{18}}$

The I_0 for the hollow parallelogram is simply the difference between the I_e for the external, and the I_i for the internal parallelogram.

Area of strip $= b \,.\, dh$; but $b = \frac{hB}{H}$

" " $= \frac{B}{H} h \,.\, dh$

second moment of strip $= \frac{B}{H} h^3 \,.\, dh$

" " triangle $= \frac{B}{H} \int_0^H h^3 \,.\, dh$

" " " $= \frac{BH^4}{4H} = \frac{BH^3}{4}$

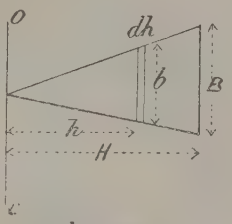


FIG. 100a.

area of triangle $= \frac{BH}{2}$

radius of gyration $= \sqrt{\frac{\frac{BH^3}{4}}{\frac{BH}{2}}} = \sqrt{\frac{H^2}{2}} = \frac{H}{\sqrt{2}}$

From the theorem on p. 76, we have—

$I_0 = I + R_0^2 A$

$I = I_0 - R_0^2 A$

$I = \frac{BH^3}{4} - \frac{4H^2}{9} \times \frac{BH}{2}$

$I = \frac{BH^3}{36}$

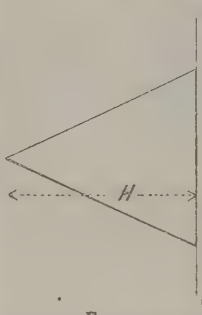
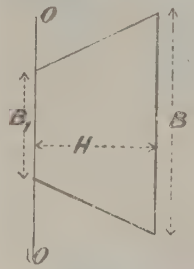
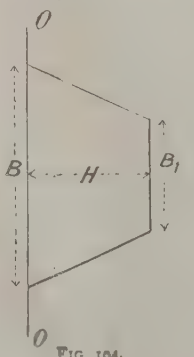
$I_0 = \frac{BH^3}{4}$

$R_0^2 = \left(\frac{2H}{3}\right)^2 = \frac{4H^2}{9}$

$A = \frac{BH}{2}$

radius of gyration $= \sqrt{\frac{\frac{BH^3}{36}}{\frac{BH}{2}}} = \sqrt{\frac{H^2}{18}} = \frac{H}{\sqrt{18}}$

SECOND MOMENT, OR MOMENT OF INERTIA (I).

		Radius of gyration k.
Triangle about an axis at the base.  FIG. 102.	$I_0 = \frac{BH^3}{12}$	$H\sqrt{\frac{1}{6}}$
Trapezium about an axis coinciding with its short base.  FIG. 103.	$I_0 = \frac{(3B + B_1)H^3}{12}$ Let $B_1 = nB$. $I_0 = \frac{BH^3}{12}(3 + n)$	$H\sqrt{\frac{n+3}{6(n+1)}}$
Trapezium about an axis coinciding with its long base.  FIG. 104.	$I_0 = \frac{(3B_1 + B)H^3}{12}$ or $\frac{BH^3}{12}(3n + 1)$ $n = \frac{B_1}{B}$	$H\sqrt{\frac{3n+1}{6(n+1)}}$

From the theorem quoted above, we have—

$$I_0 = I + R_0^2 A \qquad R_0^2 = \frac{H^2}{9}$$

$$I_0 = \frac{BH^3}{36} + \frac{H^2}{9} \times \frac{BH}{2}$$

$$I_0 = \frac{BH^3}{12}$$

Radius of gyration obtained as in the last case.

This figure may be treated as a parallelogram and a triangle about an axis passing through the apex.

For parallelogram, $I_0 = \frac{B_1H^3}{3}$

for triangle, $I_0 = \frac{(B - B_1)H^3}{4}$

for trapezium, $I_0 = \frac{B_1H^3}{3} + \frac{(B - B_1)H^3}{4}$

" " $I_0 = \frac{(3B + B_1)H^3}{12}$

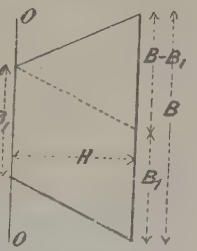


FIG. 103a

When the axis coincides with the long base, the I for the triangle $= \frac{(B - B_1)H^3}{12}$; then, adding the I for the parallelogram as above, we get the result as given.

When $n = 1$, the figures become parallelograms, and $I = \frac{BH^3}{3}$, as found above.

When $n = 0$, the figures become triangles, and the $I = \frac{BH^3}{4}$ for the first case, as found for the triangle about its apex; and $I = \frac{BH^3}{12}$ for the second case, as found for the triangle about its base.

Second Moment, or Moment of Inertia (I).

Trapezium about an axis passing through its c. of g. and parallel with the base.

Radius of gyration $= k$.

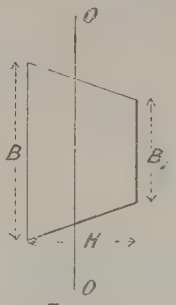


Fig. 105.

$$I = \frac{(B_1^2 + 4B_1B + B^2)H^3}{36(B_1 + B)}$$

or $$\frac{BH^3}{36}\left(\frac{n^2 + 4n + 1}{n + 1}\right)$$

$$n = \frac{B_1}{B}$$

For a close approximation, see next figure.

Approximate method for trapezium about axis passing through c. of g.

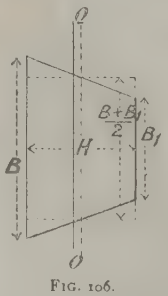


Fig. 106.

The I for dotted rectangle about an axis passing through its c. of g., is approximately the same as the I for trapezium.

For dotted rectangle—

$$I = \frac{(B + B_1)H^3}{24}$$

or if $B_1 = nB$

$$I = \frac{BH^3}{24}(n + 1)$$

From the theorem on p. 76, we have—

$$I = I_0 - R_0^2 A$$

$$I_0 = \frac{(3B_1 + B)H^3}{12} \text{ (about long base)}$$

$$R_0 = \frac{H(2B_1 + B)}{3(B_1 + B)} \text{ (see p. 60)}$$

$$A = \left(\frac{B_1 + B}{2}\right)H$$

Substituting the values in the above equation and simplifying, we get the result as given. The working out is simple algebra, but too lengthy to give here.

The I for a rectangle is $\frac{B'H^3}{12}$ (see p. 80). Putting in the value $\frac{B + B_1}{2} = B'$, we get—

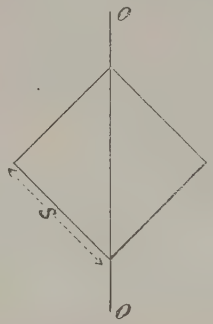
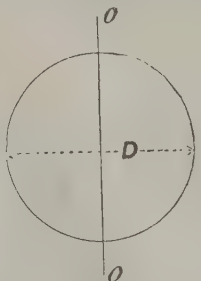
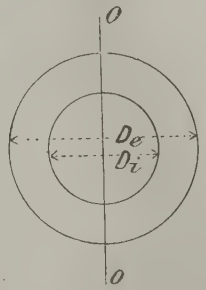
$$I = \frac{B + B_1}{2} \times \frac{H^3}{12} = \frac{(B + B_1)H^3}{24}$$

The following table shows the error involved in the above assumption; it will be seen that the error becomes serious when $n < 0{\cdot}5$:—

Value of n.	Approx. method, the correct value being 1.
0·9	1·001
0·8	1·005
0·7	1·011
0·6	1·021
0·5	1·039
0·4	1·065
0·3	1·107
0·2	1·174

The approximate method always gives too high results.

SECOND MOMENT, OR MOMENT OF INERTIA (I).

		Radius of gyration κ.
Square about its diagonal.  FIG. 107.	$I = \frac{S^4}{12}$	$\frac{S}{\sqrt{12}}$
 FIG. 108.	*Circle about a diameter.* $I = \frac{\pi D^4}{64}$	$\frac{D}{4}$
 FIG 109.	*Hollow circle about a diameter.* $\frac{\pi}{64}(D_e^4 - D_i^4)$	$\frac{\sqrt{D_e^2 + D_i^2}}{4}$

This may be taken as two triangles about their bases (see p. 84).

In this case, $B = \sqrt{2}S$

$$H = \frac{S}{\sqrt{2}}$$

$$I = 2\left(\frac{BH^3}{12}\right)$$

$$= 2\left\{\frac{\sqrt{2}S\left(\frac{S}{\sqrt{2}}\right)^3}{12}\right\}$$

$$= \frac{S^4}{12}$$

area of figure $= S^2$

radius of gyration $= \sqrt{\frac{S^4}{12S^2}} = \frac{S}{\sqrt{12}}$

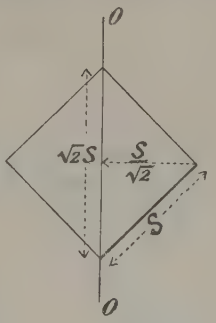


FIG. 107a

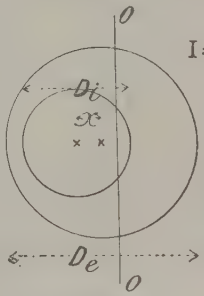
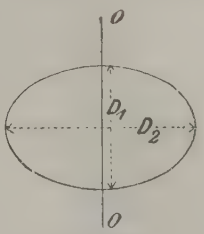
From the theorem on p. 77, we have $I_p = I_x + I_y$; in the circle, $I_x = I_y$.

Then $I_p = 2I_x$

and $I_x = \frac{I_p}{2} = \frac{\pi D^4}{2 \times 32} = \frac{\pi D^4}{64}$ (see Fig. 118).

The I for the hollow circle is simply the difference between the I for the outer and inner circles.

Second Moment, or Moment of Inertia (I).

	Radius of gyration κ
Hollow eccentric circle about a line normal to the line joining the two centres, and passing through the c. of g. of the figure.  Fig. 110. $I=\frac{\pi}{64}\left(D_e^4-D_i^4-\frac{16D_e^2D_i^2x^2}{D_e^2-D_i^2}\right)$ where x is the eccentricity. Note.—When the eccentricity is zero, *i.e.* when the outer and inner circles are concentric, the latter term in the above expression vanishes, and the value of I is the same as in the case given above for the hollow circle.	
Ellipse about minor axis.  Fig. 111. $I=\frac{\pi D_2^3 D_1}{64}$	$\frac{D_2}{4}$

The axis OO passes through the c. of g. of the figure, and is at a distance b from the centre of the outer circle, and a from the centre of the inner circle.

From the principle of moments, we have—

$$\frac{\pi}{4}D_i^2(b+x) = \frac{\pi}{4}D_e^2 b$$

$$\text{whence } b = \frac{D_i^2 x}{D_e^2 - D_i^2}$$

$$\text{also } \frac{\pi}{4}D_e^2(a-x) = \frac{\pi}{4}D_i^2 a$$

$$\text{whence } a = \frac{D_e^2 x}{D_e^2 - D_i^2}$$

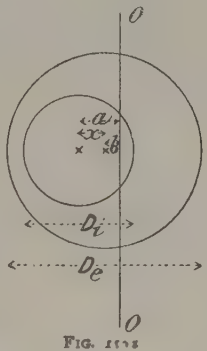


FIG. 1118

From the theorem on p. 76, we have—

$I'_e = I_e + A_e b^2$ for the outer circle about the c. of g. of figure

$$= \frac{\pi D_e^4}{64} + \frac{\pi D_e^2}{4} \times b^2$$

also $I'_i = I_i + A_i a^2$ for the inner circle about the c. of g. of figure

$$= \frac{\pi D_i^4}{64} + \frac{\pi D_i^2}{4} \times a^2$$

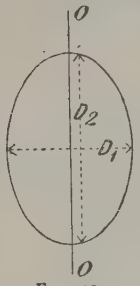
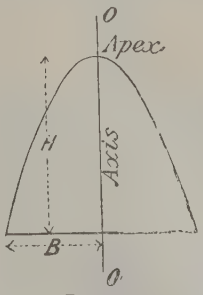
and $I = I'_e - I'_i$ for the whole figure.

Substituting the values given above, and reducing, we get the expression given on the opposite page.

The second moment, or moment of inertia, of a figure varies directly as its breadth taken parallel to the axis of revolution; hence the I for an ellipse about its minor axis is simply the I for a circle of diameter D_2 reduced in the ratio $\frac{D_1}{D_2}$

$$\text{or } \frac{\pi D_2^4}{64} \times \frac{D_1}{D_2} = \frac{\pi D_2^3 D_1}{64}$$

SECOND MOMENT, OR MOMENT OF INERTIA (I).

		Radius of gyration κ.
Ellipse about major axis.  FIG. 112.	$I = \frac{\pi D_1^3 D_2}{64}$	$\frac{D_1}{4}$
Parabola about its axis.  FIG. 113.	$I = \frac{4}{15} HB^3$	$\frac{B}{\sqrt{5}}$

And for an ellipse about its major axis, the I is that for a circle of diameter D_1 increased in the ratio $\frac{D}{D_1}$

$$\text{or } \frac{\pi D_1^4}{64} \times \frac{D_2}{D_1} = \frac{\pi D_1^3 D_2}{64}$$

$$h = H\left(1 - \frac{b^2}{B^2}\right) \text{ (see p. 69).}$$

$$\text{area of strip} = h \,.\, db$$

$$\text{second moment of strip} = b^2 \,.\, h \,.\, db$$

$$\text{,, ,,} = b^2 H\left(1 - \frac{b^2}{B^2}\right) db$$

$$\text{second moment of whole figure} = H \int_0^B \left(b^2 - \frac{b^4}{B^2}\right) db$$

$$\text{,, ,,} = H\left[\frac{b^3}{3} - \frac{b^5}{5B^2}\right]_0^B$$

$$\text{,, ,,} = H\left[\frac{B^3}{3} - \frac{B^3}{5}\right]$$

$$\text{,, ,,} = \frac{2HB^3}{15}$$

$$\text{second moment for double figure shown on opposite page} = \tfrac{4}{15} HB^3$$

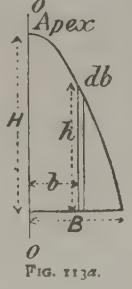
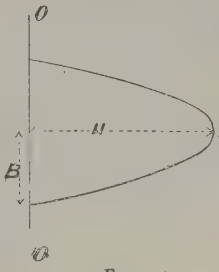
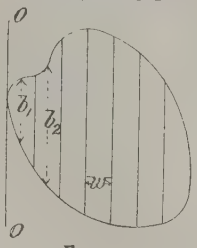


FIG. 113a.

Second Moment, or Moment of Inertia (I).

		Radius of gyration κ.
Parabola about its base.  Fig. 114.	$I = \frac{32}{105}BH^3$	$\sqrt{\frac{8}{35}}H$
Irregular figures.  Fig. 115.	$I_0 = \frac{w^3}{4}(b_1 + 9b_2 + 25b_3 + 49b_4 +$, etc.)	(See opposite page.)

$$b = \frac{B \cdot h^{\frac{1}{2}}}{H^{\frac{1}{2}}}$$

area of strip $= b \cdot dh$

second moment of strip $= (H-h)^2 b \cdot dh$

" " $= \dfrac{(H-h)^2 B h^{\frac{1}{2}} \cdot dh}{H^{\frac{1}{2}}}$

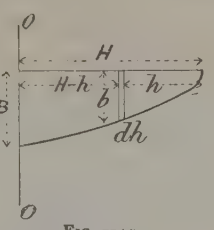


FIG. 114*a*.

second moment of whole figure $= \dfrac{B}{H^{\frac{1}{2}}} \int_0^H (H^2 h^{\frac{1}{2}} + h^{\frac{5}{2}} - 2h^{\frac{3}{2}}H)dh$

$$= \frac{B}{H^{\frac{1}{2}}} \left[\frac{2H^2 h^{\frac{3}{2}}}{3} + \frac{2h^{\frac{7}{2}}}{7} - \frac{4h^{\frac{5}{2}}H}{5} \right]_0^H$$

$$= \frac{B}{H^{\frac{1}{2}}} (\tfrac{2}{3}H^{\frac{7}{2}} + \tfrac{2}{7}H^{\frac{7}{2}} - \tfrac{4}{5}H^{\frac{7}{2}})$$

$$= B(\tfrac{2}{3}H^3 + \tfrac{2}{7}H^3 - \tfrac{4}{5}H^3)$$

$$= \tfrac{16}{105}BH^3$$

for the double figure shown on opposite page $= \tfrac{32}{105}BH^3$

Divide the figure up as shown in Fig. 87*a*.

Let the areas of the strips be a_1, a_2, a_3, a_4, etc., respectively; and their mean distances from the axis be r_1, r_2, r_3, r_4, etc., respectively.

Then $I_0 = a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2 +$, etc.

But $a_1 = wb_1$, and $a_2 = wb_2$, and so on

and $r_1 = \dfrac{w}{2}$, $r_2 = \dfrac{3w}{2}$, $r_3 = \dfrac{5w}{2}$, and so on

hence $I_0 = w\left\{b_1\left(\dfrac{w}{2}\right)^2 + b_2\left(\dfrac{3w}{2}\right)^2 + b_3\left(\dfrac{5w}{2}\right)^2 +, \text{ etc.}\right\}$

$$I_0 = \frac{w^3}{4}(b_1 + 9b_2 + 25b_3 + 49b_4 +, \text{ etc.})$$

Also $\kappa^2 = \dfrac{\dfrac{w^3}{4}(b_1 + 9b_2 + 25b_3 + 49b_4 +, \text{ etc.})}{w(b_1 + b_2 + b_3 + b_4 +, \text{ etc.}}$

$$\kappa = \frac{w}{2}\sqrt{\frac{b_1 + 9b_2 + 25b_3 +, \text{ etc.}}{b_1 + b_2 + b_3 +, \text{ etc.}}}$$

This expression should be compared with that obtained for finding the position of the c. of g. on p. 71. The comparison helps one to realize the relation between the first and second moments.

Second Moment, or Moment of Inertia (I).

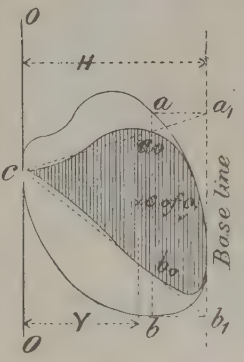


Fig. 116.

Graphic method.

Let A = shaded area;
Y = distance of c. of g. of shaded area from OO;
H = extreme dimension of figure measured normal to OO;

Then $I_0 = AYH$

Radius of gyration $= k$.

Second Polar Moments of Surfaces or Thin Laminæ.

Parallelogram about a pole passing through its c. of g.

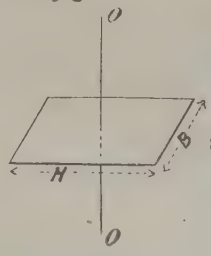


Fig. 117.

$$I_p = \frac{BH}{12}(H^2 + B^2)$$

square of side S—

$$I_p = \frac{S^4}{6}$$

$$\sqrt{\frac{H^2 + B^2}{12}}$$

$$\frac{S}{\sqrt{6}}$$

Circle about a pole passing through the c. of g.

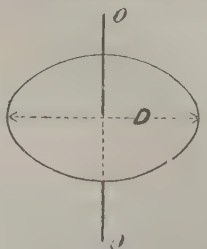


Fig. 118.

$$I_p = \frac{\pi D^4}{32}, \text{ or } \frac{\pi R^4}{2}$$

$$\frac{D}{\sqrt{8}}$$

or $\frac{R}{\sqrt{2}}$

Divide the figure up into a number of strips, as shown in Fig. 116; project each on to the base-line, *e.g.* ab projected to a_1b_1; join a_1 and b_1 to c, some convenient point on OO, cutting ab in a_0b_0, and so on with the other lines, which when joined up give the boundary of the shaded figure. Find the c. of g. of shaded figure (by cutting out in cardboard and balancing). The principle of this construction is fully explained in Chap. IX., p. 360.

See also Barker's "Graphical Calculus," p. 184, and Lineham's "Text-book of Mechanical Engineering," Appendix.

From the theorem on p. 77, we have—

$$I_p = I_y + I_x = \frac{BH^3}{12} + \frac{HB^3}{12}$$
$$= \frac{BH}{12}(H^2 + B^2).$$

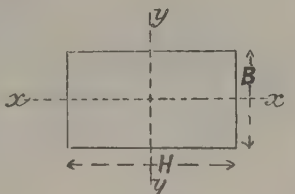


FIG. 117*a*.

Thickness of ring $= dr$

area of ring $= 2\pi r \,.\, dr$

second moment of ring $= 2\pi r \,.\, r^2 \,.\, dr$

" " circle $= 2\pi \int_0^R r^3 \,.\, dr$

$$= \frac{2\pi R^4}{4} = \frac{\pi R^4}{2}$$

" " " $\left(\text{but } R = \frac{D}{2}\right)$

$$= \frac{\pi D^4}{2 \times 16} = \frac{\pi D^4}{32}$$

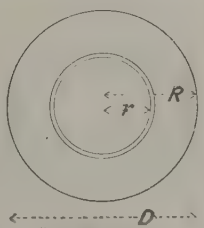


FIG. 118*a*.

Second Polar Moment, or Polar Moment of Inertia.

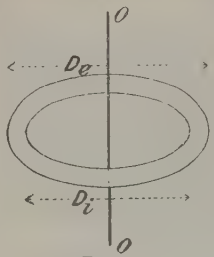


Fig. 119.

Hollow circle about a pole passing through the c. of g. and normal to the plane.

$$I_p = \frac{\pi}{32}(D_e^4 - D_i^4)$$

Radius of gyration κ.

$$\sqrt{\frac{D_e^2 + D_i^2}{8}}$$

or

$$\sqrt{\frac{R_e^2 + R_i^2}{2}}$$

Second (Polar) Moments of Solids.

Gravitational Units.

Bar of rectangular section about a pole passing through its c. of g.

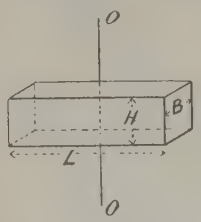


Fig. 120.

$$I_p = \frac{LBH}{12}(L^2 + B^2)\,\frac{W}{g}$$

$$\sqrt{\frac{L^2 + B^2}{12}}$$

For a circular bar of radius R—

$$I_p = \frac{L\pi R^2}{12}(L^2 + 3R^2)\,\frac{W}{g}$$

$$\sqrt{\frac{L^2 + 3R^2}{12}}$$

Cylinder about its axis.

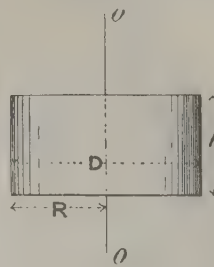


Fig. 121.

$$I_p = \frac{\pi D^4 H}{32}\frac{W}{g}, \text{ or } \frac{\pi R^4 H}{2}\frac{W}{g}$$

$$\frac{D}{\sqrt{8}}$$

or

$$\frac{R}{\sqrt{2}}$$

The I_p for the hollow circle is simply the difference between the I_p for the outer and the I_p for the inner circles.

The bar may be regarded as being made up of a great number of thin laminæ of rectangular form, of length L and breadth B, revolving about their polar axis, the radius of gyration of each being $K = \sqrt{\frac{L^2 + B^2}{12}}$ (see Fig. 117), which is the radius of gyration of the bar. The second moment of the bar will then be K^2 (weight of bar), or $\frac{LBH}{12}(L^2 + B^2)\frac{W}{g}$ Where W is the weight of 1 cubic inch or foot of the material, according to the units chosen.

The cylinder may be regarded as being made up of a great number of thin circular laminæ revolving about a pole passing through their centre, the radius of gyration of each being $K = \sqrt{\frac{D^2}{8}}$

The second moment of cylinder = K^2 (weight of cylinder)

$$= \frac{D^2}{8} \times \frac{\pi D^2 H}{4} \frac{W}{g} = \frac{\pi D^4 H}{32} \frac{W}{g}$$

Second Polar Moment, or Polar Moment of Inertia.

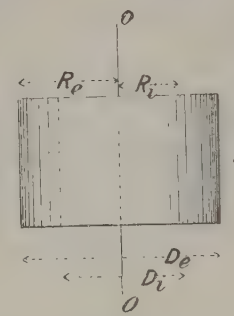


Fig. 122.

Hollow cylinder about a pole passing through its axis.

Radius of gyration K.

$$I_p = \frac{\pi H}{32}(D_e^4 - D_i^4)\frac{W}{g}, \quad \sqrt{\frac{D_e^2 + D_i^2}{8}}$$

or

$$\text{or } \frac{\pi H}{2}(R_e^4 - R_i^4)\frac{W}{g} \quad \sqrt{\frac{R_e^2 + R_i^2}{4}}$$

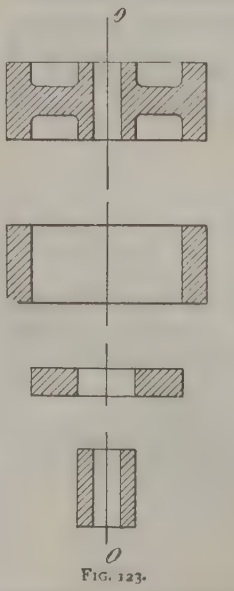


Fig. 123.

Disc flywheel.

Treat each part separately as hollow cylinders, and add the results.

The I_p for the hollow cylinder is simply the difference between the I_p for the outer and the inner cylinders.

It must be particularly noticed that the radius of gyration of a solid body, such as a cylinder, flywheel, etc., is not the radius of gyration of a plane section; the radius of gyration of a plane section is that of a thin lamina of uniform thickness, while the radius of gyration of a solid is that of a thin wedge. The radius of gyration of a solid may be found by correcting the section in this manner, and finding the I for the shaded figure treated as a plane surface.

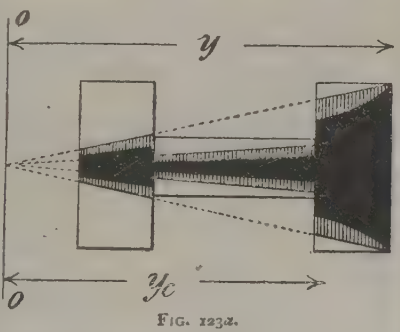


FIG. 123*a*.

The construction simply reduces the width of the solid section at each point proportional to its distance from OO; it is, in fact, the "modulus figure" (see Chap. IX.) of the section.

Let y_c = the distance of the c. of g. of the *second* modulus figure from the axis OO shown black;

A = the area of the black figure.

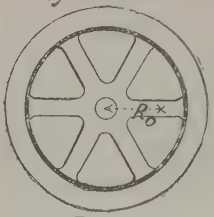
A_1 = the area of the modulus figure

then $I_1 = A_1K^2 = Ay_cy$ (see page 96)

$$K^2 = \frac{Ay_cy}{A_1}$$

$$\left.\begin{matrix}\text{The moment of inertia} \\ \text{of the wheel}\end{matrix}\right\} = \frac{\text{Weight of the wheel}}{A_1g} \times Ay_cy$$

SECOND POLAR MOMENT, OR MOMENT OF INERTIA.

	Radius of gyration κ.
Flywheel with arms.  FIG. 124. Treat the rim and boss separately as hollow cylinders, and each arm thus (assumed parallel)— For each arm (see Fig. 120) $\frac{L\ (\text{sectional area})}{12}(L^2 + B^2 + 12R_0^2)\frac{W}{g}$ where R_0 = the radius of the c. of g. of the arm. For most practical purposes the rim only is considered, and the arms and boss neglected.	
Sphere about its diameter. FIG. 125. $I_p = \frac{8}{15}\pi R^5\frac{W}{g}$, or $\frac{1}{60}\pi D^5\frac{W}{g}$	$R\sqrt{\frac{2}{5}}$ or $\frac{D}{\sqrt{10}}$

The arms are assumed to be of rectangular section; if they are not, the error involved will be exceedingly small.

The sphere may be regarded as being made up of a great number of thin circular layers of radius r_1, and radius of gyration $\frac{r_1}{\sqrt{2}}$ (see p. 96).

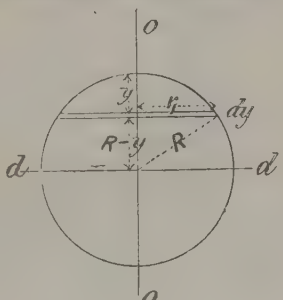


FIG. 125a.

$$r_1^2 = R^2 - (R-y)^2$$
$$= R^2 - R^2 - y^2 + 2Ry$$
$$= 2Ry - y^2$$

volume of thin layer $= \pi r_1^2 dy$

second moment of layer about OO $= \pi r_1^2 dy \times \frac{r_1^2}{2}$

$$= \frac{\pi r_1^4}{2} dy$$
$$= \frac{\pi}{2}(2Ry - y^2)^2 dy$$
$$= \frac{\pi}{2}(4R^2y^2 + y^4 - 4Ry^3)dy$$

second moment of hemisphere, *i.e.* of all layers on one side of the diameter *dd* $= \frac{\pi}{2}\int_{y=0}^{y=R}(4R^2y^2 + y^4 - 4Ry^3)dy$

$$= \frac{\pi}{2}\left[\frac{4R^2y^3}{3} + \frac{y^5}{5} - \frac{4Ry^4}{4}\right]_{y=0}^{y=R}$$
$$= \frac{\pi}{2}\left[\frac{4R^5}{3} + \frac{R^5}{5} - \frac{4R^5}{4}\right]$$
$$= \frac{\pi}{2}\left[\frac{8R^5}{15}\right] = \tfrac{4}{15}\pi R^5$$

second moment of sphere $= \frac{8}{15}\pi R^5$

SECOND POLAR MOMENT, OR POLAR MOMENT OF INERTIA.

Sphere about an external axis.

Radius of gyration κ.

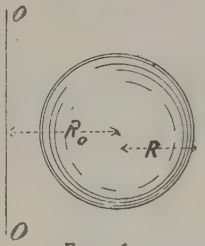


FIG. 126.

$$I_{op} = \tfrac{4}{3}\pi R^3(\tfrac{2}{5}R^2 + R_0^2)\frac{W}{g}$$

When the axis becomes a tangent, $R_0 = R$;

$$I_{op} = \tfrac{28}{15}\pi R^5\frac{W}{g}$$

Cone about its axis.

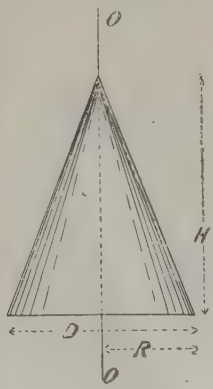


FIG. 127.

$$I_p = \frac{\pi}{10}R^4H\frac{W}{g},$$

$$\text{or } \frac{\pi}{160}D^4H\frac{W}{g}$$

i.e. $\frac{1}{5}$ of the I for the circumscribing cylinder.

$R\sqrt{\frac{3}{10}}$

$D\sqrt{\frac{3}{40}}$

From the theorem on p. 76, we have—

$$I_{op} = I_p + R_0^2 V \qquad I_p = \tfrac{8}{15}\pi R^5$$
$$= \left(\tfrac{8}{15}\pi R^5 + \tfrac{4}{3}\pi R^3 R_0^2\right) \qquad V = \tfrac{4}{3}\pi R^3$$
$$= \tfrac{4}{3}\pi R^3\left(\tfrac{2}{5}R^2 + R_0^2\right)$$

The cone may also be regarded as being made up of a great number of thin layers.

Volume of thin layer $= \pi r^2 dh$

second moment of layer about axis OO $\Big\} = \pi r^2 dh \times \frac{r^2}{2} = \frac{\pi r^4}{2} dh$

$\left(\text{but } r = \frac{Rh}{H}\right) \qquad = \frac{\pi R^4 h^4}{2H^4} dh$

second moment of cone $\Big\} = \frac{\pi R^4}{2H^4}\int_0^H h^4 . dh = \frac{\pi R^4 H^5}{10 H^4}$

$$= \frac{\pi R^4 H}{10}$$

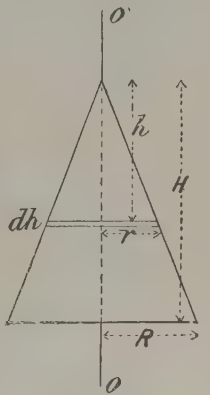


FIG. 127*a*.

CHAPTER IV.

RESOLUTION OF FORCES.

We have already explained how two forces acting on a point may be replaced by one which will have precisely the same effect on the point as the two. We must now see how to apply the principle involved to more complex systems of forces.

Polygon of Forces.—If we require to find the resultant of more than two forces which act on a point, we can do so by finding the resultant of any two by means of the parallelogram of forces, and then take the resultant of this resultant and the next force, and so on, as shown in the diagram. The resultant of 1 and 2 is marked R1.2., and so on. Then we finally get the resultant R1.2.3.4. for the whole system.

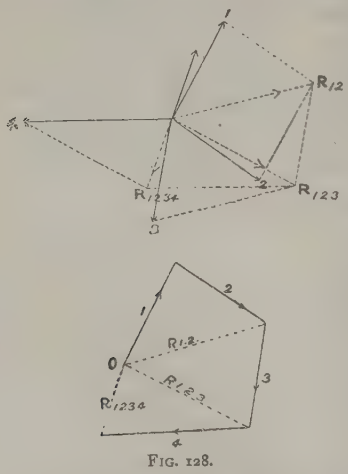


Fig. 128.

Such a method is, however, clumsy. The following will be found much more direct and convenient: Start from any point O, and draw the line 1 parallel and equal on a given scale to the force 1; from the extremity of 1 draw the line 2 equal and parallel to the force 2; then, by the triangle of forces, it will be seen that the line R1.2. is the resultant of the forces 1 and 2. From the extremity of 2 draw 3 in a similar manner, and so on with all the forces; then it will be seen that the line R1.2.3.4. represents the resultant of the forces. In using this construction, there is no need to put in the lines R1.2., etc.; in the figure they have been inserted in order to make it

clear. Hence, if any number of forces act upon a point in such a manner that lines drawn parallel and equal on some given scale to them form a closed polygon, the point is in equilibrium under the action of those forces. This is known as the theorem of the polygon of forces.

Method of lettering Force Diagrams.—In order to keep force diagrams clear, it is essential that the forces be lettered in each diagram to prevent confusion. Instead of lettering the force itself, it is very much better to letter the spaces, and to designate the force by the letters corresponding to the spaces on each side, thus: The force separating *a* from *b* is termed the force *ab*; likewise the force separating *d* from *b*, *db*.

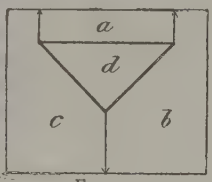


FIG. 129.

This method of notation is usually attributed to Bow; several writers, however, claim to have been the first to use it.

Funicular or Link Polygons.—When forces in equilibrium act at the corners of a series of links jointed together at their extremities, the force acting along each link can be readily found by a special application of the triangle of forces.

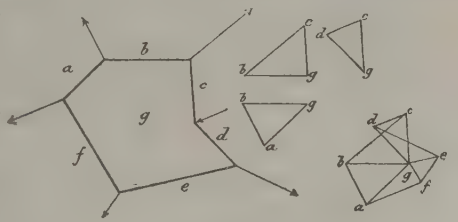


FIG. 130.

Consider the links *ag* and *bg*. There are three forces in equilibrium, viz. *ab*, *ag*, *bg*, acting at the joint. The magnitude of *ab* is known, therefore the magnitude of the other two acting on the links may be obtained from the triangle of forces shown on the right-hand side, viz. *abg*. Similarly consider all the other joints. It will be found that each triangle of forces contains a line equal in every respect to a line in the preceding triangle, hence all the triangles may be brought together to form one diagram, as shown to the extreme right hand. It should be noticed that the external forces form a closed polygon, and the forces in the bars are represented by radial lines meeting in the point or pole *g*.

It will be evident that the form taken up by the polygon depends on the magnitude of the forces acting at each joint.

Suspension Bridge.—Another special application of the triangle of forces in a funicular polygon is that of finding the forces in the chain of a suspension bridge. The platform on which the roadway is carried is supported from the chain by means of vertical ties. We will assume that the weight supported by each tie is known. The force acting on each portion of the chain can be found by constructing a triangle of forces at each joint of a vertical tie to the chain, as shown in the figure above the chain. But *bo* occurs in both triangles; hence the two triangles may be fitted together, *bo* being common to each. Likewise all the triangles of forces for all the joints may be fitted together. Such a figure is shown at the side, and is known as a ray or vector polygon. Instead,

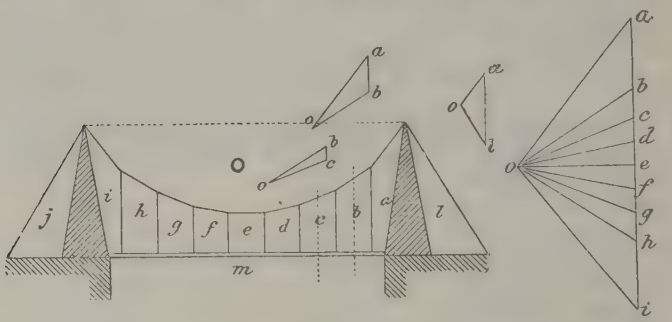


FIG. 131.

however, of constructing each triangle separately and fitting them together, we simply set off all the vertical loads *ab*, *bc*, etc., on a straight line, and from them draw lines parallel to each link of the suspension chain; if correctly drawn, all the rays will meet in a point. The force then acting on each segment of the chain is measured off the vector polygon, to the same scale as the vertical loads. In the figure the vertical loads are drawn to a scale of 1″ = 10 tons; hence, for example, the tension in the segment *ao* is 9·8 tons.

The downward pressure on the piers and the tension in the outer portion of the chain is given by the triangle *aol*.

If a chain (or rope) hangs freely without any platform suspended below, the vertical load will be simply that due to the weight of the chain itself. If the weight per foot of horizontal span were constant, it is easy to show that the curve taken up by the chain is a parabola (see p. 493). In the same chapter, the link and vector polygon construction is employed

to determine the bending moment due to an evenly distributed load. The bending moment M_x at x is there shown to be the depth D_x multiplied by the polar distance OH (Fig. 132); *i.e.*

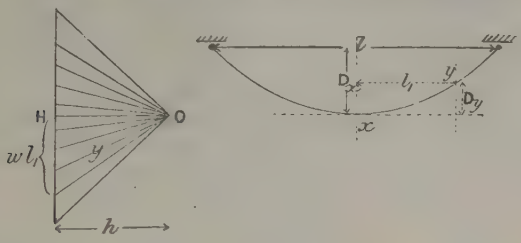


FIG. 132.

the dip of the chain at x multiplied by the horizontal component of the forces acting on the links, viz. the force acting on the link at x, or $M_x = D_x \times OH$. In the same chapter, it is also shown that with an evenly distributed load $M_x = \frac{wl^2}{8}$, where w is the load per foot run, and l is the span in feet.

$$\text{Hence } \frac{wl^2}{8} = D_x \,.\, \overline{OH} = D_x h$$

$$\text{or } h = \frac{wl^2}{8D_x}$$

where h is the tension at the bottom of the chain, viz. at x.

It will, however, be seen that the load on a freely hanging chain is not evenly distributed per foot of horizontal run, because the inclination of the chain varies from point to point. Therefore the curve is not parabolic; it is, in reality, a catenary curve. For nearly all practical purposes, however, when the dip is not great compared with the span, it is sufficiently accurate to take the curve as being parabolic.

Then, assuming the curve to be parabolic, the tension at any other point, y, is given by the length of the corresponding line on the vector polygon, which is readily seen to be—

$$T_y = \sqrt{h^2 + w^2 l_1^2}$$

The true value of the tension obtained from the catenary is—

$$T_y = h + wD_y$$

(see Unwin's "Machine Design," p. 421), which will be found to agree closely with the approximate value given above.

Data for Force Polygons.—Sometimes it is impossible to construct a polygon of forces on account of the incompleteness of the data.

In the case of the triangle and polygon of forces, the following data must be given in order that the triangle or polygon can be constructed. If there are n conditions in the completed polygon, $n - 2$ conditions must be given; thus, in the triangle of forces there are six conditions, three magnitudes and three directions: then at least four must be supplied before the triangle can be constructed, such as—

3	magnitude(s)	and	1	direction(s)
2	,,	,,	2	,,
1	,,	,,	3	,,

Likewise in a five-sided polygon, there are ten conditions, eight of which must be known before the polygon can be constructed. When the two unknown conditions refer to the same or adjacent sides, the construction is perfectly simple, but when the unknown conditions refer to non-adjacent sides, a special construction is necessary. Thus, for example, suppose when dealing with five forces, the forces 1, 2, and 4 are completely known, but only the directions, not the magnitudes, of 3 and 5 are known. We proceed thus:

Draw lines 1 and 2 in the polygon of forces, Fig. 133, in the usual way. From the extremity of 2 draw a line of indefinite

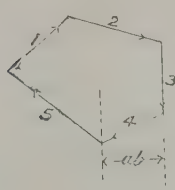


FIG. 133.

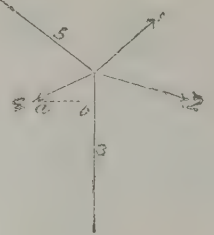


FIG. 134.

length parallel to the force 3; its length cannot yet be fixed, because we do not know its value. From the origin of 1 draw a line of indefinite length parallel to 5; its length is also not yet known. From the extremity of 4 in the diagram of forces, Fig. 134, drop a perpendicular *ab* on to 3, and in the polygon of forces, Fig. 133, draw a line parallel to 3, at a distance *ab*

from it. The point where this line cuts the line 5 is the extremity of 5. From this point draw a line parallel to 4; then by construction it will be seen that its extremity falls on the line 3, giving us the length of 3.

The order in which the forces are taken is of no importance.

Forces in the Members of a Jib Crane. CASE I. *The weight* W *simply suspended from the end of the jib.*—There is no need to construct a separate diagram of forces. Set off *bc* = W, or BC on some convenient scale,[1] and draw *ca* parallel to the tie CA; then the triangle *bac* is the triangle of forces acting on the point *b*. On measuring the force diagram, we find there is a compressive force of 15·2 tons along AB, and a tension force of 9·8 tons along AC.

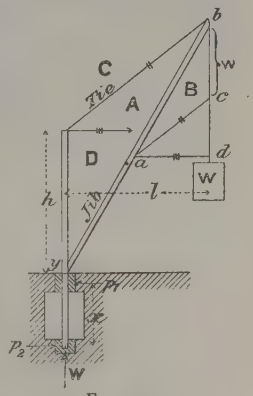


FIG. 135.

The pressure on the bottom pivot is W (neglecting the weight of the crane itself). The horizontal pull at the top of the crane-post is *ad*, or 7·9 tons; and the force (tension) acting on the post between the junction of the jib and the tie is *cd*, or 6 tons.

The bending moment at *y* will be $ad \times h$, or $W \times l$. For determining the bending stress at *y*, see Chap. IX.

Taking moments about the pivot bearing, we have—

$$p_1 x = p_2 x = Wl$$
$$\text{or } p_1 = p_2 = \frac{Wl}{x}$$

The sections of the various parts of the structure must be determined by methods to be described later on.

The weight of the structure itself should be taken into account, which can only be arrived at by a process of approximation; the dimensions and weight may be roughly arrived at by neglecting the weight of the structure in the first instance. Then, as the centre of gravity of each portion will be approximately at the middle of each length, the load W must be increased to

[1] In this case the scale is 0·1 inch = 2 tons, and W = 7 tons.

W + ½(weight of jib and tie). The downward pressure on the pivot will be W + weight of structure.

The dimensions of the structure must then be increased accordingly. In a large structure the forces should be again determined, to allow for the increased dimensions.

The bending moment on the crane-post at *y* may be very much reduced by placing a balance weight W_1 on the crane, as shown. The forces acting on the balance-weight members are found in a similar manner to that described above, and, neglecting the weight of the structure, are found to be 8·1 tons on the tie, and 4·4 tons on the horizontal strut.

The balance weight produces a compression in the upper part of the post of 6·8 tons; but, due to the tie *ac*, we had a tension of 6·0 tons, therefore there is a compression of 0·8 ton

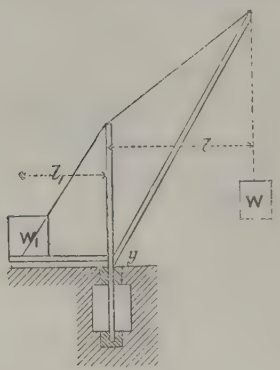


Fig. 136.

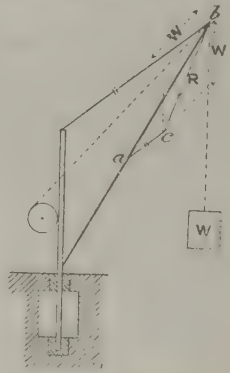


Fig. 137.

in the upper part of the crane-post. The pressure on the lower part of the crane-post and pivot is W + W_1 + weight of structure.

Then, neglecting the weight of the structure, the bending moment on the post at *y* will be—

$$Wl - W_1 l_1$$

The moment $W_1 l_1$ should be made equal to $\frac{Wl}{2}$, then the post will never be subjected to a bending moment of more than one-half that due to the lifted load, and the pressure p_1 and p_2 will be correspondingly reduced.

Case II. *The weight W suspended from a chain passing to a barrel on the crane-post.*—As both portions of the chain are

subjected to a pull W, the resultant R is readily determined. From *c* a line *ac* is drawn parallel to the tie; then the force acting down the jib is *ab* = 16·4 tons; down the tie *ac* = 4·4 tons. The bending moments on the post, etc., are determined in precisely the same manner as in Case I.

When pulley blocks are used for lifting the load, the pull in the chain between the jib pulley and the barrel will be less than W in the proportion of the velocity ratio.

The general effect of the pull on the chain is to increase the thrust on the jib, and to reduce the tension in the tie. In designing a crane, the members should be made strong enough to resist the greater of the two, as it is quite possible that a link of the chain may catch in the jib pulley, and the conditions of Case I. be realized.

Forces in the Members of Sheer Legs.—In the type of crane known as sheer legs the crane-post is dispensed with, and lateral stability is given by using two jibs or sheer legs spread out at the foot; the tie is usually brought down to the level of the ground, and is attached to a nut working in guides. By means of a horizontal screw, the sheer legs can be tilted or "derricked" at will: the end thrust on the screw is taken by a thrust block; the upward pull on the nut and guides is taken by bolts passing down to massive foundations below. The forces are readily determined by the triangle of forces.

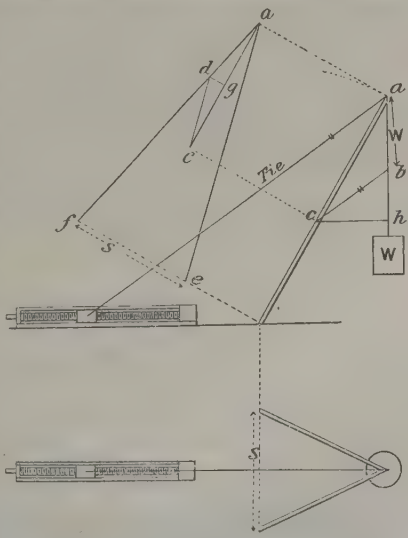


FIG. 138.

The line *bc* is drawn parallel to the tie, and represents the force acting on it; then *ac* represents the force acting down the middle line of the two sheer legs. This is shown more clearly on the projected view of the sheer legs. *cd* is then drawn parallel to the sheer leg *ae*; then *dc* represents the force acting down the sheer leg *ae*; likewise *ad* down the leg *af*, and

dg the force acting at the bottom of the sheer legs tending to make them spread; *ch* represents the thrust of the screw on the thrust block and the force on the screw, and *bh* the upward pull which has to be resisted by the nut guides and the foundation bolts.

The members of this type of structure are necessarily very

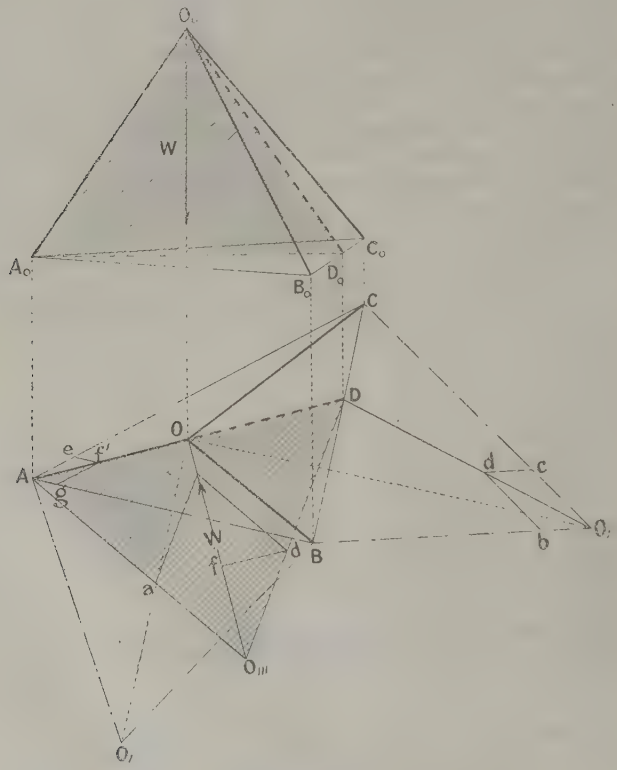


FIG. 138*a*.

heavy and long, consequently the bending stress due to their own weight is very considerable, and has to be carefully considered in the design. The problem of combined bending and compression is dealt with in Chapter XII.

Forces in a Tripod or Three Legs.—Let the lengths

of the legs be measured from a horizontal plane. The vertical height of the apex O from the plane, also the horizontal distances, AB, BC, CA, must be known.

In the plan set out the triangle ABC from the known lengths of the sides; from A as centre describe an arc of radius equal to the length of the pole A, likewise from B describe an arc of radius equal to the length of the pole B. They cut in the point o_1. From o_1 drop a perpendicular on AB and produce; similarly, by describing arcs from B and C of their respective radii, find the point o_{11}, and from it drop a perpendicular on BC, and produce to meet the perpendicular from o_1 in O, which is the apex of the tripod. The plan of the three legs can now be filled in, viz. AO, BO, CO. Produce AO to meet CB in D. From O set off the height of the apex above the plane, viz. Oo_{111}, at right angles to AO; join Ao_{111}. This should be measured to see that it checks with the length of the pole A. Join Do_{111}. From o_{111} set off a length to a convenient scale to represent W, complete the parallelogram of forces, then $o_{111}a$ gives the force acting down the leg A, and $o_{111}d$ the force acting down the imaginary leg D, shown in broken line, which lies in the plane of the triangle OBC; resolve this force down OC and OB by setting off $o_{11}d$ along o_{11}D equal to $o_{111}d$, found by the preceding parallelogram, then the force acting down the leg B is $o_{11}b$, and that down C is $o_{11}c$.

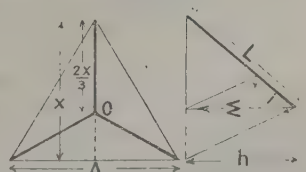
The horizontal force tending to spread the legs AO_{111} and DO_{111} is given by fd. This is set off at Af, and is resolved along AC and AB. The force acting on an imaginary tie AC is Ae, and on AB is Ag; similarly with the remaining tie.

When the three legs are of equal length and are symmetrically placed, the forces can be obtained thus—

$$x = \frac{A\sqrt{3}}{2} \qquad \frac{2x}{3} = \frac{A}{\sqrt{3}}$$

$$\frac{i}{L} = \frac{1}{3} \qquad \text{and } h = \sqrt{L^2 - \frac{A^2}{3}},$$

$$\frac{p}{W} = \frac{i}{h} = \frac{L}{3\sqrt{L^2 - \frac{A^2}{3}}}$$



Three equal legs

FIG. 138*b*.

$$p = \frac{WL}{3\sqrt{L^2 - \frac{A^2}{3}}}$$ where p is the force acting down each leg.

Forces in the Members of a Roof Truss.—Let the roof truss be loaded with equal weights at the joints, as shown; the reactions at each support will be each equal to half the total load on the structure. We shall for the present neglect the weight of the structure itself.

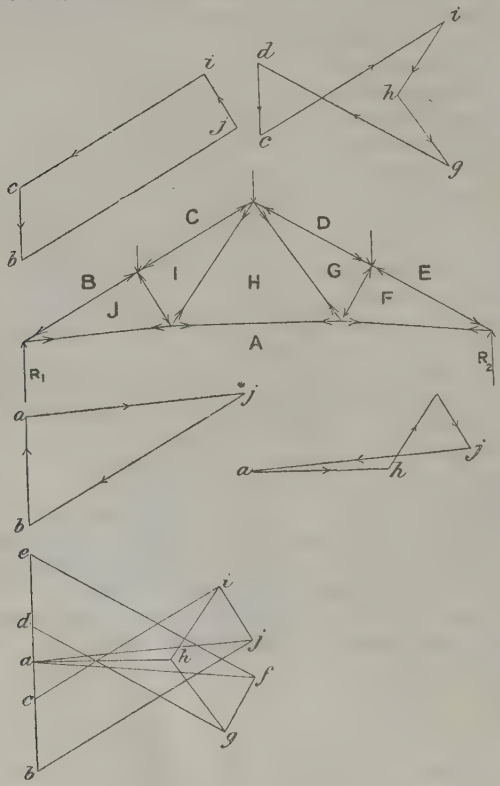


FIG. 139.

The forces acting on each member can be readily found by a special application of the polygon of forces.

Consider the joint at the left-hand support BJA or R_1. We have three forces meeting at a point; the magnitude of one, viz. R_1 or *ba*, and the direction of all are known; hence we can determine the other two magnitudes by the triangle of forces. This we have done in the triangle *ajb*.

Consider the joint BJIC. Here we have four forces meeting at a point; the magnitude of one is given, viz. *bc*, and the direction of all the others; but this is not sufficient—we must have at least six conditions known (see p. 110). On referring back to the triangle of forces just constructed, we find that the force *bj* is known; hence we can proceed to draw our polygon of forces *cbji* by taking the length of *bj* from the triangle previously constructed. By proceeding in a similar manner with every joint, we can determine all the forces acting on the structure.

On examination, we find that each polygon contains one side which has occurred in the previous polygon; hence, if these similar and equal sides be brought together, each polygon can be tacked on to the last, and so made to form one figure containing all the sides. Such a figure is shown below the structure, and is known as a "reciprocal diagram."

When determining the forces acting on the various members of a structure, we invariably use the reciprocal diagram without going through the construction of the separate polygons. We have only done so in this case in order to show that the reciprocal diagram is nothing more nor less than the polygon of forces.

We must now determine the nature of the forces, whether tensile or compressive, acting on the various members. In order to do this, we shall put arrows on the bars to indicate the *direction in which the bars resist the external forces.*

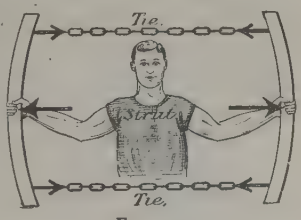


FIG. 140.

The illustration represents a man's arm stretched out, resisting certain forces. The arrows indicate the direction in which he is exerting himself, from which it will be seen that when the arrows on his arms point outwards his arms are in compression, and when in the reverse direction, as in the chains, they are in tension; hence, when we ascertain the directions in which a bar is resisting the external forces acting on it, we can at once say whether the bar is in tension or compression, or, in other words, whether it is a tie or a strut.

We know, from the triangle and polygon of forces, that the arrows indicating the directions in which the forces act follow round in the same rotary direction; hence, knowing the direction of one of the forces in the polygon, we can immediately find the direction of the others. Thus at the joint BJA we

know that the arrow points upwards from *b* to *a*; then, continuing round the triangle, we get the arrow-heads as shown. Transfer these arrows to the bars themselves *at the joint in question;* then, if an arrow points outwards at one end of a bar, the arrow at the other end must also point outwards; hence we can at once put in the arrow at the other end of the bar, and determine whether it is a strut or tie. When the arrows point outwards the bar is a strut, and when inwards a tie. Each separate polygon has been thus treated, and the arrow-heads transferred to the structure. But arrow-heads must *not* be put on the reciprocal diagram; if they are they will cause hopeless confusion. With a very little practice, however, one can run round the various sections of the reciprocal diagram by eye, and put the arrow-heads on the structure without making a single mark on the diagram. If a mistake has been made anywhere, it is certain to be detected before all the bars have been marked. If the beginner experiences any difficulty, he should make separate rough sketches for each polygon of forces, and mark the arrow-heads on each side. At some joints, where there are no external forces, the direction of the arrows will not be evident at first; they must not be taken from other polygons, but from the arrow-heads on the structure itself *at the joint in question.* For example, the arrows at the joints ABJ and BJIC are perfectly readily obtained, the direction being started by the forces AB and BC, but at the joint JIA the direction of the arrow on the bars JI and JA are known *at the joint;* either of these gives the direction for starting round the polygon *ahij.*

The following bars are struts: BJ, IC, GD, FE, JI, GF.

The following bars are ties: JA, IH, HA, HG, FA.

Some more examples of reciprocal diagrams will be given in the chapter on "Framework Structures."

CHAPTER V.

MECHANISMS.

PROFESSOR KENNEDY[1] defines a machine as "a combination of resistant bodies, whose relative motions are completely constrained, and by means of which the natural energies at our disposal may be transformed into any special form of work." Whereas a mechanism consists of a combination of simple links, arranged so as to give the same relative motions as the machine, but not necessarily possessing the resistant qualities of the machine parts; thus a mechanism may be regarded as a skeleton form of a machine.

Constrained and Free Motion.—Motion may be either constrained or free. A body which is free to move in any direction relatively to another body is said to have *free* motion, but a body which is constrained to move in a definite path is said to have *constrained* motion. Of course, in both cases the body moves in the direction of the resultant of all the forces acting upon it; but in the latter case, if any of the forces do not act in the direction of the desired path, they automatically bring into play constraining forces in the shape of stresses in the machine parts. Thus, in the figure, let *ab* be a crank which revolves about *a*, and let the force *bc* in the direction of the connecting-rod act on the pin at *b*. Then, if *b* were free, it would move off in the direction of the dotted line, but as *b* must move in a circular path, a force must act along the crank in order to prevent it following the dotted line. This force acting along the crank is readily found by resolving *bc* in

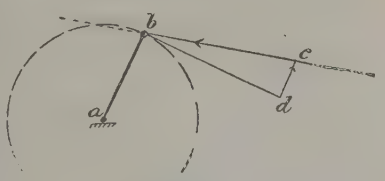


FIG. 141.

[1] "Mechanics of Machinery," p. 2.

a direction normal to the crank, viz. *bd*, *i.e.* in the direction in which *b* is moving, and along the crank, viz. *dc*, which in this instance is a compression. Hence the path of *b* is determined by the force acting along the connecting-rod and the force acting along the crank.

The constraining forces always have to be supplied by the parts of the machine itself. Machine design consists in arranging suitable materials in suitable form to supply these constraining forces.

The various forms of constrained motion we shall now consider.

Plane Motion.—When a body moves in such a manner that any point of it continues to move in one plane, such as in revolving shafts, wheels, connecting-rods, cross-heads, links, etc., such motion is known as plane motion. In plane motion a body may have either a motion of translation in any direction in a given plane or a motion of rotation about an axis.

Screw Motion.—When a body has both a motion of rotation and a translation perpendicular to the plane of rotation, a point on its surface is said to have a *screw* motion, and when the velocity of the rotation and translation are kept constant, the point is said to describe a helix, and the amount of translation corresponding to one complete rotation is termed the *pitch* of the helix or screw.

Spheric Motion.—When a body moves in such a manner that every point in it remains at a constant distance from a fixed point, such as when a body slides about on the surface of a sphere, the motion is said to be spheric. When the sphere becomes infinitely great, spheric motion becomes plane motion.

Relative Motion.—When we speak of a body being in motion, we mean that it is shifting its position relatively to some other body. This, indeed, is the only conception we can have of motion. Generally we speak of bodies as being in motion relatively to the earth, and, although the earth is going through a very complex series of movements, it in nowise affects our using it as a standard to which to refer the motions of bodies; it is evident that the relative motion of two bodies is not affected by any motions which they may have in common. Thus, when two bodies have a common motion, and at the same time are moving relatively to one another, we may treat the one as being stationary, and the other as moving relatively to it; that is to say, we may subtract their common

motion from each, and then regard the one as being at rest. Similarly, we may add a common motion to two moving bodies without affecting their relative motion. We shall find that such a treatment will be a great convenience in solving many problems in which we have two bodies, both of which are moving relatively to one another and to a third. As an example of this, suppose we are studying the action of a valve gear on a marine engine; it is a perfectly simple matter to construct a diagram showing the relative positions of the valve and piston. Precisely the same relations will hold, as regards the valve and piston, whether the ship be moving forwards or backwards, or rolling. In this case we, in effect, add or subtract the motion of the ship to the motion of both the valve and the piston.

Velocity.—Our remarks in the above paragraph, as regards relative motion, hold equally well for relative velocity.

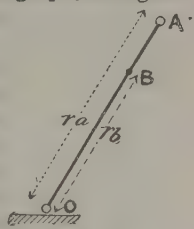


FIG. 142.

Many problems in mechanisms resolve themselves into finding the velocity of one part of a mechanism relatively to that of another. The method to be adopted will depend upon the very simple principle that the linear velocity of any point in a rotating body varies directly as the distance of that point from the axis or centre of rotation. Thus, when the link OA rotates about O, we have—

$$\frac{\text{velocity of A}}{\text{velocity of B}} = \frac{V_a}{V_b} = \frac{r_a}{r_b}$$

If the link be rotating with an angular velocity ω radians per second (see p. 4), then the linear velocity of a, viz. $V_a = \omega r_a$, and of b, $V_b = \omega r_b$, but the angular velocity of every point in the link is the same.

As the link rotates, every point in it moves at any given instant in a direction normal to the line drawn to the centre of rotation, hence at each instant the point is moving in the direction of the tangent to the path of the point, and the centre about which the point is rotating lies on a line drawn normal to the tangent of the curve at that point. This property will enable us to find the centre about which a body having plane motion is rotating. The plane motion of a body is completely known when we know the motion of any two points in the body. If the paths of the points be circular and concentric, then the centre of rotation will be the same for all positions of

the body. Such a centre is termed a "permanent" or "fixed" centre; but when the centre shifts as the body shifts, its centre at any given instant is termed its "instantaneous" or "virtual" centre.

Instantaneous or Virtual Centre.—Complex plane motions of a body can always be reduced to one very simply expressed by utilizing the principle of the virtual centre. For example, let the link *ab* be part of a mechanism having a complex motion. The paths of the two end points, *a* and *b*, are known, and are shown dotted. In order to find the relative velocities of the two points, we draw tangents to the paths at *a* and *b*, which give us the directions in which each is moving at the instant. From the points *a*, *b* draw normals *aa'* and *bb'* to the tangents; then the centre about which *a* is moving at the instant lies somewhere on the line *aa'*, likewise with *bb'*; hence the centre about which both points are revolving at the instant, must be at the intersection of the two lines, viz. at O. This

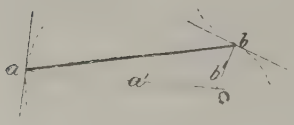


Fig. 143.

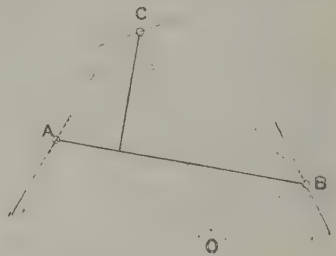


Fig. 144.

point is termed the *virtual* or *instantaneous* centre, and the whole motion of the link at the instant is the same as if it were attached by rods to the centre O. As the link has thickness normal to the plane of the paper, it would be more correct to speak of O as the plan of the virtual *axis*. If the bar had an arm projecting as shown in Fig. 144, the path of the point C could easily be determined, for every point in the body, at the instant, is describing an arc of a circle round the centre O; thus, in order to determine the path of the point C, all we have to do is to describe a small arc of a circle passing through C, struck from the centre O with the radius OC.

The radii OA, OB, OC are known as the virtual radii of the several points.

If the tangents to the point-paths at A and B had been parallel, the radii would not meet, except at infinity. In that

case, the points may be considered to be describing arcs of circles of infinite radius, *i.e.* their point-paths are straight parallel lines.

If the link AB had yet another arm projecting as shown in the figure, the end point of which coincided with the virtual centre O, it would, at the instant, have no motion at all relatively to the plane, *i.e.* it is a fixed point. Hence there is no reason why we should not regard the virtual centre as a point in the moving body itself.

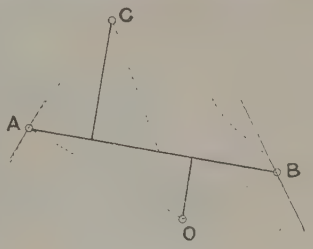


FIG. 145.

It is evident that there cannot be more than one of such fixed points, or the bar as a whole would be fixed, and then it could not rotate about the centre O.

It is clear, from what we have said on relative motion, that if we fixed the bar, which we will term *m* (Fig. 146), and move the plane, which we will term *n*, the relative motion of the two would be precisely the same. We shall term the virtual centre of the bar *m* relatively to the plane *n*, O*mn*.

Centrode and Axode.—As the link *m* moves in such a manner that its end joints *a* and *b* follow the point-paths, the virtual centre O*mn* also shifts relatively to the plane, and traces out the curve as shown in Fig. 146. This curve is simply the point-path of the virtual centre, or the virtual axis. This curve is known as the *centrode*, or *axode*.

Now, if we fix the link *m*, and move the plane *n* relatively to it, we shall, at any instant, obtain the same relative motion, therefore the position of the virtual centre will be the same in both cases. The centrodes, however, will not be the same, but as they have one point in common, viz. the virtual centre, they will always touch at this point, and as the motions of the two bodies continue, the two centrodes will roll on one another.

This rolling action can be very clearly seen in the simple four-bar mechanism shown in Fig. 147. The point A moves in the arc of a circle struck from the centre D, hence AD is normal to the tangent to the point-path of A; hence the virtual centre lies somewhere on the line AD. For a similar reason, it lies somewhere on the line BC; the only point common to the two is their intersection O, which is therefore their virtual centre. If the virtual centre, *i.e.* the intersection of the two

bars, be found for several positions of the mechanism, the centrodes will be found to be ellipses.

As the mechanism revolves, the two ellipses will be found to

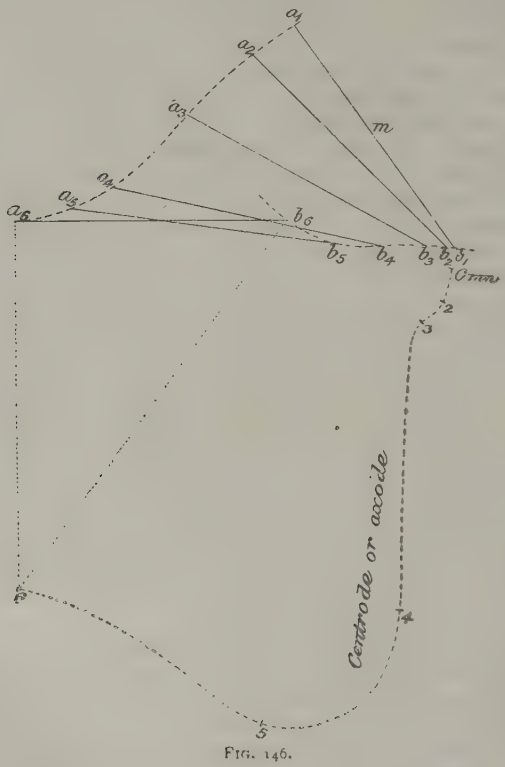


FIG. 146.

roll on one another, because A, B and C, D are the foci of the two ellipses. That such is the case can easily be proved experimentally, by a model consisting of two ellipses cut out of suitable material and joined by cross-bars AD and BC; it will be found that they will roll on one another perfectly.

Hence we see that, if we have given a pair of centrodes for two bodies, we can, by making the one centrode roll on the other, completely determine the relative motion of the two bodies.

Position of Virtual Centre.—We have shown above that when two point-paths of any body are known, we can

readily find the position of the virtual centre. In the case of most mechanisms, however, we can determine the virtual centres without first constructing the point-paths. We will show this by taking one or two simple cases. In the four-bar mechanism shown in Fig. 148, it is evident that if we consider *d* as stationary, the virtual centre O*ad* will be at the joint of *a* and *d*, and the velocity of any point in *a* relatively to any point in *d* will be proportional to the distance from this joint; likewise with O*dc*. Then, if we consider *b* as fixed, the virtual centre of *a* and *b* will also be at their joint. By similar reasoning, we have the virtual centre O*bc*. Again, let *d* be fixed, and consider the motion of *b* relatively to *d*. The point-path of one end of *b*, viz. O*ab*, describes the arc of a circle about O*ad*, therefore the virtual centre lies on *a* produced; for a similar reason, the virtual centre lies on *c* produced hence it must be at O*bd*, the meet of the two lines.

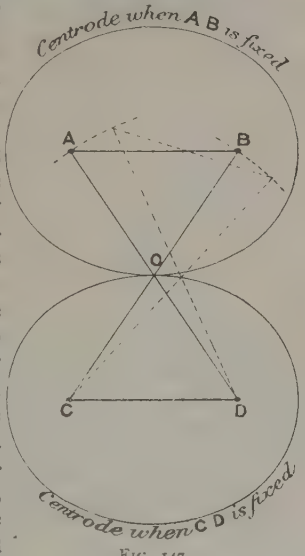


FIG. 147.

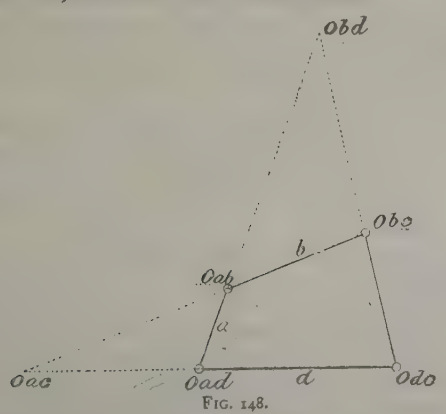


FIG. 148.

In a similar manner, consider the link *c* as fixed; then, for

the same reason as was given above for *b* and *d*, the virtual centre of *a* and *c* lies at the meet of the two lines *b* and *d*, viz. O*ac*.

If the mechanism be slightly altered, as shown in Fig. 149, we shall get one of the virtual centres at infinity, viz. O*cd*.

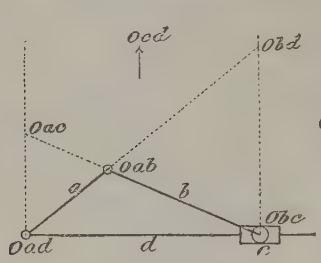


Fig. 149.

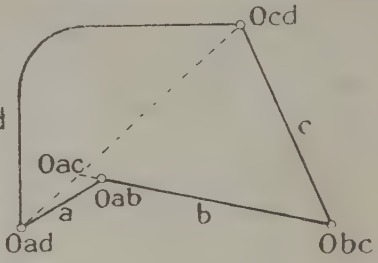


Fig. 150.

The mechanism shown in Fig. 149 is kinematically similar to the mechanism in Fig. 150. Instead of *c* sliding to and fro in guides, a link of any length may be substituted, and the fixed link *d* may be carried round in order to provide a centre from which *c* shall swing. Then it is evident that the joint O*bc* moves in the arc of a circle, and if *c* be infinitely long it moves in a straight line in precisely the same manner as the sliding link *c* in Fig. 149.

The only virtual centre that may present any difficulty in finding is O*ac*. Consider the link *c* as fixed, then the bar *d* swings about the centre O*cd*; hence every point in it moves in a path at right angles to a line drawn from that point to O*cd*. Hence the virtual centre lies on the line O*cd*, O*ad*; also, for reasons given below, it lies on the prolongation of the bar *b*, viz. O*ac*.

Three Virtual Centres on a Line.—By referring to the figures above, it will be seen that there are always three virtual centres on each line. In Figs. 149, 150, it must be remembered that the three virtual centres O*ad*, O*ac*, O*cd* are on one line; also O*bc*, O*bd*, O*cd*.

The proof that the three virtual centres corresponding to the three contiguous links must lie on one line is quite simple, and as this property is of very great value in determining the positions of the virtual centres for complex mechanisms, we will give it here. Let *b* (Fig. 151) be a body moving relatively to *a*, and let the virtual centre of its motion relative to *a* be O*ab*; likewise let O*ac* be the virtual centre of *c*'s motion relative to *a*. If we

want to find the velocity of a point in *b* relatively to a point in *c*, we must find the virtual centre, O*bc*. Let it be at O: then, considering it as a point of *b*, it will move in the arc 1.1 struck from the centre O*ab*; but considering it as a point in *c*, it will move in the arc 2.2 struck from the centre O*ac*. But the tangents of these arcs intersect at O, therefore the point O has a motion in two directions at the same time, which is impossible. In the same manner, it may be shown that the virtual centre O*bc* cannot lie anywhere but on the line joining O*ab*, O*ac*, for at that point only will the tangents to the point-paths at O coincide; therefore the three virtual centres must lie on one straight line.

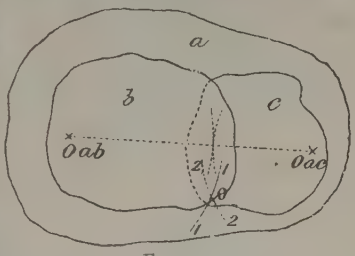


FIG. 151.

Relative Linear Velocities of Points in Mechanisms.—Once having found the virtual centre of any two bars of a mechanism, the finding of the velocity of any point in one bar relatively to that of any other point is a very simple matter, for their velocities vary directly as their virtual radii.

In the mechanism shown, let the bar *d* be fixed; to find the relative velocities of the points 1 and 2, we have—

$$\frac{\text{velocity } 1}{\text{velocity } 2} = \frac{O\,bd\ 1}{O\,bd\ 2} = \frac{r_1}{r_2}$$

Similarly—

$$\frac{\text{velocity } 1}{\text{velocity } 3} = \frac{r_1}{r_3}$$

and $$\frac{\text{velocity } 3}{\text{velocity } 4} = \frac{r_3}{r_4}$$

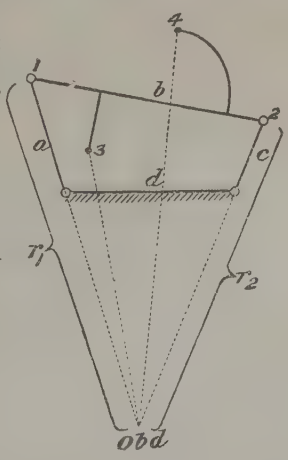


FIG. 152.

The relative velocities are not affected in the slightest degree by the *shape* of the bars.

When finding the velocity of a point on one bar relatively to the velocity of a point on another bar, it must be remembered

that at any instant the two bars move as though they had one point in common, viz. their virtual centre.

As an instance of points on non-adjacent bars, we will proceed to find the velocity of the point 8 (Fig. 153) relatively to that of point 9. By the method already explained, the virtual centre O*ac* is found, which may be regarded as a point in the bar *c* pivoted at O*cd*; likewise as a point in the bar *a* pivoted

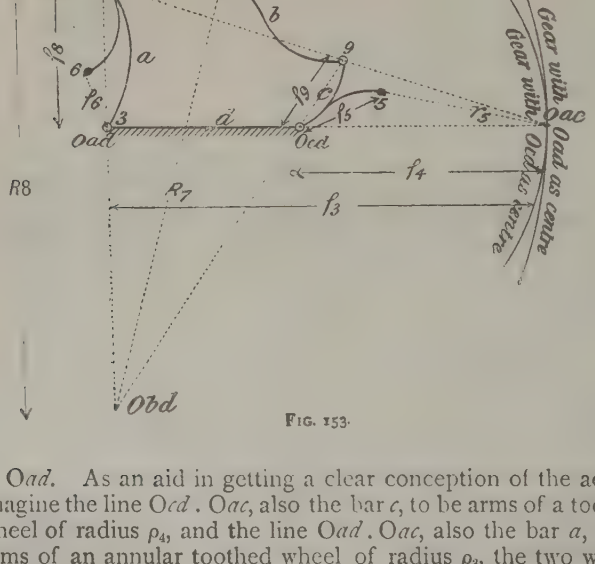


FIG. 153.

at O*ad*. As an aid in getting a clear conception of the action, imagine the line O*cd* . O*ac*, also the bar *c*, to be arms of a toothed wheel of radius ρ_4, and the line O*ad* . O*ac*, also the bar *a*, to be arms of an annular toothed wheel of radius ρ_3, the two wheels are supposed to be in gear, and to have the common point O*ac*, therefore their peripheral velocities are the same. Denoting the angular velocity of *a* as ω_a, and the linear velocity of the point 8 as V_8, etc., we have—

$$\omega_a\rho_3 = \omega_c\rho_4, \text{ and } \omega_a = \frac{\omega_c\rho_4}{\rho_3} \text{ also } V_8 = \omega_a\rho_8 \qquad V_9 = \omega_c\rho_9$$

Substituting the value of ω_a, we have—

$$\frac{V_8}{V_9} = \frac{\omega_c\rho_4\rho_8}{\omega_a\rho_3\rho_9} = \frac{\rho_4\rho_8}{\rho_3\rho_9} = \frac{1{\cdot}29 \times 0{\cdot}76}{2{\cdot}29 \times 0{\cdot}42} = 1{\cdot}02$$

or $$\frac{V_8}{V_9} = \frac{Obd\,.\,8}{Obd\,.\,9} = \frac{2{\cdot}20}{2{\cdot}16} = 1{\cdot}02$$

Similarly, if we require the velocity of the point 6 relatively to that of point 5—

$$\frac{V_6}{V_8} = \frac{\rho_6}{\rho_8} \qquad V_6 = \frac{V_8\rho_6}{\rho_8}$$

$$\frac{V_5}{V_9} = \frac{\rho_5}{\rho_9} \qquad V_5 = \frac{V_9\rho_5}{\rho_9}$$

$$\text{whence } \frac{V_6}{V_5} = \frac{V_8}{V_9} \cdot \frac{\rho_6\rho_9}{\rho_8\rho_5} = \frac{\rho_4\rho_8\rho_6\rho_9}{\rho_3\rho_9\rho_8\rho_5} = \frac{\rho_4\rho_6}{\rho_3\rho_5}$$

$$\text{or } \frac{V_8}{V_{oac}} = \frac{\rho_8}{\rho_3} \text{ and } \frac{V_5}{V_{oac}} = \frac{\rho_5}{\rho_4}$$

$$\frac{V_6}{V_5} = \frac{\rho_4\rho_6}{\rho_3\rho_5} = \frac{1{\cdot}29 \times 0{\cdot}3}{2{\cdot}29 \times 0{\cdot}47} = 0{\cdot}36$$

Likewise—

$$\frac{\text{velocity } 7}{\text{velocity } 8} = \frac{R_7}{R_8} = \frac{2{\cdot}31}{2{\cdot}20} = 1{\cdot}05$$

$$\text{hence } \frac{\text{velocity } 6}{\text{velocity } 7} = \frac{\rho_6 \times \text{velocity } 8 \times R_8}{\rho_8 \times R_7 \times \text{velocity } 8}$$

$$= \frac{\rho_6 \times R_8}{\rho_3 \times R_7} = \frac{0{\cdot}3 \times 2{\cdot}20}{0{\cdot}76 \times 2{\cdot}31} = 0{\cdot}38$$

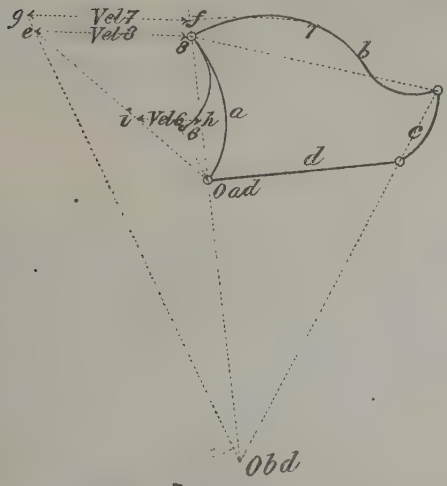


FIG. 154.

This can be arrived at much more readily by a graphical process; thus (Fig. 154): With O*ad* as centre, and ρ_6 as

radius, set off $Oad.h = \rho_6$ along the line joining Oad to 8; set off a line hi to a convenient scale in any direction to represent the velocity of 6. From Oad draw a line through i, and from 8 draw a line $8e$ parallel to hi; this line will then represent the velocity of the point 8 to the same scale as 6, for the two triangles $Oad.8.e$ and $Oad.h.i$ are similar; therefore—

$$\frac{8e}{hi} = \frac{Oad.8}{Oad.h} = \frac{\rho_8}{\rho_6} = \frac{\text{velocity } 8}{\text{velocity } 6} = \frac{0{\cdot}80}{0{\cdot}32} = 2{\cdot}5$$

From the centre Obd and radius R_7, set off $Obd.f = R_7$; draw $f.g$ parallel to $e.8$, and from Obd draw a line through e to meet this line in g; then $fg = V_7$, for the two triangles $Obd.f.g$ and $Obd.8.e$ are similar; therefore—

$$\frac{fg}{8e} = \frac{Obd.f}{Obd.8} = \frac{R_7}{R_8} = \frac{\text{velocity } 7}{\text{velocity } 8} = \frac{2{\cdot}3}{2{\cdot}2} = 1{\cdot}05$$

$$\text{and } \frac{\text{velocity } 6}{\text{velocity } 7} = \frac{ih}{fg} = \frac{0{\cdot}32}{0{\cdot}84} = 0{\cdot}38$$

The same graphical process can be readily applied to all cases of velocities in mechanisms.

Relative Angular Velocities of Bars in Mechanisms.—Every point in a rotating bar has the same angular velocity. Let a bar be turning about a point O in the bar with an angular velocity ω; then the linear velocity V_a of a point A situated at a radius r_a is—

$$V_a = \omega r_a, \text{ and } \omega = \frac{V_a}{r_a}$$

In order to find the relative angular velocities of any two links, let the point A (Fig. 155) be first regarded as a point in the bar a, and let its radius about Oad be r_A. When the point A is regarded as a point in the bar b, we shall term it B, and its radius about Obd, r_B. Let the linear velocity of A be V_A, and that of B be V_B, and the angular velocity of A be ω_A, and of B be ω_B. Then $V_A = \omega_A r_A$, and $V_B = \omega_B r_B$.

But $V_A = V_B$ as A and B are the same point; hence—

$$\omega_A r_A = \omega_B r_B$$

$$\text{or } \frac{\omega_A}{\omega_B} = \frac{r_B}{r_A} = \frac{(Obd)B}{(Oad)A}$$

This may be very easily obtained graphically thus: Set off a line A*e* in any direction from A, whose length on some given scale is equal to ω_A; join *e*.O*bd*; from O*ad* draw O*ad*.*f* parallel

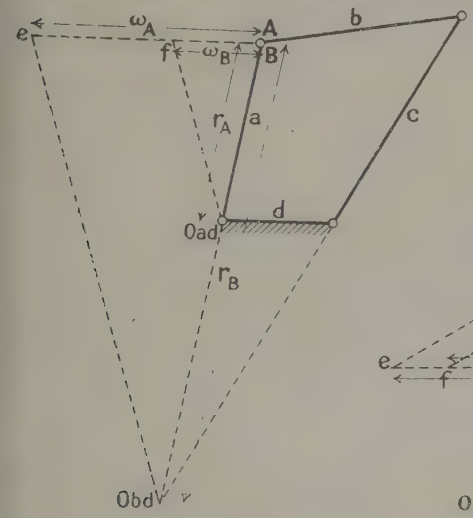


Fig. 155.

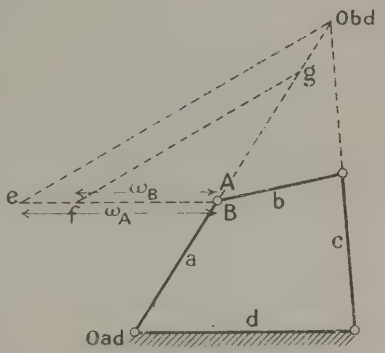


Fig. 156.

to *e*.O*bd*. Then A*f* = ω_B, because the two triangles A.*f*.O*ad* and A.*e*.O*bd* are similar. Hence—

$$\frac{Ae}{Af} = \frac{(Obd)B}{(Oad)A} = \frac{\omega_A}{\omega_B}$$

In Fig. 156, the distance A*g* has been made equal to A.O*ad*, and *gf* is drawn parallel to *e*.O*bd*. The proof is the same as in the last case. When *a* is parallel to *c*, the virtual centre is at infinity, and the angular velocity of *b* becomes zero.

When finding the relative angular velocity of two non-adjacent links, such as *a* and *c*, we proceed thus: For convenience we have numbered the various points instead of using the more cumbersome virtual centre nomenclature (Figs. 157 and 158). The radius 1.6 we shall term $r_{1.6}$, and so on.

Then, considering points 1 and 2 as points of the bar *b*, we have—

$$\frac{V_1}{V_2} = \frac{r_{1.6}}{r_{2.6}}$$

Then, regarding point 1 as a point in bar *c*, and regarding point 2 as a point in bar *a*—

$$V_1 = \omega_c \times r_{1,4} \qquad V_2 = \omega_a r_{2,3}$$

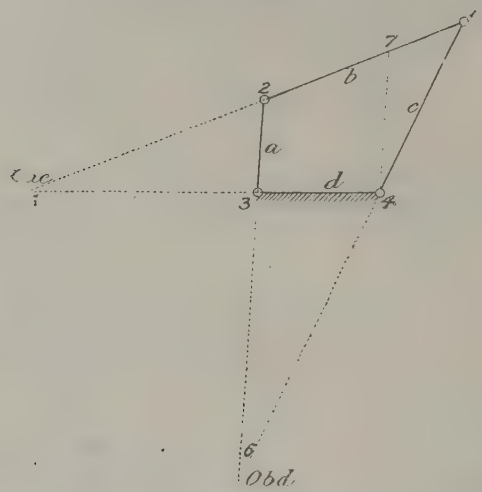


FIG. 157.

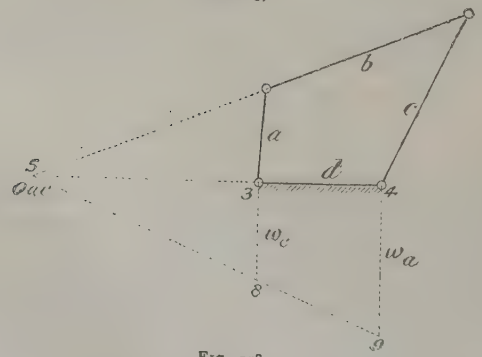


FIG. 158.

Then, substituting the values of V_1 and V_2 in the equation above, we have—

$$\frac{r_{1,4}}{r_{2,3}} = \frac{\omega_c \times r_{1,4}}{\omega_a \times r_{2,3}}, \text{ or } \frac{\omega_c}{\omega_a} = \frac{r_{1,4} \times r_{2,3}}{r_{2,3} \times r_{1,4}}$$

Draw 4.7 parallel to 2.3; then, by the similar triangles 1.2.6 and 1.7.4—

$$\frac{r_{2.6}}{4.7} = \frac{r_{1.6}}{r_{1.4}}$$

$$\text{and } r_{2.6} \times r_{1.4} = r_{1.6} \times 4.7$$

Substituting this value above, we have—

$$\frac{\omega_c}{\omega_a} = \frac{r_{1.6} \times r_{2.3}}{r_{1.6} \times 4.7} = \frac{r_{2.3}}{4.7}, \text{ or } = \frac{5.3}{5.4} \text{ by similar triangles}$$

Thus, if the length $r_{2.3}$ represents the angular velocity of *c*, and a line be drawn from 4 to meet the opposite side in 7, 4.7 represents on the same scale the angular velocity of *a*. Or it may conveniently be done graphically thus: Set off from 3 a line in any direction whose length 3.8 represents the angular velocity of *c*; from 4 draw a line parallel to 3.8; from 5 draw a line through 8 to meet the line from 4 in 9. Then 4.9 represents the angular velocity of *a*, the proof of which will be perfectly obvious from what has been shown above.

When *b* is parallel to *d*, the virtual centre O*ac* is at infinity, and the angular velocity of *a* is then equal to the angular velocity of *c*.

Steam-engine Mechanism.—On p. 126 we showed how a four-bar mechanism may be developed into the ordinary steam-engine mechanism, which is then often called the "slider-crank chain."

This mechanism appears in many forms in practice, but some of them are so disguised that they are not readily recognized. We will proceed to examine it first in its most familiar form, viz. the ordinary steam-engine mechanism.

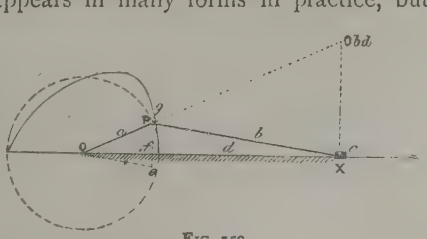


FIG. 159.

Having given the speed of the engine in revolutions per minute, and the radius of the crank, the velocity of the crank-pin is known, and the velocity of the cross-head at any instant is readily found by means of the principles laid down above. We have shown that—

$$\frac{\text{velocity P}}{\text{velocity X}} = \frac{Obd.\text{P}}{Obd.\text{X}}$$

From O draw a line parallel to the connecting-rod, and from P drop a perpendicular to meet it in *e*. Then the triangle O*Pe* is similar to the triangle P.O*bd*.X; hence—

$$\frac{OP}{Pe} = \frac{O\,bd.P}{O\,bd.X} = \frac{\text{velocity of pin}}{\text{velocity of cross-head}}$$

But the velocity of the crank-pin may be taken to be constant. Let it be represented by the radius of the crank-circle OP; then to the same scale P*e* represents the velocity of the cross-head. Set up $fg = Pe$ at several positions of the crank-pin, and draw a curve through them; then the ordinates of this curve represent the velocity of the cross-head at every point in the stroke, where the radius of the crank-circle represents the velocity of the crank-pin.

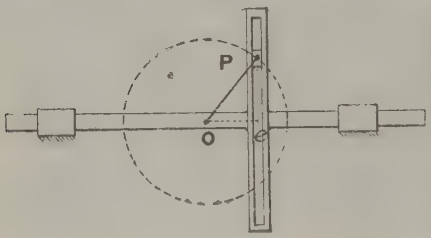


FIG. 160.

When the connecting-rod is of infinite length, or in the case of such a mechanism as that shown in Fig. 160, the line *gc* (Fig 159) is always parallel to the axis, and consequently the crosshead-velocity diagram becomes a semicircle.

An analytical treatment of these problems will be found in the early part of the next chapter.

Another problem of considerable interest in connection with the steam-engine is that of finding the journal velocity, or the velocity with which the various journals or pins rub on their brasses. The object of making such an investigation will be more apparent after reading the chapter on friction.

Let it be required to find the velocity of rubbing of (1) the crank-shaft in its main bearings; (2) the crank-pin in the big end brasses of the connecting-rod; (3) the gudgeon-pin in the small end brasses.

Let the radius of the crank-shaft journal be r_c, that of the crank-pin be r_p, and the gudgeon-pin r_g.

Let the number of revolutions per minute (N) be 160.

Let the radius of the crank be 1·25 feet.

Let the radii of all the journals be 0·25 foot. We have taken them all to be of the same size for the sake of comparing

the velocities, although the gudgeon-pin would usually be considerably smaller.

(1) $V_c = 2\pi r_c N$

or $\omega_a r_c = 250$ feet per minute in round numbers

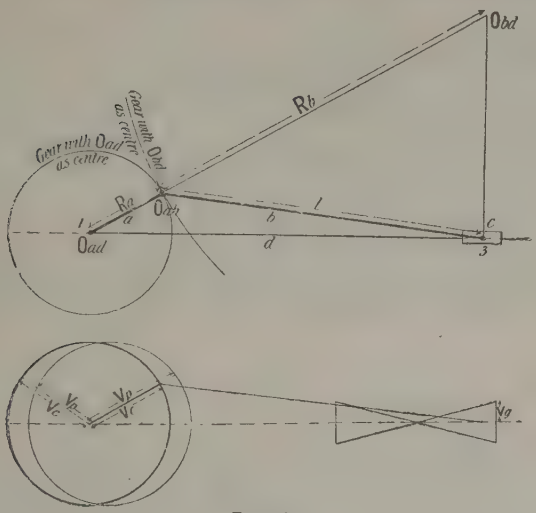


Fig. 161.

(2) We must solve this part of the problem by finding the relative angular velocity of the connecting-rod and the crank. Knowing the angular velocity of *a* relatively to *d*, we obtain the angular velocity of *b* relatively to *d* thus: The virtual centre *Oab* may be regarded as a part of the bar *a* pivoted at *Oad*, also as a part of the bar *b* rotating for the instant about the virtual centre *Obd*; then, by the gearing conception already explained, we have—

$$\frac{\omega_b}{\omega_a} = \frac{R_a}{R_b}, \text{ or } \omega_b = \frac{\omega_a R_a}{R_b} = \frac{V_2}{R_b}$$

When the crank-arm and the connecting-rod are rotating in the *opposite* sense, the rubbing velocity—

$$V_p = r_p(\omega_a + \omega_b) = r_p\omega_a\left(1 + \frac{R_a}{R_b}\right)$$

This has its maximum value when $\frac{R_a}{R_b}$ is greatest, *i.e.* when

R_b is least and is equal to the length of the connecting-rod, *i.e.* at the extreme "in" end of the stroke. Let the connecting-rod be n cranks long; then this expression becomes—

$$V_p = r_p\omega_a\left(1 + \frac{1}{n}\right)$$

which gives for the example taken—

$$= 314 \text{ feet per minute}$$

taking $n = 4$.

But when the crank-arm and the connecting-rod are rotating in the *same* sense, the rubbing velocity becomes—

$$V = r_p\omega_a\left(1 - \frac{R_a}{R_b}\right)$$

The polar diagram shows how the rubbing velocity varies at the several parts of the stroke.

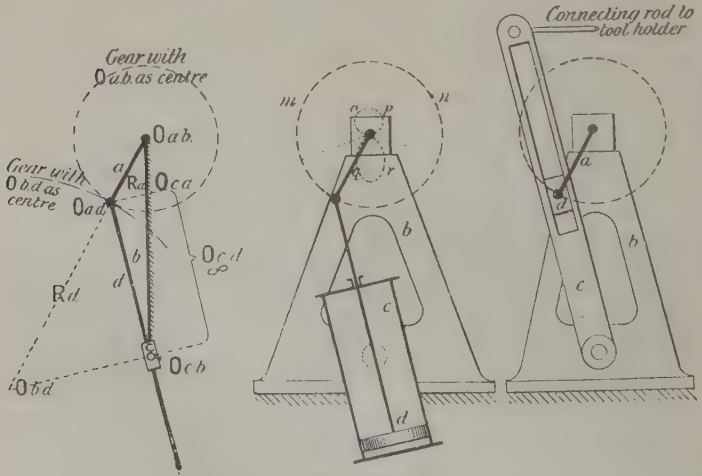


FIG. 161a.

(3) Since the gudgeon-pin itself does not rotate, the rubbing velocity is simply due to the angular velocity of the connecting-rod.

$$V_p = \omega_b r_g = \frac{\omega_a R_a r_g}{R_b}$$

which has its maximum value when R_b is least, viz. at the extreme end of the "in" stroke, and is then 63 feet per minute in the example we have taken.

By taking the same mechanism, and by fixing the link *b* instead of *d*, we get another familiar form, viz. the oscillating cylinder engine mechanism. On rotating the crank the link *d* becomes the connecting-rod, in reality the piston rod in this case, and the link *c* oscillates about its centre, which was the gudgeon-pin in the ordinary steam-engine, but in this case it is the cylinder trunnion, and the link *c* now becomes the cylinder of the engine. Another slight modification of the same inversion of the mechanism is one form of a quick-return motion used on shaping machines. In Fig. 161*a* we show the two side by side, and in the case of the engine we give a polar diagram to show the angular velocity of the link *c* at all parts of the stroke when *a* rotates uniformly.

We shall again make use of the gearing conception in the solution of this problem, whence we have—

$$\omega_a R_a = \omega_d R_d, \text{ and } \omega_d = \frac{\omega_a R_a}{R_d}$$

Taking the circle *mn* to represent the constant angular velocity of the crank, the polar curves *op*, *qr* represent to the same scale the angular velocity of the oscillating link *c* for corresponding positions of the crank. From these diagrams it will be seen that the swing to and fro of the cylinder is not accomplished in equal times. The inequality so apparent to an observer of the oscillating engine is usefully applied as a quick-return motion on shaping machines. The cutting stroke takes place during the slow swing of *c*, *i.e.* when the crank-pin is traversing the upper portion of its arc, and the return stroke is quickly effected while the pin is in its lower position. The ratio of the mean time occupied in the cutting stroke to that of the return stroke is termed the "ratio of the gear," which is readily determined. The link *c* is in its extreme position when the link *a* is at right angles to it; the cutting angle is $360 - \theta$, and the return angle θ (Fig. 161*b*).

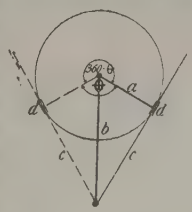


FIG. 161*b*.

$$\text{The ratio of the gear } R = \frac{360 - \theta}{\theta}$$

$$\text{or } \theta(R + 1) = 360^\circ$$

$$\text{and } a = b \cos \frac{\theta}{2}$$

Let R = 2; $\theta = 120^\circ$; $b = 2a$.
R = 3; $\theta = 90^\circ$; $b = 1{\cdot}42a$.

Another form of quick-return motion is obtained by fixing the link *a*. When the link *d* is driven at a constant velocity, the link *b* rotates rapidly during one part of its revolution and slowly during the other part. The exact speed at any instant can be found by the method already given for the oscillating cylinder engine.

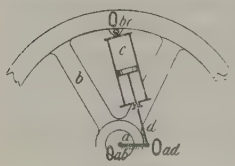


FIG. 161*c*.

The ratio R has the same value as before, but in this case we have $a = d \cos \frac{\theta}{2}$; therefore *d* must be made equal to $2a$ for a ratio of 2, and $1{\cdot}42a$ for a ratio of 3. This mechanism has also been used for a steam-engine, but it is best known as Rigg's hydraulic engine (Fig. 161*c*). This special form was adopted on account of its lending itself readily to a variation of the stroke of the piston as may be required for various powers. This variation is accomplished by shifting the point O*ad* to or from the centre of the flywheel O*ab*.

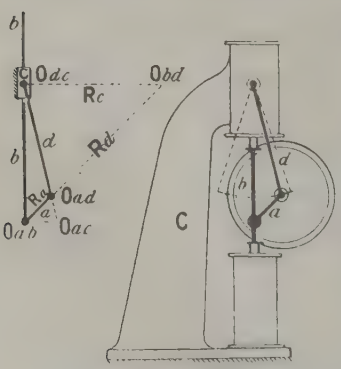


FIG. 161*d*.

A very curious development of the steam-engine mechanism is found in Stannah's pendulum pump (Fig. 161*d*). The link C is fixed, and the link *d* simply rocks to and fro; the link *a* is the flywheel, and the pin O*ab* is attached to the rim and works in brasses fitted in an eye in the piston-rod.

The velocity of the point O*ab* is the same as that of any other point in the link *b*. When C is fixed, the velocity of *b* relatively to C is the same as the velocity of C relatively to *b* when *b* is fixed, whence from p. 133 we have—

$$\frac{V_{Oab}}{V_{Oad}} = \frac{R_c}{R_d} = \frac{V_{Oab}}{\omega_a R_a}$$

$$\frac{V_{Oab}}{R_c} = \frac{\omega_a R_d}{R_d} = \omega_a$$

The Principle of Virtual Velocities applied to Mechanisms.—If a force acts on any point of a mechanism and overcomes a resistance at any other point, the work done at the two points must be equal if friction be neglected.

In Fig. 162, let the force P act on the link *a* at the point 2. Find the magnitude of the force R at the point 4 required to keep the mechanism in equilibrium.

If the bar *a* be given a small shift, the path of the point 2

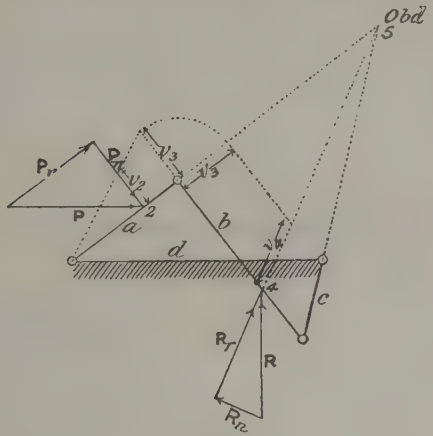


FIG. 162.

will be normal to the link *a*, and the path of the point 4 will be normal to the radius 5.4.

Resolve P along P*r* parallel to *a*, which component, of course, has no turning effort on the bar; also along the normal P*n* in the direction of motion of the point 2.

Likewise resolve R along the radius R*r*, and normal to the radius R*n*.

Now we must find the relative velocity of the points 2 and 4 by methods previously explained, and shown by the construction on the diagram. Then, as no work is wasted in friction, we have—[1]

[1] The small simultaneous displacements are proportional to the velocities.

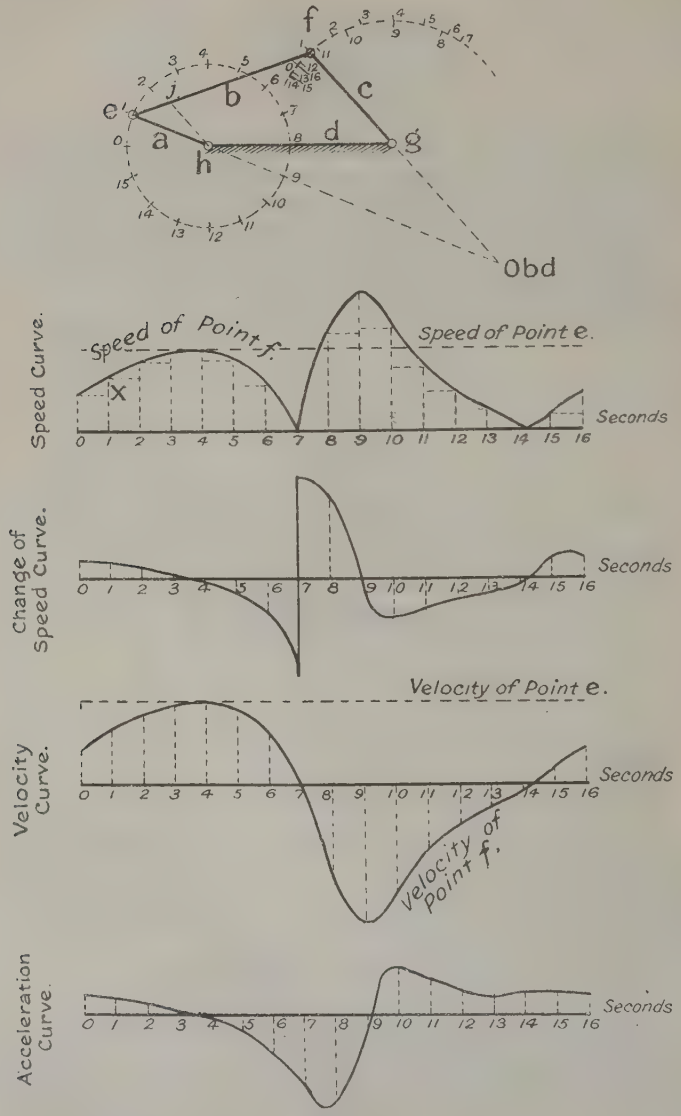

f
e'
a
b
c
d
g
h
j
Obd
Speed Curve.
Speed of Point f.
Speed of Point e.
x
Seconds
Change of Speed Curve.
Seconds
Velocity Curve.
Velocity of Point e.
Seconds
Velocity of point f.
Acceleration Curve.
Seconds

FIG. 163.

$$P_n V_2 = R_n V_4 \qquad \text{and } R_n = \frac{P_n V_2}{V_4}$$

Velocity and Acceleration Curves.—Let the link *a* revolve with a constant angular velocity. Curves are constructed to show the velocity of the point *f* relatively to the constant velocity of the point *e*.

Divide the circle that *e* describes into any convenient number of equal parts (in this case 16). The point *f* will move in the arc of a circle struck from the centre *g*; then, by means of a pair of compasses opened an amount equal to *ef*, from each position of *e* set off the corresponding position of *f* on the arc struck from the centre *g*. By joining up *he*, *ef*, *fg*, we can thus get every position of the mechanism; but only one position is shown in the figure for clearness. Then, in order to find the relative velocity of *e* and *f*, we produce the links *a* and *c* to obtain the virtual centre O*bd*. This will often come off the paper. We can, however, very easily get the relative velocities by drawing a line *hj* parallel to *c*. Then the triangles *hej* and O*bd.e.f* are similar, therefore—

$$\frac{he}{hj} = \frac{Obd.e}{Obd.f} = \frac{\text{velocity of } e}{\text{velocity of } f}$$

The velocity of *e* is constant; let the constant length of the link *a*, viz. *he*, represent it; then, from the relation above, *hj* will represent on the same scale the velocity of *f*.

Set off on a straight line the distances on the *e* curve 0.1, 1.2, 2.3, etc., as the base of the speed curve. At each point set up ordinates equal to *hj* for each position of the mechanism. On drawing a curve through the tops of these ordinates, we get a complete speed curve for the point *f* when the crank *a* revolves uniformly. The speed curve for the point *e* is a straight line parallel to the base.

In constructing the change of speed curve, each of the divisions 0.1, 1.2, 2.3, etc., represents an interval of one second, and if horizontals be drawn from the speed curve as shown, the height X represents the increase in the speed during the interval 0.1, *i.e.* in this case one second; then on the change of speed diagram the height X is set up in the middle of each space to show the mean change in speed that the point *f* has undergone during the interval 0.1, and so on for each space. A curve drawn through the points so obtained is the rate of change of speed curve for the point *f*.

The velocity curve is obtained in the same way as the speed curve, but it indicates the direction of the motion as well as its speed, and similarly in the case of the acceleration curve.

Let the curve (Fig. 164) represent the velocity of any point as it moves through space. Let the time-interval between the two dotted lines be dt, and the change of velocity of the point while passing through that space be dv. Then the acceleration during the interval is $\frac{dv}{dt}$, *i.e.* $\frac{\text{change of velocity}}{\text{interval of time}}$, or the change of velocity in the given interval.

By similar triangles, we have $\frac{dv}{dt} = \frac{zy}{xy}$.

When $dt = 1$ second, dv = mean acceleration.

Hence, make $xy_0 = 1$ on the time scale, then z_0y_0, the subnormal, gives us the acceleration measured on the same scale as the velocity.

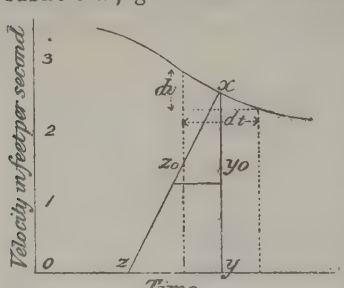


FIG. 164.

The sub-normals to the curve above have been plotted in this way to give the acceleration curve from 7 to 16. The scale of the acceleration curve will be the same as that of the velocity curve.

The reader is recommended to refer to Barker's "Graphical Calculus" and Duncan's "Practical Curve Tracing" (Longmans).

Velocity Diagrams for Mechanisms.—Force and reciprocal diagrams are in common use by engineers for finding the forces acting on the various members of a structure, but it is rare to find such diagrams used for finding the velocities of points and bars in mechanisms. We are indebted to Professor R. H. Smith for the method. (For fuller details, readers should refer to his own treatise on the subject.[1])

Let ABC represent a rigid body having motion parallel to the plane of the paper; the point A of which is moving with a known velocity V_A as shown by the arrow; the angular velocity ω of the body must also be known. If ω be zero, then every

[1] "Graphics," by R. H. Smith, Bk. I. chap. ix.; or "Kinematics of Machines," by R. J. Durley (J. Wiley & Sons).

point in the body will move with the same velocity as V_A. From A draw a line at right angles to the direction of motion as indicated by V_A, then the body is moving about a centre situated somewhere on this line, but since we know V_A and ω, we can find the virtual centre P, since $\omega R_A = V_A$. Join PB and PC, which are virtual radii, and from which we know the direction and velocities of B and C, because each point moves in a path at right angles to its radius, and its velocity is proportional to the length of the radius; thus—

$$V_B = \omega.PB, \text{ and } V_C = \omega.PC$$

The same result can be arrived at by a purely graphical process; thus—

From any pole *p* draw (1) the ray *pa* to represent V_A; (2) a ray at right angles to PB; (3) a ray at right angles to PC. These rays give the directions in which the points are moving.

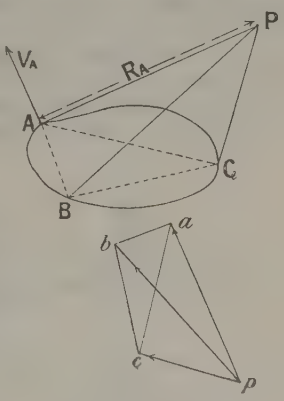


FIG. 165.

We must now proceed to find the magnitude of the velocities.

From *a* draw *ab* at right angles to AB; then *pb* gives the velocity of the point B. Likewise from *b* draw *bc* at right angles to BC, or from *a* draw *ac* at right angles to AC; then *pc* gives the velocity of the point C.

The reason for this construction is that the rays *pa*, *pb*, *pc* are drawn respectively at right angles to PA, PB, and PC, *i.e.* at right angles to the virtual radii; therefore the rays indicate the directions in which the several points move. The ray-lengths, too, are proportional to their several velocities, since the motion of B may be regarded as being compounded of a translation in the direction of V_A and a spin, in virtue of which the point moves in a direction at right angles to AB; the component *pa* represents its motion in the direction V_A, and *ab* its motion at right angles to AB, whence *pb* represents the velocity of B in magnitude and direction. Similarly, the point C partakes of the general motion *pa* in the direction V_A, and, due to the spin of the body, it moves in a direction at right angles to AC, viz. *ac*, whence *pc* represents the velocity of C. The point *c* can be equally well obtained by drawing *bc* at right angles to BC.

The triangular connecting-rod of the Musgrave engine can be readily treated by this construction. A is the crank-pin, whose velocity and direction of motion are known. The pistons are attached to the corners of the triangle by means of short connecting-rods, a suspension link DE serves to keep the connecting-rod in position, the direction in which D moves is at right angles to DE. Produce DE and AF to meet in P, which is the virtual centre of AD and FE. Join PB and PC. From the pole *p* draw the ray *pa* to represent the velocity of the point A, also draw rays at right angles to PB, PC, PD. From *a* draw a line at right angles to AD, to meet the ray at

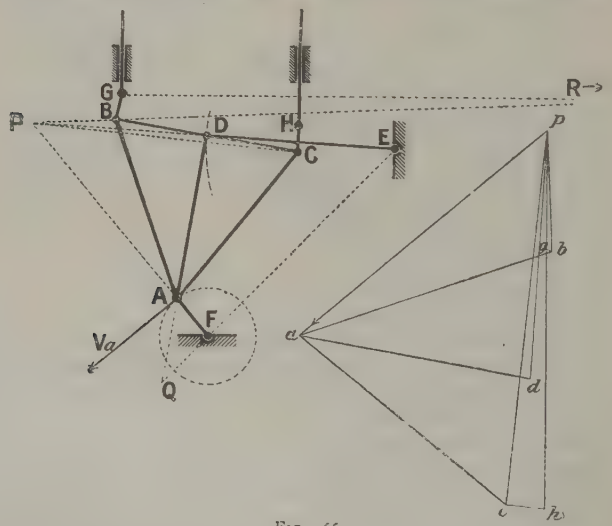


Fig. 166.

right angles to PD in *d*, also a line from *a* at right angles to AB, to meet the corresponding ray in *b*. Similarly, a line from *a* at right angles to AC, to meet the ray in *c*. Then *pb*, *pc*, *pd* give the velocities of the points B, C, D respectively. The velocity of the pistons themselves is obtained in the same manner.

As a check, we will proceed to find the velocities by another method. The mechanism ADEF is simply the four-bar mechanism previously treated; find the virtual centre of DE and AF, viz. Q; then (see Fig. 153)—

$$\frac{V_A}{V_D} = \frac{AF \,.\, EQ}{ED \,.\, QF}$$

The points C and B may be regarded as points on the bar AD, whence—

$$V_C = V_D \frac{PC}{PD}, \text{ and } V_B = V_D \frac{PB}{PD}, \text{ and } V = V_E \frac{RG}{RB}$$

Let the radius of the crank be 16", and the revolutions per minute be 80; then—

$$V_A = \frac{3{\cdot}14 \times 2 \times 16 \times 80}{12} = 670 \text{ feet per minute}$$

Then we get—

$V_B = 250$ feet per minute
$V_C = 790$ " "
$V_D = 515$ " "
$V_G = 246$ " "
$V_H = 792$ " "

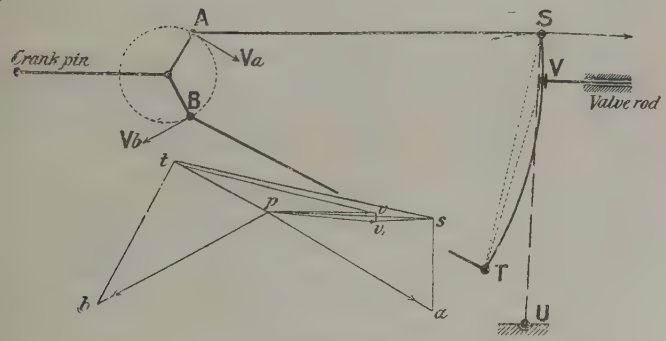


FIG. 167.

By taking one more example, we shall probably cover most of the points that are likely to arise in practice. We have selected that of a link-motion. Having given the speed of the engine and the dimensions of the valve-gear, we proceed to find the velocity of the slide-valve.

In the diagram A and B represent the centres of the eccentrics; the link is suspended from S. The velocity of the points A and B is known from the speed of the engine, and the direction of motion is also known of A, B, and S. Choose a pole *p*, and draw a ray *pa* parallel to the direction of the motion of A, and make its length equal on some given scale to

the velocity of A : likewise draw pb for the motion of B. Draw a ray from p parallel to the direction of motion of S, *i.e.* at right angles to US; through a draw a line at right angles to AS, to cut this ray in the point s. From s draw a line at right angles to ST ; from b draw a line at right angles to BT ; where this line cuts the last gives us the point t. Join pt, ps, which give respectively the velocities of T and S. From t draw a line at right angles to TV, and from S a line at right angles to SV ; they meet in v_1 : then pv_1 is the velocity of a point on the link in the position of V. But since V is guided to move in a straight line, from v_1 draw a line parallel to a tangent at V, and from p a line parallel to the valve-rod, meeting in v ; then pv is the required velocity of the valve, and vv_1 is the velocity of "slip" of the die in the link.

Cams.—When designing automatic and other machinery it often happens that the desired motion of a certain portion of the machine cannot readily be secured by the use of ordinary mechanisms such as cranks, links, wheels, etc.; cams must then be resorted to, but unless they are carefully constructed they often give trouble.

A rotating cam usually consists of a non-circular disc formed in such a manner that it imparts the desired reciprocating motion, in its own plane of rotation, to a body or follower which is kept in contact with the periphery of the disc.

Another form of cam consists of a cylindrical surface which rotates about its own axis, and has one or both edges of its curved surface specially formed to give a predetermined motion to a body which is kept in contact with them thus, causing it to slide to and fro in a direction parallel, or nearly so, to the axis of the shaft.

Generally speaking, it is a simple matter to design a cam to give the desired motion, but it is a mistake to assume that *any* conceivable motion whatever can be obtained by means of a cam. Many cams which work quite satisfactorily at low speeds entirely fail at high speeds on account of the inertia of the follower and its attachments. The hammering action often experienced on the valve stems of internal combustion engines is a familiar example of the trouble which sometimes arises from this cause.

Design of Cams.—(i) *Constant Velocity Cams.*—In dealing with the design of cams it will be convenient to take definite numerical examples. In all cases we shall assume that they rotate at a constant angular velocity. Let it be required to

design a cam to impart a reciprocating motion of 2 inches stroke to a follower moving at uniform speed (*a*) in a radial path, (*b*) in a segment of a circle.

In Fig. 168 three cams of different dimensions are shown, each of which fulfils condition (*a*), but it will shortly be shown that the largest will give more satisfactory results than the smaller cams. The method of construction is as follows :— Take the base circle *abcd*, say 3 inches diameter. Make *ce* = 2 inches, that is, the stroke of the follower. Draw the semicircle *efg* and divide it into a convenient number of equal parts—say

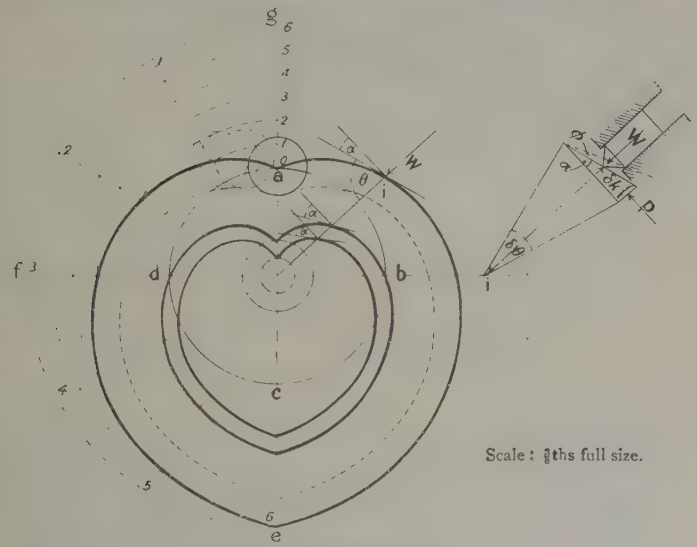


Fig. 168.

6—and draw the radii. Divide *ag*, the path of the follower, into the same number of equal parts, and from each division draw circles to meet the respective radii as shown, then draw the profile of the cam through these points. It will be obvious from the construction that the follower rises and falls equal amounts for equal angles passed through by the cam, and since the latter rotates at a constant angular speed, the follower therefore rises and falls at a uniform speed, the total lift being *ce* or *ag*. The two inner cams are constructed in a similar manner, but with base circles of 1 inch and ½ inch respectively. The form of the cam is the well-known Archimedian spiral.

The dotted profile shows the shape of the cam when the follower is fitted with a roller. The diameter of the roller must never be altered for any given cam or the timing will be upset.

In Fig. 169 a constant velocity cam is shown which raises its follower through 2 inches in $\frac{1}{3}$ of a revolution *a* to *e*, keeps

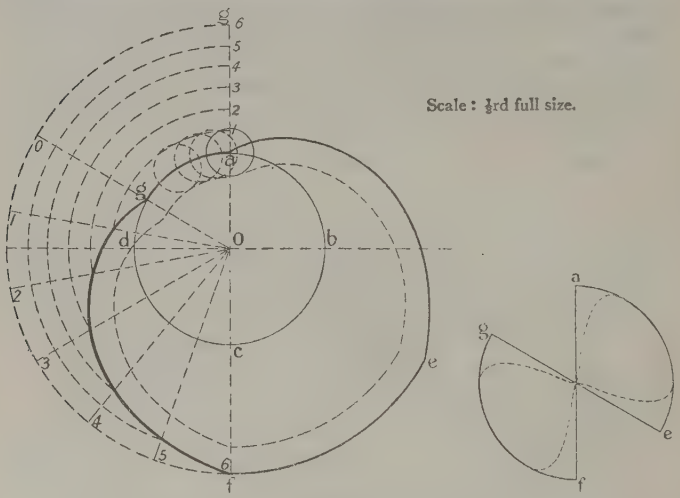


Fig. 169.

it there for $\frac{1}{6}$ of a revolution *e* to *f*, and then lowers it at a constant velocity in $\frac{1}{3}$ of a revolution *f* to *g* where it rests for the remaining $\frac{1}{6}$, *g* to *a*. The construction will be readily followed from the figure.

When the follower is attached to the end of a radius bar the point in contact with the cam moves in a circular arc *ag*, Fig. 170. In constructing the cam the curved path is reproduced at each interval and the points of intersection of these paths and the circles give points on the cam. The profile shown in broken lines is the form of the cam when the radius bar is provided with a roller.

The cams already dealt with raise and lower the follower at a constant velocity. At two points, *a* and *e*, Figs. 168 and 170, and at four points *a*, *e*, *f*, *g*, Fig. 169, the velocity of the follower, if it could be kept in contact with the cam, would be instantaneously changed and would thereby require an infinitely great force, which is obviously impossible; hence such cams

cannot be used in practice unless modified by "easing off" at the above-mentioned points, but even then the acceleration of the follower may be so great at high speeds that the cam face soon wears irregularly and causes the follower to run in an unsatisfactory manner. By still further easing the cam may be made to work well, but by the time all this easing has been

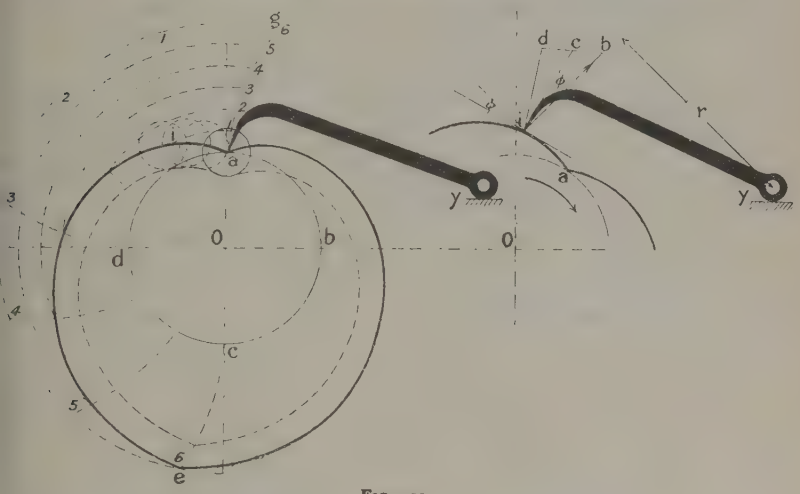


FIG. 170.

accomplished the cam practically becomes a simple harmonic cam.

(ii) *Constant acceleration or gravity cam.*—This cam, as its name suggests, accelerates the follower in exactly the same manner as a body falling freely under gravity, hence there is a constant pressure on the face of the cam when lifting the follower, but none when falling. The space through which a falling body moves is given by the well-known relation—

$$S = \tfrac{1}{2}gt^2$$

The total spaces fallen through in the given times are proportional to the values of S given in the table, that is, proportional to the squares of the angular displacements of the radius vectors.

Time, t	1	2	3	4	5	6
Space, S	1	4	9	16	25	36
Velocity, v . . .	1	2	3	4	5	6

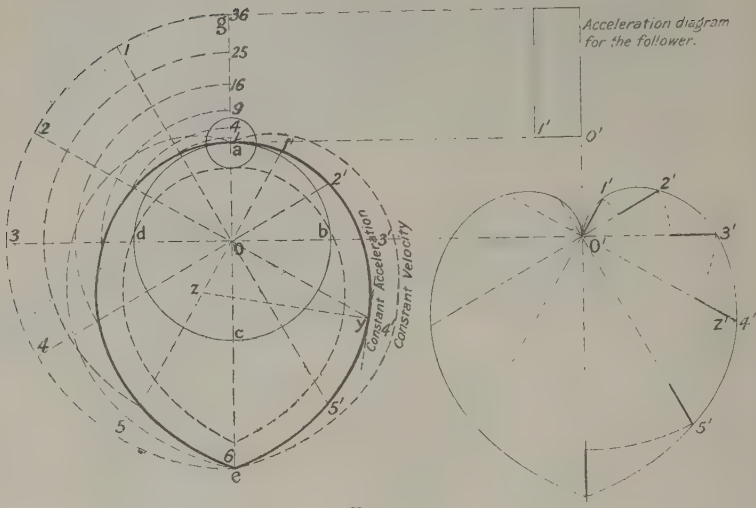


FIG. 171.

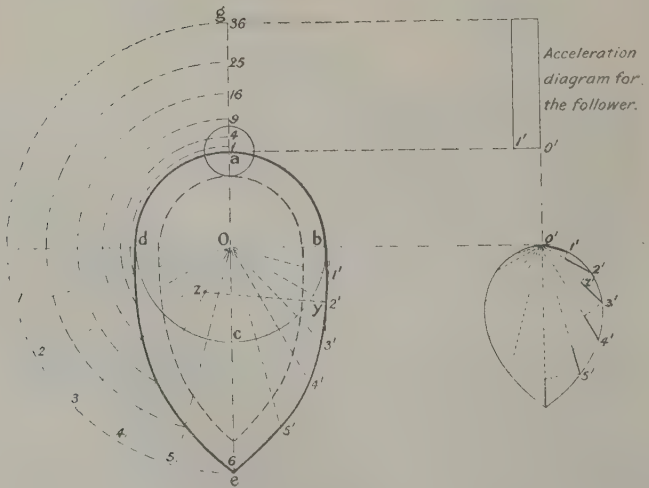


FIG. 172.

In Fig. 171 the length of the radius vectors which fall outside the circle *abcd* are set off proportional to the values of S given in the table. For the sake of comparison a "constant velocity" cam profile is shown in broken line. A cam of this design must also be "eased" off at *e* or the follower will leave the cam face at this point. As a matter of fact a true gravity cam is useless in practice; for this reason designers will do well to leave it severely alone.

In Fig. 172 the cam possesses the same properties, but the follower is idle during one half the time—*dab*—and is then accelerated at a constant rate during *be* for a quarter of a revolution and finally is allowed to fall with a constant retardation during *ed*, the last quarter of a revolution.

(iii) *Simple harmonic cam.*—In a simple harmonic cam the motion of the follower is precisely the same as that of a crosshead which is moved by a crank and infinitely long connecting rod, or its equivalent—the slotted crosshead, see Fig. 160. There would be no reason for using such a cam rather than a crank or eccentric if the motion were required to take place in one complete revolution of the cam, but in

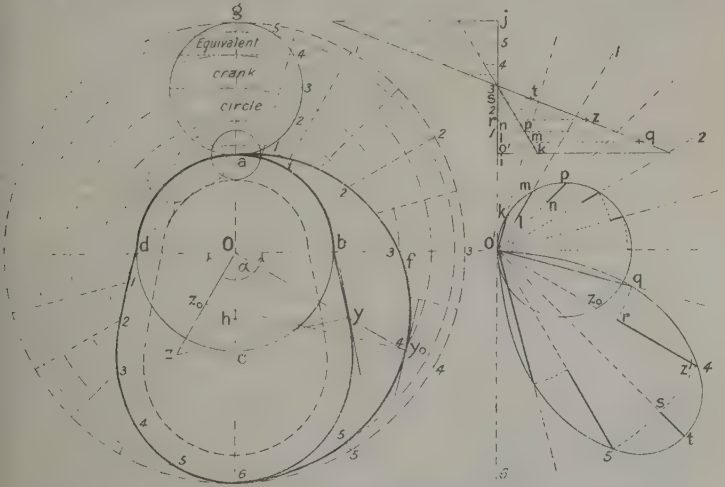


FIG. 173.

the majority of cases the s. h. m. is required to take place during a portion only of the revolution and the follower is stationary during the remainder of the time.

In the cam *abed* shown in Fig. 173 the follower is at rest for one half the time, during *dab*, and has s. h. m. for the remainder of the time, *bed*. The circle at the top of the figure represents the path of an imaginary crank pin, the diameter of the circle being equal to the stroke of the follower. To construct such a cam the semi-circumference of the equivalent crank circle is divided up into a number of equal parts, in this case six, to represent the positions of the imaginary crank pin at equal intervals of time. These points are projected on to the line *ag*, the path of the follower, and give its corresponding positions. From the latter points circles are drawn to cut the corresponding radii of the cam circle. If a cam be required to give s. h. m. to the follower without any period of rest the same construction may be used, the cam itself (only one half of which, *afe*, is shown) then becoming approximately a circle with its centre at *h*. The distance *ho* being equal to the radius of the imaginary crank viz. $\frac{ag}{2}$. If the cam were made truly circular, as in Fig. 174, it is evident that *oh*, the radius of the crank, and *ih*, the equivalent connecting rod, would be of constant length, hence the rotation of the cam imparts the same motion to the roller as a crank and connecting rod of the same proportions.

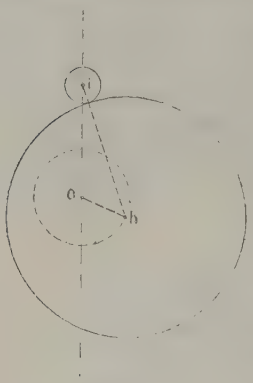


Fig 174.

Size of Cams.—The radius of a cam does not in any way affect the form of motion it imparts to the follower, but in many instances it greatly affects the sweetness of running. A cam of large diameter will, as a rule, run much more smoothly than one of smaller diameter which is designed to give precisely the same motion to the follower. The first case to be considered is that of a follower moving in a radial path in which guide-bar friction is neglected.

In Fig. 168 suppose the cam to be turned round until the follower is in contact at the point *i* (for convenience the follower has been tilted while the cam remains stationary). Draw a tangent to the cam profile at *i* and let the angle between the tangent and the radius be θ. The follower is, for the instant, acted upon by the equivalent of an inclined plane on the cam face whose angle of inclination is $\alpha = 90 - \theta$.

The force acting normally to the radius, $p = W \tan (\alpha + \phi)$ (see p. 291), where ϕ is the friction angle for the cam face and follower, and W is the radial pressure exerted by the follower. When $\alpha + \phi = 90°$ the follower will jamb in the guides and consequently the cam will no longer be able to lift it, and as a matter of fact in practice unless $\alpha + \phi$ is much less than 90° the cam will not work smoothly. Let the cam rotate through a small angle $\delta\theta$. Then a point on the cam at a radius ρ will move through a small arc of length $\delta S = \rho\delta\theta$. During this interval the follower will move through a radial distance δh.

Hence, $$\tan \alpha = \frac{\delta h}{\rho \delta \theta}$$

Thus for any given movement of the follower, the angle α, which varies nearly as the tangent for small angles, varies inversely as the radius of the cam ρ at the point of contact. By referring to Fig. 168 it will be seen that the angle α is much greater for the two smaller cams than for the largest. It has already been shown that the tendency to jamb increases as α increases, whence it follows that a cam of small radius has a greater tendency to jamb its follower than has a cam of larger radius.

The friction angle ϕ is usually reduced by fitting a roller to the follower.

When the friction between the guides and follower is considered, we have, approximately,

$$p = (W + p \tan \phi_1)(\tan \{\alpha + \phi\})$$

where ϕ_1 is the friction angle between the guide and follower.

The best practical way of reducing the guide friction of the follower is to attach it to a radius arm.

When the follower moves in a radial path the cam, if symmetrical in profile, will run equally well, or badly, in both directions of rotation, but it will work better in one direction and worse in the other when the follower is attached to a radius arm.

In Fig. 170 let the cam rotate until the follower is in contact at the point *i*, draw a tangent to the profile at this point, also a line making an angle ϕ, the friction angle, with it. The resultant force acting on the follower is *ib* where *id* is the pressure acting normally to the radius of the follower arm *iy* and *ic* is drawn normal to the tangent at the point *i*. The line *db* is drawn normal to *id*. The force *ib* has a clockwise moment $ib \times r$ about the pin *y* tending to lift the follower, if the slope of the cam is such that *bi* passes through *y* the cam

will jamb, likewise if in this particular case the moment of the force about *y* is contra-clockwise, no amount of turning effort on the cam will lift the follower.

Velocity ratio of Cams and Followers.—The cam A shown in Fig. 175 rotates about the centre O*ab*. Since the follower *c* moves in a straight guide, the virtual centre O*bc* is on a line drawn at right angles to the direction of motion of *c* and at infinity. Draw a tangent to the cam profile at the point of contact *y* which gives the direction in which sliding is taking place at the instant. The virtual centre of A and C therefore lies on a line passing through the point of contact *y* and normal to the tangent, it also lies on the line O*ab* O*bc*,

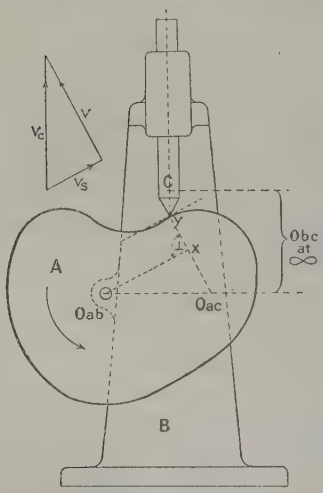


Fig. 175.

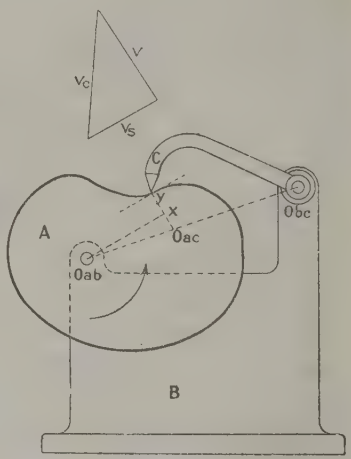


Fig. 176.

therefore it is on the intersection, viz. O*ac*. The virtual radius of the cam is O*ab* . *x*, *i.e.* the perpendicular distance of O*ab* from the normal at the point of contact. Let the angular velocity of the cam be ω_A, then $\omega_A \times Oab \,.\, x = V$ where V is the component of C's velocity in the direction *yx*. In the triangle of velocities, V_c represents the velocity of *c* in the guides, and V_s the velocity of sliding. This triangle is similar to the triangle O*ab* . O*ac* . *x* . hence

$$\frac{V_c}{V} = \frac{Oab \,.\, Oac}{Oab \,.\, x}$$

Substituting the value of V we have

$$\frac{V_c}{\omega_A \cdot Oab \cdot x} = \frac{Oab \cdot Oac}{Oab \cdot x}$$

$$\text{and } \frac{V_c}{\omega_A} = Oab \cdot Oac$$

In the case in which the follower is attached to an arm as shown in the accompanying figure, Fig. 176, the virtual centres are obtained in a similar manner, and

$$\omega_c Obc \cdot Oac = \omega_a Oab \cdot Oac$$

$$\frac{\omega_c}{\omega_a} = \frac{Oab \cdot Oac}{Obc \cdot Oac}$$

we also have the velocity V_c of the point of the follower in contact with the cam,

$$V_c = \omega_c Obc \cdot y$$

$$\frac{\omega_c}{\omega_a} = \frac{Oab \cdot Oac}{Obc \cdot Oac} = \frac{V_c}{Obc \cdot y \times \omega_a}$$

$$\frac{V_c}{\omega_a} = \frac{Oab \cdot Oac \cdot Obc \cdot y}{Obc \cdot Oac} = Oab \cdot Oac\left(\frac{Obc \cdot y}{Obc \cdot Oac}\right)$$

When the follower moves in a radial path the virtual centre is at infinity and the quantity in the bracket becomes unity; the expression then becomes that already found for this particular case.

In Figs. 171, 172, 173, polar velocity diagrams are given for the follower; the velocity has been found by the method given above, a tangent has been drawn to the cam profile at *y*, the normal to which cuts a radial line at right angles to the radius O*y* in the point *z*, the length O*z* is then transferred to the polar velocity diagram, viz. O′*z*′, in some cases enlarged for the sake of clearness.

In Figs. 168, 170, the velocity of the follower is zero at *a* and *e*, and immediately after passing these points the velocity is finite; hence if the follower actually moved as required by such cams, the change of velocity and the acceleration would be infinite at *a* and *e*, which is impossible; hence if such cams be used they must be eased off at the above-mentioned points.

In Fig. 169 there are four such points; the polar diagram

indicates the manner in which the velocity changes; the dotted lines show the effect of easing off the cam at the said points.

The cams shown in Figs. 171, 172, are constructed to give a constant acceleration to the follower, or to increase the velocity of the follower by an equal amount in each succeeding interval of time; hence the polar velocity diagram for such cams is of the same form as a constant velocity cam, viz. an Archimedian spiral.

Having constructed the polar velocity diagram, it is a simple matter to obtain the acceleration diagram from it, since the acceleration is the change of velocity per second. Assume, in the first place, that the time interval between any two adjacent radius vectors on the polar velocity diagram is one second, then the acceleration of the follower—to a scale shortly to be determined—is given by the difference in length between adjacent radius vectors as shown by thickened lines in Figs. 171, 172, 173.

The construction of the acceleration diagram for the simple harmonic cam is given in Fig. 173. The radius vector differences on the polar diagram give the mean accelerations during each interval; they are therefore set off at the middle of each space on the base line *ij*, and at right angles to it. It should be noted that these spaces are of equal length and, moreover, that *the same interval of time* is taken for the radius vector differences in both cams. In the case of the cam which keeps the follower at rest for one half the time, the lifting also has to be accomplished in one half the time, or the velocity is doubled; hence since the radial acceleration varies as the square of the velocity of the imaginary crank pin, the acceleration for the half-time cam is four times as great as that for the full time cam. Similar acceleration diagrams are obtained by a different process of reasoning in Chapter VI., p. 180. Analytical methods are also given for arriving at the acceleration in such a case as that now under consideration.

The particular case shown in Fig. 173 is chosen because the results obtained by the graphic process can be readily checked by a simple algebraic expression shortly to be given. In the above-mentioned chapter it is shown that when a crank makes N revolutions per minute, and the stroke of the crosshead, which is equivalent to the travel of the follower in the case of a simple harmonic cam, is 2R (measured in feet), the maximum force in pounds weight, acting along the centre line

of the mechanism, required to accelerate a follower of W pounds is $0{\cdot}00034WRN^2$ for the case in which the follower is lifted to its full extent in half a revolution of the cam, such as *afe* (only one half of the cam is shown). But when there is an idle period, such as occurs with the cam *abe*, Fig. 173, the velocity with which the follower is lifted is greater than before in the ratio $\frac{180}{\alpha}$ where α is the angle passed through by the cam while lifting the follower. Since the radial acceleration varies as the square of the velocity, we have for such a case—

$$\left.\begin{array}{l}\text{The maximum force}\\ \text{in pounds weight}\\ \text{required to accele-}\\ \text{rate the follower}\end{array}\right\} = \frac{0{\cdot}00034WRN^2 \times 180^2}{\alpha^2} = \frac{11{\cdot}03WRN^2}{\alpha^2}$$

The pressure existing between the follower and the face of a simple harmonic cam, when the follower falls by its own weight is—

maximum pressure, $$W + \frac{11{\cdot}03WRN^2}{\alpha^2}$$

minimum pressure, $$W - \frac{11{\cdot}03WRN^2}{\alpha^2}$$

When the follower is just about to leave the cam face—

$$W = \frac{11{\cdot}03WRN^2}{\alpha^2}$$

and the speed at which this occurs is—

$$N = \frac{0{\cdot}3\alpha}{\sqrt{R}}$$

Hence a simple harmonic cam must never run at a higher speed than is given by this expression, unless some special provision is made to prevent separation. A spring attached either directly to the follower or by means of a lever is often used for this purpose. When directly attached to the follower, the spring should be so arranged that it exerts its maximum effort when the follower is just about to leave the cam face, which should therefore be not less than $\frac{11{\cdot}03WRN^2}{\alpha^2} - W$, when the follower simply rests on the cam, and $\frac{11{\cdot}03WRN^2}{\alpha^2} + W$, when the follower is below the cam.

The methods for finding the dimensions of such springs are given in Chapter XIV.

Instead of a spring, a grooved cam (see Fig. 177), which is capable of general application, or a cam with double rollers

FIG. 177.

FIG. 178.

on the follower (see Fig. 178), can be used to prevent separation, but when wear takes place the mechanism is apt to be noisy, and the latter is only applicable to cases where the requisite movement can be obtained in 180° movement of the cam. When the follower is attached to an arm, the weight of the arm and its attachments W_a must be reduced to the equivalent weight acting on the follower.

Let K = the radius of gyration of the arm about the pivot.
r_c = the radius of the c. of g. of the arm about the pivot.
r = the radius of the arm itself from the follower to the pivot.
W_e = the equivalent weight acting on the follower when considering inertia effects.
W_c = Ditto, when considering the dead weight of the arm.

Then $W_e = \frac{W_a K}{r}$ and $W_c = \frac{W_a r_c}{r}$

Then $\frac{11{\cdot}03WRN^2}{a^2}$ becomes $\frac{11{\cdot}03W_aKRN^2}{ra^2}$

and W becomes $\frac{W_a r_c}{r}$ in the expressions above for the case when the follower is attached to a radius arm.

General Case of Cam Acceleration.—In Fig. 179, velocity and acceleration diagrams have been worked out for such a cam as that shown in Fig. 175. The polar velocity diagram has been plotted about a zero velocity circle, in which the radius vectors outside the circle represent velocities of the follower when it is rising, and those inside the circle when it is falling; they have afterwards been transferred to a straight base, and by means of the method given on p. 141, the acceleration diagram has also been constructed. It will be seen that even with such a simple-looking cam the accele-

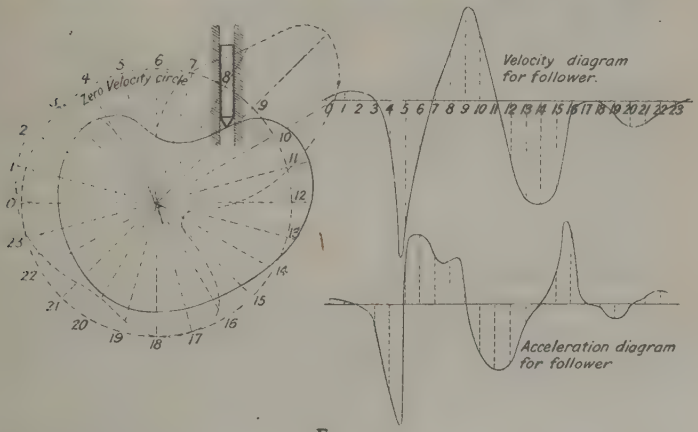


FIG. 179.

ration of the follower varies considerably in amount and rapidly, and the sign not unfrequently changes.

Scale of Acceleration Diagrams.—Let the drawing of the cam be $\frac{1}{n}$th full size, and let the length of the radius vectors on the polar velocity diagram be m times the corresponding length obtained from the cam drawing. In Fig. 171, $m = 3$. In Fig. 172, $m = 1$. In Fig. 173, $m = 2$.

Let the cam make N revolutions per minute, or $\frac{2\pi N}{60} = \frac{N}{9{\cdot}55}$ radians per second.

The velocity of the follower at any point where the length of the radius vector is $O'z'$ (see Fig. 173) measured in feet, is

$$V_o = \frac{Nn \,.\, O'z'}{9{\cdot}55m} \text{ feet per second.}$$

One foot length on the polar diagram represents $\frac{Nn}{9{\cdot}55m}$ feet per second

„ inch „ „ „ $\frac{Nn}{114{\cdot}6m}$ feet per second

Let the angle between the successive radius vectors be θ degrees, and let the change of velocity during one of the intervals be δV. Then each portion of the circle subtending θ represents an interval of time $\delta t = \frac{60\theta}{360N} = \frac{\theta}{6N}$ seconds.

The acceleration of the follower is—

$$\frac{\delta V}{\delta t} = \frac{\delta l \times N^2 \times n \times 6}{9{\cdot}55 \times m \times \theta} = \frac{\delta l \times N^2 \times n}{1{\cdot}59 \times m \times \theta}$$ feet per second per second.

Where δl is the difference in length of two adjacent radius vectors, measured in feet. The numeral in the denominator becomes 19·1 if δl is measured in inches.

The force due to the acceleration of the follower is in all cases found by multiplying the above expressions by $\frac{W}{g}$, or the equivalent when a rocking arm is used.

Useful information on the designing and cutting of cams will be found in pamphlet No. 9 of "Machinery's" Reference Series, published by "Machinery," 27, Chancery Lane; "A Method of Designing Cams," by Frederick Grover, A.M.I.C.E.; *Proceedings I.C.E.*, Vol. cxcii. p. 258.

The author wishes to acknowledge his indebtedness to Mr. Frederick Grover, of Leeds, for many valuable suggestions on cams.

Toothed Gearing.—In the figure let A and B be the end elevations of two shafts. A rotates at a given speed; it is required to drive B at some predetermined speed of rotation. If the shafts be provided with circular discs *a* and *b* of suitable diameters, and whose peripheries are kept in contact, the shaft A will drive B as desired so long as there is sufficient friction at the line of contact, but the latter condition is only realized when the twisting moment is small, and no slipping takes place. In order to prevent slipping each wheel

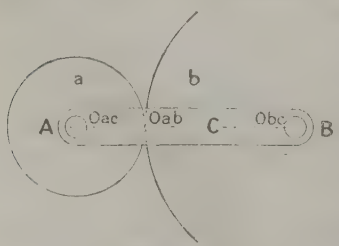


FIG. 180.

must be provided with teeth which gear with one another, and the form of which is such that the relative speeds of the two shafts are maintained at every instant.

The circles representing the two discs in Fig. 180 are known as the *pitch circles* in toothed gearing. The names given to the various portions of the teeth are given in Fig. 181.

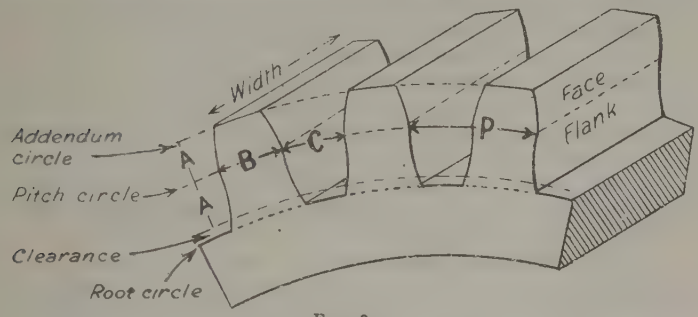


FIG. 181.

The pitch P is measured on the pitch circle. The usual proportions are A = 0·3P, B = 0·47 to 0·48P, C = 0·53 to 0·52P. The clearance at the bottom of the teeth is 0·1P, thus the total depth of the tooth is 0·7P. The width is chosen to suit the load which comes on each tooth—for light wheels it is often as small as 0·5P and for heavy gearing it gets up to 5P. Most books on machine design assume that the whole load is concentrated on the corner of the tooth, and the breadth B calculated accordingly, but gearing calculated on this basis is far heavier than necessary.

Velocity Ratio.—Referring to Fig. 180, let there be sufficient friction at the line of contact to make the one wheel revolve without slipping when the other is rotated, if this be so the linear velocity of each rim will be the same.

Let the radius of a be r_a, of b be r_b;
the angular velocity of a be ω_a, of b be ω_b;
N_a = the number of revolutions of a in a unit of time;
N_b = the number of revolutions of b in a unit of time.

$$\text{Then } \omega_a = \frac{2\pi r_a N_a}{r_a} = 2\pi N_a, \text{ and } \omega_b = 2\pi N_b$$

$$v = \omega_a r_a = \omega_b r_b$$

$$\text{or } \frac{r_b}{r_a} = \frac{\omega_a}{\omega_b} = \frac{2\pi N_a}{2\pi N_b} = \frac{N_a}{N_b}$$

or the revolutions of the wheels are inversely proportional to their respective radii.

The virtual centres of a and c, and of b and c, are evidently at their permanent centres, and as the three virtual centres must lie in one line (see p. 126), the virtual centre Oab must lie on the line joining the centres of a and b, and must be a point (or axis) common to each. The only point which fulfils these conditions is Oab, the point of contact of the two discs. To insure that the velocity ratio at every instant shall be constant, the virtual centre Oab must always retain its present position. We have shown that the direction of motion of any point in a body moving relatively to another body is normal to the virtual radius; hence, if we make a projection or a tooth, say on A (Fig. 182), the direction of motion of any point d relatively to B, will be a normal to the line drawn from d through the virtual centre Oab.

Likewise with any point in B relatively to A. Hence, if a projection on the one wheel is required to fit into a recess in the other, a normal to their surfaces at the point of contact must pass through the virtual centre Oab. If such a normal do not pass through Oab, the velocity ratio will be altered, and if Oab shifts about as the one wheel moves relatively to the other, the motion will be jerky.

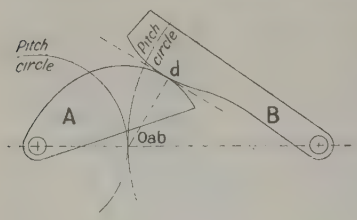


FIG. 182.

Hence, in designing the teeth of wheels, we must so form them that they fulfil the condition that the normal to their profiles at the point of contact must pass through the virtual centre of the one wheel relatively to the other, *i.e.* the point where the two pitch circles touch one another, or the point where the pitch circles cut the line joining their centres. An infinite number of forms might be designed to fulfil this condition; but some forms are more easily constructed than others, and for this reason they are chosen.

The forms usually adopted for the teeth of wheels are the cycloid and the involute, both of which are easily constructed and fulfil the necessary conditions.

If the circle e rolls on either the straight line or the arc of a circle f, it is evident that the virtual centre is at their point of contact, viz. Oef; and the path of any point d in the circle

moves in a direction normal to the line joining d to Oef, or normal to the virtual radius. When the circle rolls on a straight line, the curve traced out is termed a cycloid (Fig. 183); when on the outside of a circle, the curve traced out is

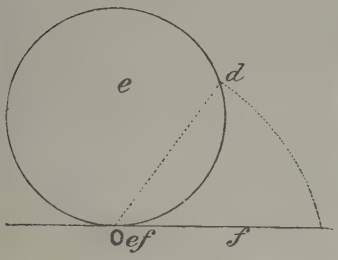


Fig. 183.

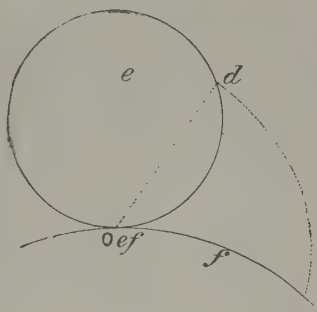


Fig. 184.

termed an epicycloid (Fig. 184); when on the inside of a circle, the curve traced out is termed a hypocycloid (Fig. 185).

If a straight line f (Fig. 186) be rolled without slipping on the arc of a circle, it is evident that the virtual centre is at their point of contact, viz. Oef, and the path of any point d in the line moves in a direction normal to the line f, or normal to the virtual radius. The curve traced out by d is an involute. It may be described by wrapping a piece of string round a circular disc and attaching a pencil at d; as the string is unwound d moves in an involute.

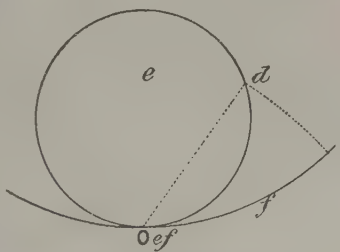


Fig. 185

When setting out cycloidal teeth, only small portions of the cycloids are actually used. The cycloidal portions can be obtained by construction or by rolling a circular disc on the pitch circle. By reference to Fig. 187, which represents a model used to demonstrate the theory of cycloidal teeth, the reason why such teeth gear together smoothly will be evident.

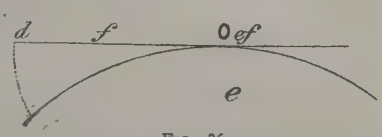


Fig. 186.

A and B are parts of two circular discs of the same diameter as the pitch circles; they are arranged on spindles, so that when the one revolves the other turns by the friction at the line of contact. Two small discs or rolling circles are provided with

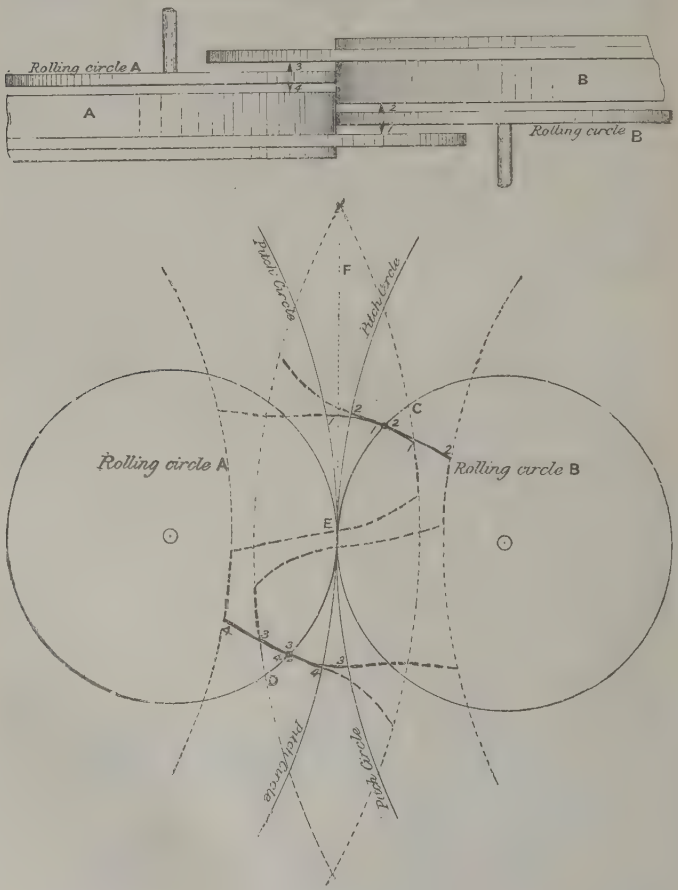


FIG. 187.

double-pointed pencils attached to their rims; they are pressed against the large discs, and turn as they turn. Each of the large discs, A and B, is provided with a flange as shown. Then, when these discs and the rolling circles all turn together,

the pencil-point 1 traces an epicycloid on the inside of the flange of A, due to the rolling of the rolling circle on A, in exactly the same manner as in Fig. 184; at the same time the pencil-point 2 traces a hypocycloid on the side of the disc B, as in Fig. 185. Then, if these two curves be used for the profiles of teeth on the two wheels, the teeth will work smoothly together, for both curves have been drawn by the same pencil when the wheels have been revolving smoothly. The curve traced on the flange of A by the point 1 is shown on the lower figure, viz. 1.1.1; likewise that traced on the disc B by the point 2 is shown, viz. 2.2.2. In a similar manner, the curves 3.3.3, 4.4.4, have been obtained. The full-lined curves are those actually drawn by the pencils, the remainder of the teeth are dotted in by copying the full-lined curves.

In the model, when the curves have been drawn, the discs are taken apart and the flanges pushed down flush with the inner faces of the discs, then the upper and lower parts of the curves fit together, viz. the curve drawn by 3 joins the part drawn by 2; likewise with 1 and 4.

From this figure it will also be clear that the point of contact of the teeth always lies on the rolling circles, and that contact begins at C and ends at D. The double arc from C to D is termed the "arc of contact" of the teeth. In order that two pairs of teeth may always be in contact at any one time, the arc CD must not be less than the pitch. The direction of pressure between the teeth when friction is neglected is evidently in the direction of a tangent to this arc at the point of contact. Hence, the greater the angle the tangent makes to a line EF (drawn normal to the line joining the centres of the wheels), the greater will be the pressure pushing the two wheels apart, and the greater the friction on the bearings; for this reason the angle is rarely allowed to be more than 30°. The effect of friction is to increase this angle during the arc of approach by an amount equal to the friction angle between the surfaces of the teeth in contact, and to diminish it by the same amount during the arc of recess. The effect of friction in reducing the efficiency is consequently more marked during approach than during recess, for this reason the teeth of the wheels used in watches and clocks are usually made of such a shape that they do not rub during the arc of approach. In order to keep this angle small, a large rolling circle must be used.

In many instances the size of the rolling circle has to be carefully considered. A large rolling circle increases the path

of contact and tends to make the gear run smoothly with a small amount of outward thrust, but as the diameter of the rolling circle is increased the thickness of the tooth at the flank is decreased and consequently weakened. If the diameter of the rolling circle be one half that of the pitch circle the flanks become radial, and if larger than that the flanks are undercut. Hence, in cases where the strength of the teeth is the ruling factor, the diameter of the rolling circle is never made less than one half the diameter of the smallest wheel in the train.

Generally speaking the same rolling circle is used for all the wheels required to gear together, but in special cases, where such a practice might lead to undercut flanks in the

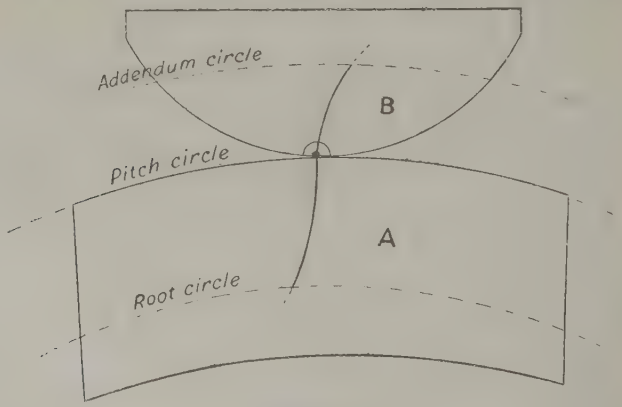
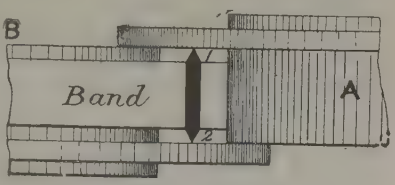


Fig. 188.

small wheels of a train, one rolling circle may be used for generating the faces of the teeth on a wheel A, and the flanks of the teeth they gear with on a wheel B, and another size of rolling circle may be used for the flanks of A and the faces of B. In some instances the teeth are made shorter than the standard proportions in order to increase their strength.

In setting out the teeth of wheels in practice, it is usual to make use of wooden templates. The template A (Fig. 188) is made with its inner and outer edges of the same radius as the pitch circle, and the edge of the template B is of the same radius as the rolling circle. A piece of lead pencil is attached by means of a clip to the edge of the template, and having its point *exactly* on the circumference of the circle. The template

A is kept in place on the drawing paper by means of a heavy weight or screws, a pencil run round the convex edge gives the pitch circle. The template B is placed with the pencil point just touching the pitch circle; it is then rolled, without slipping, on the edge of the template A and the pencil traces out the epicycloid required for the face of a tooth. The template A is then shifted until its concave edge coincides with the pitch circle. The template B is then placed with the pencil point on the pitch line, and coinciding with the first point of the epicycloid, then when rolled upon the inside edge of the template A, the pencil traces out the hypocycloid which gives the profile of the flank of the tooth. A metal template is then carefully made to exactly fit the profile of one side of the tooth thus obtained, and the rest of the teeth are set out by means of it. It is well known that cycloidal teeth do not work well in practice in cases where it is difficult to ensure ideal conditions as regards constancy of shaft centres, perfection



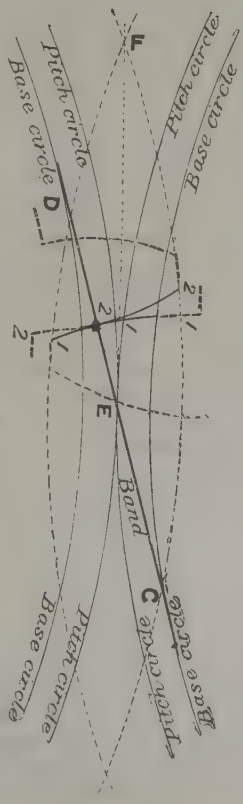


FIG. 189.

of workmanship, freedom from grit, etc. For this reason involute teeth are almost universally used for engineering purposes.

The model for illustrating the principle of involute teeth is shown in Fig. 189. Here again A and B are parts of two circular discs connected together with a thin cross-band which rolls off one disc on to the other, and as the one disc turns it makes the other revolve in the opposite direction. The band is provided with a double-pointed pencil, which is pressed against two flanges on the discs; then, when the discs turn, the pencil-points describe involutes on the two flanges, in exactly the same manner as that described on p. 164.

Then, from what has been said on cycloidal teeth, it is evident that if such curves be used as profiles for teeth, the two wheels will gear smoothly together, for they have been drawn by the same pencil as the wheels revolved smoothly together.

The point of contact of the teeth in this case always lies on the band; contact begins at C, and ends at D. The arc of contact here becomes the straight line CD. In order to prevent too great pressure on the axles of the wheels, the angle DEF seldom exceeds $15\frac{1}{2}^\circ$; this gives a base circle $\frac{63}{65}$ of the pitch circle. In special cases where pinions are required with a small number of teeth, this angle is sometimes increased to 20°.

Involute teeth can be set off by templates similar to those shown in Fig. 188, but instead of the template A being made to fit the pitch circle, it is curved to fit the base circle, and the template B is simply a straight-edge with a pencil attached and having its point on the edge itself.

If for any reason the distance between the centres of two involute gear wheels be altered by a small amount, the teeth will still work perfectly, provided the path of contact is not less than the pitch. By reference to Fig. 189 it will be evident that altering the wheel centres only alters the diameters of the pitch circles, but does not affect the diameters of the base circles upon which the velocity ratio entirely depends. This is a very valuable property of involute teeth, and enables them to be used in many places where the wheel centres cannot for many reasons be kept constant. If the same angle DEF, Fig. 189, be used in setting out the teeth, all involute wheels of the same pitch will gear with one another.

The portions of the flanks inside the base circles are made radial.

Readers interested in mechanical devices for drawing the teeth of wheels and the pistons for rotary pumps and blowers should refer to a paper by Dr. Hele-Shaw, F.R.S., in the *British Association Report* for 1898, an extract of which will be found in Dunkerley's "Mechanism" (Longmans). The general question of the design of toothed gearing will also be found in standard books on machine design. Readers should also refer to Anthony's "Essentials of Gearing" (D. C. Heath & Co., Boston, U.S.A.), and the series of pamphlets published by "Machinery," 27, Chancery Lane. No. 1, Worm Gearing; No. 15, Spur Gearing; No. 20, Spiral Gearing.

Velocity Ratio of Wheel Trains.—In most cases the problem of finding the velocity ratio of wheel trains is

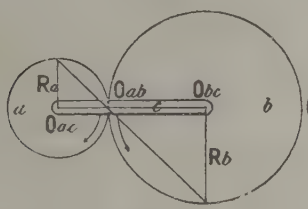


FIG. 190.

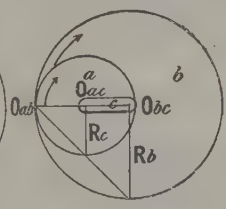


FIG. 191.

easily solved, but there are special cases in which difficulties may arise. The velocity ratio may have a positive or a negative value, according to the *form* of the wheels used; thus if *a* in Fig. 190 have a clockwise or + rotation, *b* will have an anti-clockwise or − rotation; but in Fig. 191 both wheels rotate in the same sense, since an annular wheel, *i.e.* one with internal teeth, rotates in the reverse direction to that of a wheel with external teeth. In both cases the velocity ratio is—

$$V_r = \frac{R_b}{R_a} = \frac{T_b}{T_a} = \frac{Oab \,.\, Obc}{Oab \,.\, Oac} = \frac{N_a}{N_b}$$

where T_a is the number of teeth in *a*, and T_b in *b*, and N_a is the number of revolutions per minute of *a* and N_b of *b*.

In the case of the three simple wheels in Fig. 192, we have the same peripheral velocity for all of them; hence—

$$\omega_a R_a = -\omega_b R_b = \omega_c R_c$$

$$\text{or } \frac{\omega_a}{\omega_c} = \frac{R_c}{R_a} = \frac{T_c}{T_a} = \frac{N_a}{N_c}$$

and the first and last wheels rotate in the same sense. The same velocity ratio could be obtained with two wheels only, but then we should have the sense of rotation reversed, since—

$$\omega_a R_a = -\omega_b R_b$$

$$\text{or } \frac{\omega_a}{\omega_b} = -\frac{R_b}{R_a}$$

Thus the second or "idle" wheel simply reverses the sense of rotation, and does not affect the velocity ratio. The velocity ratio is the same in Figs. 191, 192, 193, but in the second and third cases the sense of the last wheel is the same as that of the first. When the radius line R_c of the last wheel falls on the same side of the axle frame as that of the first wheel, the two rotate in the same sense; but if they fall on opposite sides, the wheels rotate in opposite senses. When the second wheel is compound, *i.e.* when two wheels of different sizes are fixed to one another and revolve together, it is no longer an idle wheel, but the sense of rotation is not altered. If it is desired to get the same velocity ratio with an idle wheel in the train as with a compound wheel, the wheel c must be altered in the proportion $\frac{b}{b'}$, where b is the driven and b' the driver.

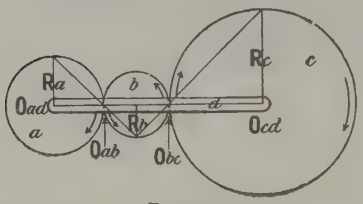


FIG. 192.

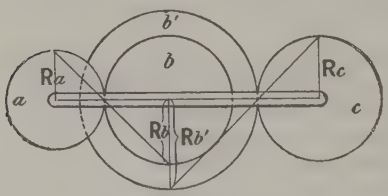


FIG. 193.

The velocity ratio V_r of this train is obtained thus—

$$\omega_a R_a = -\omega_b R_b$$

$$\text{and } -\omega_{b'} R_{b'} = \omega_c R_c$$

Substituting the value of $-\omega_{b'}$ or ω_b, we get—

$$\omega_a R_a = \frac{\omega_c R_c R_b}{R_{b'}}$$

$$\text{and } V_r = \frac{\omega_a}{\omega_c} = \frac{R_c R_b}{R_a R_{b'}} = \frac{T_c T_b}{T_a T_{b'}} = \frac{N_a}{N_c}$$

Thus, taking *a* as the driving wheel, we have for the velocity ratio—

$$\frac{\text{Revolutions of driving wheel}}{\text{Revolutions of the last wheel in train}} = \frac{\text{product of the radii or number of teeth in driven wheels}}{\text{product of the radii or number of teeth in driving wheels}}$$

The same relation may be proved for any number of wheels in a train.

If C were an annular wheel, the virtual centres would be as shown. R_c is on the opposite side of *d* to R_a; therefore the wheel *c* rotates in the opposite sense to *a* (Fig. 194).

In some instances the wheel C rotates on the same axle as *a*; such a case as this is often met with in the feed arrangements of a drilling-machine (Fig. 195). The wheel A fits loosely on the outside of the threads of the screwed spindle S, and is driven by means of a feather which slides in a sunk keyway, the wheels B and B′ are both keyed to the same shaft; C, however, is a nut which works freely on the screw S. Now, if A and C make the same number of revolutions per minute, the screw will not advance, but if C runs faster than A, the screw will advance; the number of teeth in the several wheels are so arranged that C shall do so. For example—Let A have 30 teeth, B, 20, B′, 21, C, 29, and the screw have four threads per inch: find the linear advance of the screw per revolution of S. For one revolution of A the wheel C makes $\frac{30 \times 21}{20 \times 29} = \frac{630}{580} = 1{\cdot}086$ revolution. Thus, C makes 0·086 revolution relatively to A per revolution of the spindle, or it advances the screw $\frac{0{\cdot}086}{4} = 0{\cdot}021$ inch per revolution of S.

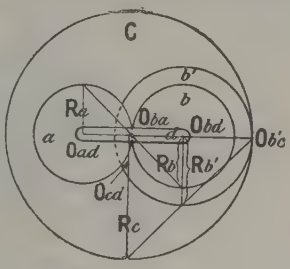


FIG. 194.

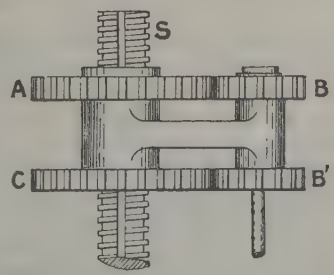


FIG. 195.

Change Speed Gears.—The old-fashioned back gear

so common on machine tools is often replaced by more convenient methods of changing the speed. The advent of motor-cars has also been responsible for many very ingenious devices for rapidly changing speed gears. The sliding key arrangement of Lang is largely used in many change-speed gears; in this arrangement all the wheels are kept continuously in mesh, and the two which are required to transmit the power are thrown into gear by means of a sliding cotter or key. In Fig. 196 the wheels on the shaft A are keyed, whereas the wheels on the hollow shaft B are loose, the latter wheels are each provided with six keyways, a sliding key or cotter which passes through slots in the shaft engages with two of these keyways in one of the desired wheels. The wheel to be driven is

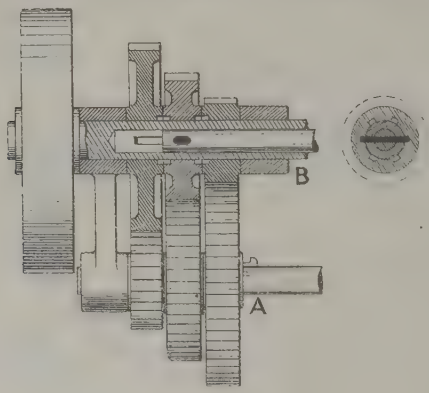


FIG. 196.

determined by the position of the key, which is shifted to and fro by means of a rod which slides freely in the hollow shaft. The bosses of the wheels are counterbored to such an extent that when the key is shifted from the one wheel to the next both keyways are clear of the key, and consequently both wheels are free. The sliding rod is held in position by a suitable lever and locking gear, which holds it in any desired position.

Epicyclic Trains.—In all the cases that we have considered up to the present, the axle frame on which the wheels are mounted is stationary, but when the frame itself moves, its own rotation has to be added to that of the wheels. In the mechanism of Fig. 197, if the bar *c* be fixed, and the wheel *a* be rotated in clockwise fashion, the point *x* would approach *c*, and the wheel *b* would rotate in contra-clockwise fashion.

If a be fixed, the bar c must be moved in contra-clockwise fashion to cause c and x to approach, but b will still continue to move in contra-clockwise fashion. Let c be rotated through one complete revolution in contra-clockwise fashion; then b will make $-N$ revolutions due to the teeth, where $N = \frac{T_a}{T_b}$, and at the same time it will make -1 revolution due to its bodily rotation round a, or the total revolutions of b will be $-N-1$ or $-(N+1)$ revolutions relative to a, the fixed wheel.

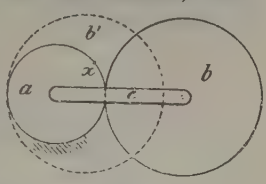


FIG. 197.

The $-$ sign is used because both the arm and the wheel rotate in a contra-clockwise sense. But if b had been an annular wheel, as shown by a broken line, its rotation would have been of the opposite sense to that of c; consequently, in that case, b would make $N-1$ revolutions to one of c.

If either idle or compound wheels be introduced, as in Fig. 198, we get the revolutions of each wheel as shown for each revolution of the arm e, where $v_{rb} = \frac{T_b}{T_a}$, and $v_{rc} = \frac{T_c}{T_a}$, when there is an idle wheel between, or $v_{rc} = \frac{T_b \cdot T_c}{T_a T_b'}$ when there is a compound wheel.

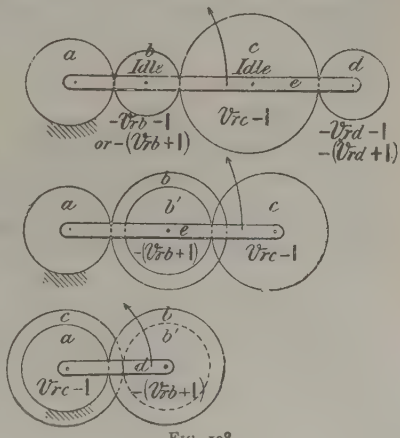


FIG. 198.

In the last figure the wheel c is mounted loosely on the same axle as a and d. In this arrangement neither the velocity ratio nor the sense is altered.

The general action of all epicyclic trains may be summed up thus: The number of revolutions of any wheel of the train for one revolution of the arm is the number of revolutions that the wheel would make if the arm were fixed, and the first wheel were turned through one revolution, $+1$ for wheels that rotate in the same sense as the arm, and -1 for wheels that rotate in the opposite sense to the arm.

In the case of simple, *i.e.* not bevil trains, it should be remembered that wheels on the nth axle rotate in the same sense as the arm when n is an *even* number, and in the opposite sense to that of the arm when n is an *odd* number, counting the axle of the fixed wheel as "one." Hence we have—

n	Sense of rotation of nth wheel.
Even	Same as arm.
Odd	Same as arm if V_r is less than 1.
	Opposite to arm if V_r is greater than 1.

Epicyclic Bevil Trains.—When dealing with bevil trains the sense of rotation of each wheel must be carefully

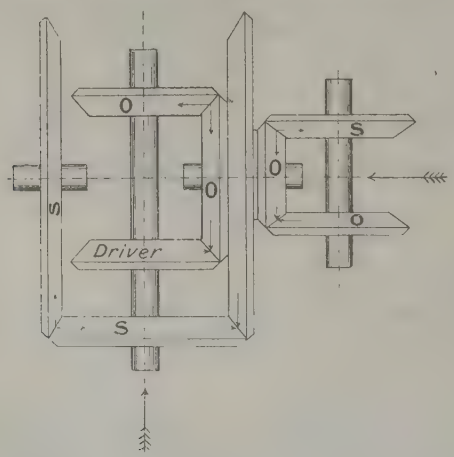


Fig. 199.

considered, and apparently no simple rule can be framed to cover every case; in Fig. 199 the several wheels are marked S for same and O for opposite senses of rotation—arrows on the wheels are of considerable assistance in ascertaining the sense of rotation. The larger arrows indicate the direction in which the observer is looking.

The bevil train shown in Fig. 200 is readily dealt with, thus let T_a, T_c, etc., represent the number of teeth in *a* and *c* respectively—*b* is an idle wheel, and consequently T_b does not affect the velocity ratio. Fix D, and rotate *a* contra-clockwise through one revolution, then *c* makes $\frac{T_a}{T_c}$ revolutions in a clockwise direction. Fix *a* and rotate the arm D through one clockwise revolution. Then since the tooth of *b*, which meshes with the stationary tooth of *a*, may be regarded as the fulcrum of a lever; hence the tooth of *b*, which meshes with *c*, moves in the same direction as D, and at twice the speed. Hence the number of revolutions of *c* is $\frac{T_a}{T_c} + 1$ for one revolution of D. The problem may also be dealt with in the same manner as the simple epicyclic train. The number of revolutions of any wheel of the train for one revolution of the arm is the number of revolutions that the wheel would make if the arm were fixed and the first wheel were turned through one revolution, + 1 for wheels that rotate in the same sense as the arm, and − 1 for wheels that rotate in the opposite sense to the arm.

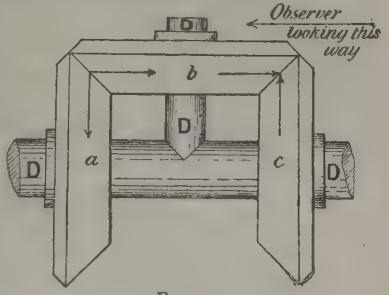


FIG. 200.

For elementary bevil trains, such as that shown in Fig. 200, wheels on the *n*th axle rotate in the opposite sense to the arm when *n* is an *even* number, and in the same sense when *n* is an odd number. Hence we have for elementary bevil trains—

n	Sense of rotation of *n*th wheel.
Odd	Same as arm.
Even	Same as arm if V_r is less than 1.
	Opposite to arm if V_r is greater than 1.

In all cases the actual number of revolutions per minute of the several wheels calculated for one revolution of the arm must be multiplied by the number of revolutions per minute of the arm N_d, also, if the wheel *a* rotates, its revolutions per minute must be added, with due regard to the sign, to the revolutions per minute of each wheel calculated for a stationary wheel.

The following table may be of assistance in this connection.

N_a	N_d	N_c when $\frac{T_a}{T_c} = 1$.	N_c when $\frac{T_a}{T_c} = 1{\cdot}5$.
		$V_{rc} + 1 = 2$.	$V_{rc} + 1 = 2{\cdot}5$.
100	0	-100	-150
100	5	$-100+2\times 5=-90$	$-150+2{\cdot}5\times 5=-137{\cdot}5$
-100	5	$100+2\times 5=110$	$150+2{\cdot}5\times 5=162{\cdot}5$
100	-5	$-100-2\times 5=-110$	$-150-2{\cdot}5\times 5=-162{\cdot}5$
100	20	$-100+2\times 20=-60$	$-150+2{\cdot}5\times 20=-100$
100	50	$-100+2\times 50=0$	$-150+2{\cdot}5\times 50=-25$
100	60	$-100+2\times 60=20$	$-150+2{\cdot}5\times 60=0$
100	70	$-100+2\times 70=40$	$-150+2{\cdot}5\times 70=25$
100	100	$-100+2\times 100=100$	$-150+2{\cdot}5\times 100=100$
-100	100	$100+2\times 100=300$	$150+2{\cdot}5\times 100=400$
100	-100	$-100-2\times 100=-300$	$-150-2{\cdot}5\times 100=-400$
40	30	$-40+2\times 30=20$	$-60+2{\cdot}5\times 30=15$
40	0	-40	-60
0	5	$2\times 5=10$	$2{\cdot}5\times 5=12{\cdot}5$
0	-100	$2(-100)=-200$	$2{\cdot}5(-100)=-250$

Humpage's Gear.—This compound epicyclic bevil train is used by Messrs. Humpage, Jacques, and Pedersen, of Bristol, as a variable-speed gear for machine tools (see *The Engineer*, December 30, 1898).

The number of teeth in the wheels are: A = 46, B = 40, B_1 = 16, C = 12, E = 34. The wheels A and C are loose on the shaft F, but E is keyed. The wheel A is rigidly attached to the frame of the machine, and C is driven by a stepped pulley; the arm *d* rotates on the shaft F; the two wheels B and B_1 are fixed together. Let *d* make one complete clockwise revolution; then the other wheels will make—

$$\text{Revs. of B on own axle} = -\frac{T_a}{T_b} = -\frac{46}{40} = -1{\cdot}15$$

$$\text{,, C absolute} = \frac{T_a}{T_c} + 1 = \frac{46}{12} + 1 = 4{\cdot}83$$

$$\text{Revs. E} = \text{revs. of B} \times \frac{T_{b_1}}{T_e} + 1 = -1{\cdot}15 \times \tfrac{16}{34} + 1$$
$$= -0{\cdot}541 + 1 = 0{\cdot}459$$

Whence for one revolution of E, C makes $\frac{4{\cdot}8\dot{3}}{0{\cdot}459} = 10{\cdot}53$ revolutions.

The + sign in the expressions for the speed of C and E is

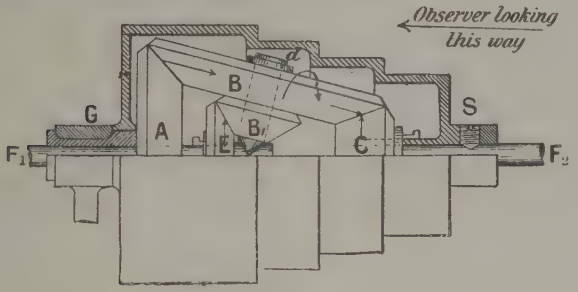


FIG. 201.

on account of these wheels of the epicyclic train rotating in the same sense as that of the arm *d*.

As stated above the *n*th wheel in a *bevil* train of this type rotates in the same sense as the arm when *n* is odd, and in the opposite sense when *n* is even; hence the sign is + for odd axes, and − for even axes, always counting the first as "one."

It may help some readers to grasp the solution of this problem more clearly if we work it out by another method. Let A be free, and let *d* be prevented from rotating; turn A through one −revolution; then—

$$\left.\begin{matrix}\text{Revs.}\\ \text{of E}\end{matrix}\right\} = \frac{\text{product of teeth in drivers}}{\text{product of teeth in driven wheels}} = -\frac{T_a \times T_{b_1}}{T_b \times T_e} = -0{\cdot}541$$

$$\left.\begin{matrix}\text{Revs.}\\ \text{of C}\end{matrix}\right\} = \frac{T_a}{T_e} = 3{\cdot}8\dot{3}$$

Hence, when A is fixed by clamping the split bearing G, and *d* is rotated, the train becomes epicyclic, and since C and E are on *odd* axes of *bevil* trains, they rotate in the same sense as the arm; consequently, for reasons already given, we have—

$$\frac{\text{Revs. E}}{\text{Revs. C}} = \frac{N_e}{N_a} = \frac{-0{\cdot}541 + 1}{3{\cdot}8\dot{3} + 1} = \frac{1}{10{\cdot}53}$$

Particular attention must be paid to the sense of rotation. Bevil gears are more troublesome to follow than plain gears; hence it is well to put an arrow on the drawing, showing the direction *in which the observer is supposed to be looking.*

This mechanism can also be used as a simple reduction gear—in which case the shaft F_1 is not continuous with F_2. The wheel C is then keyed to F_2, or drives it through the set screw S. Since E is keyed to F_1 we have $\frac{\text{Revs. E}}{\text{Revs. C}} = \frac{\text{Revs. } F_1}{\text{Revs. } F_2}$. Thus, if F_2 is coupled to a motor running at 1053 revs. per min., the low speed shaft F_1 will run at 100 revs. per min. for the set of wheels mentioned above.

CHAPTER VI.

DYNAMICS OF THE STEAM-ENGINE.

Reciprocating Parts.—On p. 133 we gave the construction for a diagram to show the velocity of the piston at each part of the stroke when the velocity of the crank-pin was assumed to be constant. We there showed that, for an infinitely long connecting-rod or a slotted cross-head (see Fig. 160), such a diagram is a semicircle when the ordinates represent the velocity of the piston, and the abscissæ the distance it has moved through. The radius of the semicircle represents the constant velocity of the crank-pin. We see from such a diagram that the velocity of the reciprocating parts is zero at each end of the stroke, and is a maximum at the middle; hence during the first half of the stroke the velocity is increased, or the reciprocating parts are accelerated, for which purpose energy has to be expended; and during the second half of the stroke the velocity is decreased, or the reciprocating parts are retarded, and the energy expended during the first half of the stroke is given back. This alternate expenditure and paying back of energy very materially affects the smoothness of running of high-speed engines, unless some means are adopted for counteracting these disturbing effects.

We will first consider the case of an infinitely long connecting-rod, and see how to calculate the pressure at any part of the stroke required to accelerate and retard the reciprocating parts.

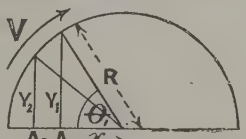


FIG. 202.

The velocity diagram for this case is given in Fig. 202. Let R represent V, the linear velocity of the crank-pin, assumed constant; then the ordinates Y_1, Y_2 represent to the same scale the velocity of the piston when it is at the positions A_1, A_2 respectively, and $\frac{V_1}{V} = \frac{Y_1}{R}$

Let the total weight of the reciprocating parts = W. Then—

The kinetic energy of the reciprocating parts at A_1 $= \frac{WV_1^2}{2g}$

$$= \frac{WV^2Y_1^2}{2gR^2}$$

Likewise at $A_2 = \frac{WV^2Y_2^2}{2gR^2}$

The increase of kinetic energy during the interval A_1A_2 $= \frac{WV^2}{2gR^2}(Y_1^2 - Y_2^2)$

This energy must have come from the steam or other motive fluid in the cylinder.

Let P = the pressure on the piston required to accelerate the moving parts.

Work done on the piston in accelerating the moving parts during the interval $= P(x_2 - x_1)$

But $x_1^2 + Y_1^2 = x_2^2 + Y_2^2 = R^2$

hence $Y_1^2 - Y_2^2 = x_2^2 - x_1^2$

and—

Increase of kinetic energy of the reciprocating parts during the interval $= \frac{WV^2}{2gR^2}(x_2^2 - x_1^2)$

then $P(x_2 - x_1) = \frac{WV^2}{2gR^2}(x_2^2 - x_1^2)$

and $P = \frac{WV^2}{2gR^2}(x_2 + x_1)$

$$= \frac{WV^2x}{gR^2}$$

where x is the mean distance $\frac{x_1 + x_2}{2}$ of the piston from the

middle of the stroke; and when $x = R$ at the beginning and end of the stroke, we have—

$$P = \frac{WV^2}{gR}$$

We shall term P the "total acceleration pressure." Thus with an infinitely long connecting-rod the pressure at the end of the stroke required to accelerate or retard the reciprocating parts is equal to the centrifugal force (see p. 19), assuming the parts to be concentrated at the crank-pin, and at any other part of the stroke distant x from the middle the pressure is less in the ratio $\frac{x}{R}$.

Another simple way of arriving at the result given above is as follows: If the connecting-rod be infinitely long, then it always remains parallel to the centre line of the engine; hence the action is the same as if the connecting-rod were rigidly attached to the cross-head and piston, and the whole rotated together as one solid body, then each point in the body would describe the arc of a circle, and would be subjected to the centrifugal force $C = \frac{WV^2}{gR}$, but we are only concerned with the component along the centre line of the piston, marked P in the diagram. It will be seen that P vanishes in the middle of the stroke, and increases directly as the distance from the middle, becoming equal to C at the ends of the stroke.

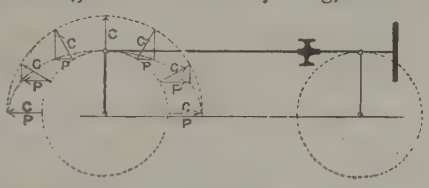


FIG. 203.

When the piston is travelling towards the middle of the stroke the pressure P is positive, and when travelling away from the middle it is negative. Thus, in constructing a diagram to show the pressure exerted at all parts of the stroke, we put the first half above, and the second half below the base-line. We show such a diagram in Fig. 204. The height of any point in the sloping line *ab* above the base-line represents the pressure

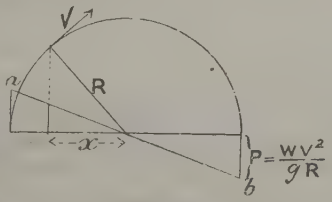


FIG. 204.

at that part of the stroke required to accelerate or retard the moving parts. It is generally more convenient to express the pressure in pounds per square inch, p, rather than the total pressure P; then $\frac{P}{A} = p$, where A = the area of the piston. We will also put $w = \frac{W}{A}$, where w is the weight of the reciprocating parts per square inch of piston. It is more usual to speak of the speed of an engine in revolutions per minute N, than of the velocity of the crank-pin V in feet per second.

$$V = \frac{2\pi RN}{60}$$

$$\text{and } p = \frac{w_2^{\bullet}\pi^2 R^2 N^2}{60^2 gR} = 0{\cdot}00034 w R N^2$$

N.B.—The radius of the crank R is measured in feet.

In arriving at the value of w it is usual to take as reciprocating parts—the piston-head, piston-rod, tail-rod (if any), cross-head, small end of connecting-rod and half the plain part of the rod. A more accurate method of finding the portion of the connecting-rod to be included in the reciprocating parts is to place the rod in a horizontal position with the small end resting on the platform of a weighing machine or suspended from a spring balance, the reading gives the amount to be included in the reciprocating parts. When air-pumps or other connections are attached to the cross-head, they may approximately be taken into account in calculating the weight of the reciprocating parts; thus—

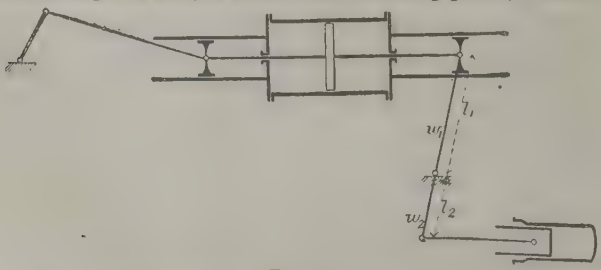


Fig. 205.

$$w = \frac{\text{weight of } \left\{\text{piston} + \text{piston and tail-rods} + \text{both cross-heads} + \text{small end of con. rod} + \frac{\text{plain part}}{2} + \text{air-pump plunger}\left(\frac{l_2}{l_1}\right)^2 + \frac{w_1}{4} + \frac{w_2}{2}\left(\frac{l_2}{l_1}\right)^2\right\}}{\text{area of piston}}$$

The kinetic energy of the parts varies as the square of the velocity; hence the $\left(\frac{l_2}{l_1}\right)^2$

Values of w in pounds per square inch of piston :—

Steam engines with no air pump or other attachments 2 to 4
" " attachments . . . . . 3 to 6

In compound and triple expansion engines the reciprocating parts are frequently made of approximately the same total weight in each cylinder for balancing purposes, in such cases w is often as high as 6 lbs. for the H.P. cylinder and as low as 1 lb. for the L.P.

Influence of Short Connecting-rods.—In Fig. 159 a velocity diagram is given for a short connecting-rod; reproducing a part of the figure in Fig. 206, we have—

Short rod—

$$\frac{\text{cross-head velocity}}{\text{crank-pin velocity}} = \frac{OX}{OC_1}$$

Infinite rod—

$$\frac{\text{cross-head velocity}}{\text{crank-pin velocity}} = \frac{OX_1}{OC_1}$$

$$\frac{\text{velocity of cross-head with short rod}}{\text{velocity of cross-head with long rod}} = \frac{OX}{OX_1}$$

But at the "in" end of the stroke $\frac{OX}{OX_1} = \frac{XC}{X_1C_1} = \frac{L+R}{L}$

$$= 1 + \frac{R}{L}$$

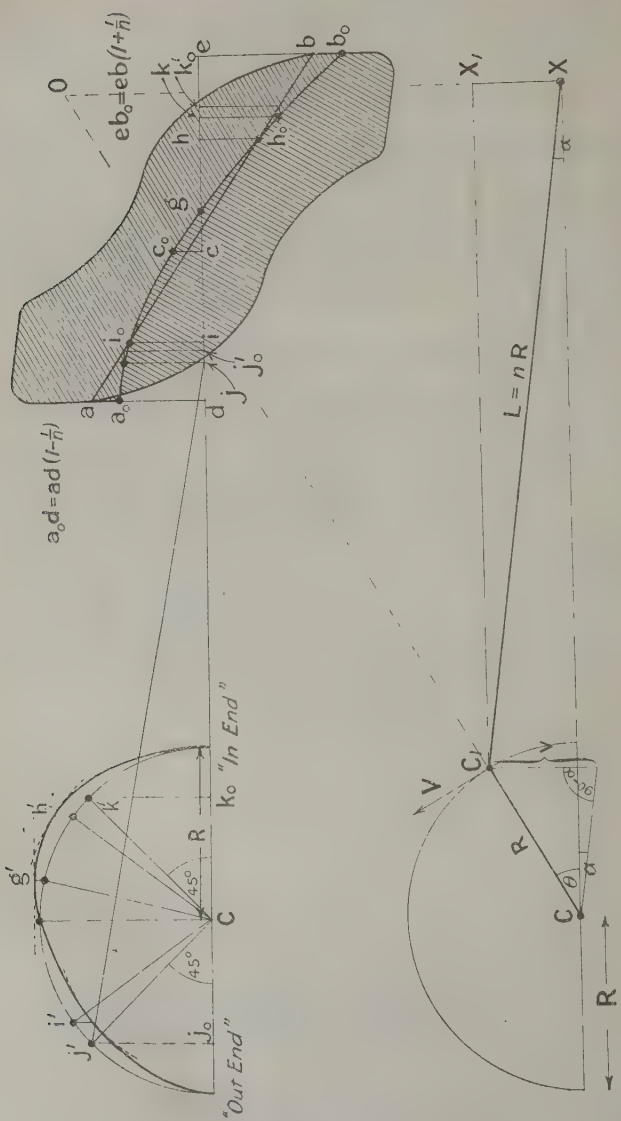
and at the "out" end of the stroke $= 1 - \frac{R}{L}$

Thus if the connecting-rod is n cranks long, the pressure at the "in" end is $\frac{1}{n}$ greater, and at the "out" end $\frac{1}{n}$ less, than if the rod were infinitely long.

The value of p at each end of the stroke then becomes—

$$p = 0{\cdot}00034wRN^2\left(1 + \frac{1}{n}\right) \text{ for the "in" end}$$

$$p = 0{\cdot}00034wRN^2\left(1 - \frac{1}{n}\right) \text{ for the "out" end}$$



O
$eb_o = eb(1+\frac{1}{n})$
$a_od = ad(1-\frac{1}{n})$
a
a_o
i_o
c_o
g_o
h_o
k_o
k'_o
e
b
b_o
h
c
i
j
j'_o
d
X_1
X
α
$L = nR$
k_o "In End"
"Out End"
j_o
i'
j'
g'
h'
k'
R
C
45°
45°
C_1
V
R
θ
α
90°
C
R

FIG. 206

The line *ab* is found by the method described on p. 133. Set off $aa_o = \frac{ad}{n}$, also $bb_o = \frac{eb}{n}$, and $cc_o = \frac{eb}{n}$, *c* is the position of the cross-head when the crank is at right angles to the centre line, *i.e.* vertical in this case. The acceleration is zero where the slope of the velocity curve is zero, *i.e.* where a tangent to it is horizontal. Draw a horizontal line to touch the curve, viz. at g'. As a check on the accuracy of the work, it should be noticed that this point very nearly indeed corresponds to the position in which the connecting-rod is at right angles to the crank;[1] the crosshead is then at a distance $R(\sqrt{n^2+1} - n)$, or very nearly $\frac{R}{2n}$, from the middle of the stroke. The point g' having been found, the corresponding position of the cross-head *g* is then put in. At the instant when the slope of the short-rod velocity curve is the same as that of the long-rod velocity curve, viz. the semicircle (see page 134), the accelerations will be the same in both cases. In order to find where the accelerations are the same, draw arcs of circles from C as centre to touch the short-rod curve, and from the points where they touch draw perpendiculars to cut the circle at the points h' and i'. The corresponding positions of the cross-head are shown at *h* and *i* respectively. In these positions the acceleration curves cross one another, viz. at h_o and i_o. It will shortly be shown that when the crank has passed through 45° and 135° the acceleration pressure with the short rod is equal to that with the infinitely long rod. From k' drop a perpendicular to k_o, set off $k_o k'_o = L$, also $k'k = L$, and from the point where the perpendicular from k'_o cuts *ab* draw a horizontal to meet the perpendicular from *k*, where they cut is another point on the acceleration curve for the short rod; proceed similarly with *j*. We now have eight points on the short-rod acceleration curve through which a smooth curve may be drawn, but for ordinary purposes three are sufficient, say a_o, c_o, b_o.

The acceleration pressure at each instant may also be arrived at thus—

Let θ = the angle turned through by the crank starting from the "in" end;

V = the linear velocity of the crank-pin, assumed constant and represented by R;

v = the linear velocity of the cross-head.

[1] *Engineering*, July 15, 1892, p. 83; also June 2, 1899.

Then—

$$\frac{v}{V} = \frac{\sin(\theta + \alpha)}{\sin(90 - \alpha)} \text{ (see Fig. 206)}$$

$$v = V\left(\frac{\sin\theta\cos\alpha + \cos\theta\sin\alpha}{\cos\alpha}\right)$$

In all cases in practice the angle α is small, consequently $\cos\alpha$ is very nearly equal to unity; even with a very short rod the average error in the final result is well within one per cent. Hence—

$$v = V(\sin\theta + \cos\theta\sin\alpha) \text{ nearly}$$

We also have—

$$\frac{L}{R} = n = \frac{\sin\theta}{\sin\alpha}$$

$$\sin\alpha = \frac{\sin\theta}{n}$$

Substituting this value—

$$v = V\left(\sin\theta + \frac{\cos\theta\sin\theta}{n}\right)$$

$$\text{or } v = V\left(\sin\theta + \frac{\sin 2\theta}{2n}\right)$$

$$\frac{dv}{d\theta} = V\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

The acceleration of the cross-head $f_a = \frac{dv}{dt} = \frac{dv}{d\theta}\cdot\frac{d\theta}{dt}$

Let θ = the angle in radians turned through in the time t.

Then $\theta = \frac{\text{arc}}{\text{radius}}$

$$= \frac{V \cdot t}{R}, \text{ and } \frac{d\theta}{dt} = \frac{V}{R}$$

whence $f_a = \frac{dv}{d\theta}\cdot\frac{V}{R}$

Substituting the value of $\frac{dv}{d\theta}$ found above, we have—

$$f_a = \frac{V^2}{R}\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

and the acceleration pressure, when the crank has passed through the angle θ from the "in" end of the stroke, is—

$$p = \frac{w\mathrm{V}^2}{g\mathrm{R}}\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$\text{or } p = 0{\cdot}00034w\mathrm{RN}^2\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

When $\theta = 45°$ and $135°$, $\cos 2\theta = 0$ and the expression becomes the same as that for a rod of infinite length. When $\theta = 0$ and $180°$ the quantity in brackets becomes $1 + \frac{1}{n}$ and $1 - \frac{1}{n}$.

Correction of Indicator Diagram for Acceleration Pressure.—An indicator diagram only shows the pressure of the working fluid in the cylinder; it does not show the real pressure transmitted to the crank-pin because some of the energy is absorbed in accelerating the reciprocating parts during the first part of the stroke, and is therefore not available for driving the crank, whereas, during the latter part of the stroke, energy is given back from the reciprocating parts, and there is excess energy over that supplied from the working fluid. But, apart from these effects, a single indicator diagram does not show the impelling pressure on a piston at every portion of the stroke. The impelling pressure is really the difference between the two pressures on both sides of the piston at any one instant, hence the impelling pressure must be measured between the top line of one diagram and the bottom line of the other, as shown in full lines in Fig. 207.

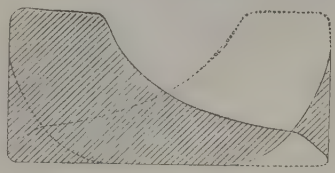


Fig. 207.

The diagram for the return stroke is obtained in the same manner. The two diagrams are set out to a straight base in Fig. 208, the one above the line and the other below. On the same base line the acceleration diagram is also given to the same scale. The real pressure transmitted along the centre line of the engine is given by the vertical height of the shaded figures. In the case of a vertical engine the acceleration line is shifted to increase the pressure on the downward stroke and decrease it on the upward stroke by an amount w,

see Fig. 209. The area of these figures is not altered in any way by the transformations they have passed through, but it should be checked with the area of the indicator diagrams in order to see that no error has crept in.

When dealing with engines having more than one cylinder, the question of scales must be carefully attended to; that is, the heights of the diagrams must be corrected in such a manner that the mean height of each shall be proportional to the total effort exerted on the piston.

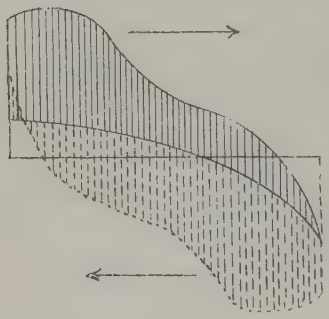


FIG. 208.

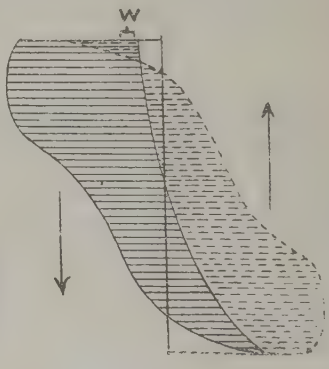


FIG. 209.

Let the original indicator diagrams be taken with springs of the following scales, H.P. $\frac{1}{x}$, I.P. $\frac{1}{y}$, L.P. $\frac{1}{z}$. Let the areas of the pistons (allowing for rods) be H.P. X, I.P. Y, L.P. Z. Let all the pistons have the same stroke. Suppose we find that the H.P. diagram is of a convenient size, we then reduce all the others to correspond with it. If, say, the intermediate piston were of the same size as the high-pressure piston, we should simply have to alter the height of the intermediate diagram in the ratio of the springs; thus—

$$\left.\begin{matrix}\text{Corrected height of I.P. diagram} \\ \text{if pistons were of same size}\end{matrix}\right\} = \left\{\begin{matrix}\text{actual height of} \\ \text{intermediate} \\ \text{diagram}\end{matrix}\right\} \times \frac{\frac{1}{x}}{\frac{1}{y}}$$

$$= \text{actual height} \times \frac{y}{x}$$

But as the cylinders are not of the same size, the height of the diagram must be multiplied by the ratio of the two areas; thus—

$$\left.\begin{array}{l}\text{Height of intermediate diagram}\\ \text{corrected for scales of springs}\\ \text{and for areas of pistons}\end{array}\right\} = \left\{\begin{array}{l}\text{actual height of}\\ \text{intermediate}\\ \text{diagram}\end{array}\right\} \times \frac{y}{x} \times \frac{Y}{X}$$

$$= \text{actual height} \times \frac{yY}{xX}$$

Similarly for the L.P. diagram—

$$\left.\begin{array}{l}\text{Height of L.P. diagram corrected}\\ \text{for scales of springs and for}\\ \text{areas of pistons}\end{array}\right\} = \left\{\begin{array}{l}\text{actual height of}\\ \text{L.P. diagram}\end{array}\right\} \times \frac{zZ}{xX}$$

It is probably best to make this correction for scale and area after having reduced the diagrams to the form given in Fig. 208.

Pressure on the Crank-pin.—The diagram given in

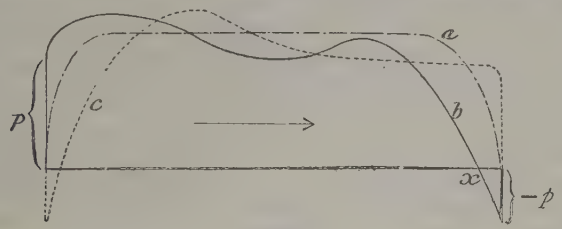


FIG. 210.

Fig. 208 represents the pressure transmitted to the crank-pin at all parts of the stroke. The ideal diagram would be one in which the pressure gradually fell to zero at each end of the stroke, and was constant during the rest of the stroke, such as *a*, Fig. 210.

The curve *b* shows that there is too much compression resulting in a negative pressure $-p$ at the end of the stroke; at the point *x* the pressure on the pin would be reversed, and, if there were any "slack" in the rod-ends, there would be a knock at that point, and again at the end of the stroke, when the pressure on the pin is suddenly changed from $-p$ to $+p$. These defects could be remedied by reducing the amount of compression and the initial pressure, or by running the engine at a higher speed.

The curve (*c*) shows that there is a deficiency of pressure at

the beginning of the stroke, and an excess at the end. The defects could be remedied by increasing the initial pressure and the compression, or by running the engine at a lower speed.

For many interesting examples of these diagrams, as applied to steam engines, the reader is referred to Rigg's "Practical Treatise on the Steam Engine;" also a paper by the same author, read before the Society of Engineers; and to Haeder and Huskisson's "Handbook on the Gas Engine," Crosby Lockwood, for the application of them to gas and oil engines.

Cushioning for Acceleration Pressures.—In order to counteract the effects due to the acceleration pressure, it is usual in steam-engines to close the exhaust port before the end of the stroke, and thus cause the piston to compress the exhaust steam that remains in the cylinder. By choosing the point at which the exhaust port closes, the desired amount of compression can be obtained which will just counteract the acceleration pressure. In certain types of vertical single-acting high-speed engines, the steam is only admitted on the downstroke; hence on the upstroke some other method of cushioning the reciprocating parts has to be adopted. In the well-known Willans engine an air-cushion cylinder is used; the required amount of cushion at the top of the cylinder is obtained by carefully regulating the volume of the clearance space. The pistons of such cylinders are usually of the trunk form; the outside pressure of the atmosphere, therefore, acts on the full area of the underside, and the compressed air cushion on the annular top side.

Let A = area of the underside of the piston in square inches;
A_a = area of the annular top side in square inches;
W = total weight of the reciprocating parts in lbs.;
c = clearance in feet at top of stroke.

At the top, *i.e.* at the "in end," of the stroke we have—

$$P_1 = 0{\cdot}00034WRN^2\left(1 + \frac{1}{n}\right) - W + 14{\cdot}7A$$

Assuming isothermal compression of the air, and taking the pressure to be atmospheric at the bottom of the stroke, we have—

$$14{\cdot}7A_a(2R + c) = P_1c$$

$$\text{whence } c = \frac{29{\cdot}4A_aR}{P_1 - 14{\cdot}7A_a}$$

Or for adiabatic compression—

$$14{\cdot}7A_a(2R + c)^{1{\cdot}41} = P_1c^{1{\cdot}41}$$

$$c = \frac{2R}{\left(\frac{P_1}{14{\cdot}7A_a}\right)^{\frac{1}{1{\cdot}41}} - 1}$$

in the expression for c given above.

The problem of balancing the reciprocating parts of gas and oil engines is one that presents much greater difficulties than in the steam-engine, partly because the ordinary cushioning method cannot be adopted, and further because the effective pressure on the piston is different for each stroke in the cycle. Such engines can, however, be partially balanced by means of helical springs attached either to the cross-head or to a tail-rod, arranged in such a manner that they are under no stress when the piston is at the middle of the stroke, and are under their maximum compression at the ends of the stroke. The weight of such springs is, however, a great drawback; in one instance known to the author the reciprocating parts weighed about 1000 lbs. and the springs 800 lbs.

Polar Twisting-Moment Diagrams. — From the diagrams of real pressures transmitted to the crank-pin that

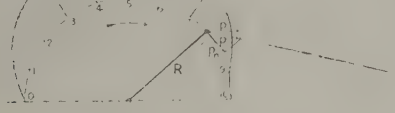


FIG. 211.

we have just constructed, we can readily determine the twisting moment on the crank-shaft at each part of the revolution.

In Fig. 211, let p be the horizontal pressure taken from such a diagram as Fig. 209. Then p_1 is the pressure transmitted along the rod to the crank-pin. This may be resolved in a direction parallel to the crank and normal to it (p_n); we need not here concern ourselves with the pressure acting along the crank, as that will have no turning effect. The twisting moment on the shaft is then p_nR; R, however, is constant, therefore the twisting moment is proportional to p_n. By setting off values of p_n radially from the crank-circle we get a diagram showing the

twisting moment at each part of the revolution. p_n is measured on the same scale, say $\frac{1}{x}$, as the indicator diagram; then, if A be the area of the piston in square inches, the twisting moment in pounds feet $= p_n x \mathrm{AR}$, where p_n is measured in inches, and the radius of the crank R is expressed in feet.

When the curve falls inside the circle it simply indi-

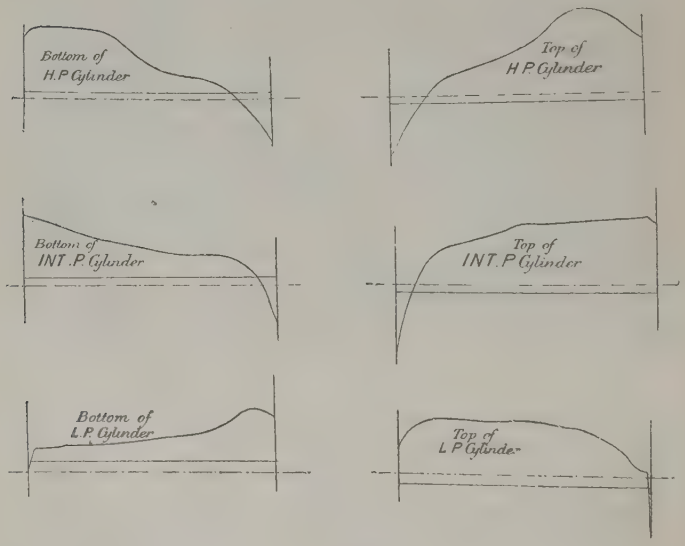

Bottom of H.P Cylinder
Top of H.P Cylinder
Bottom of INT.P Cylinder
Top of INT.P Cylinder
Bottom of L.P Cylinder
Top of L.P Cylinder

FIG. 212.

cates that there is a deficiency of driving effort at that place, or, in other words, that the crank-shaft is driving the piston.

In Fig. 212 indicator diagrams are given, which have been set down in the manner shown in Fig. 208, and after correcting for inertia pressure they have been utilized for constructing the twisting moment diagram shown in Fig. 213. The diagrams were taken from a vertical triple-expansion engine made by Messrs. McLaren of Leeds, and by whose courtesy the author now gives them.

The dimensions of the engine were as follows :—

Diameter of cylinders—

High pressure	... ...	9·01 inches.
Intermediate	... ...	14·25 ,,
Low pressure	... ...	22·47 ,,
Stroke		2 feet.

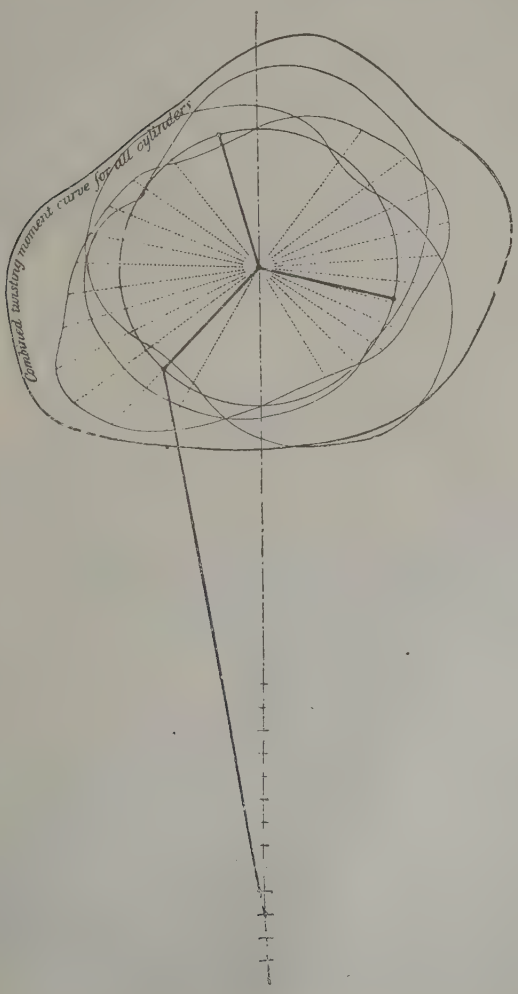


FIG. 213.

The details of reducing the indicator diagrams have been omitted for the sake of clearness; the method of reducing them has been fully described.

Twisting Moment on a Crank-shaft.—In some instances it is more convenient to calculate the twisting moment on the crank-shaft when the crank has passed through the angle θ from the inner dead centre than to construct a diagram.

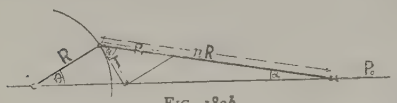


FIG. 182*b*.

Let P_0 = the effort on the piston-rod due to the working fluid and to the inertia of the moving parts;

P_1 = the component of the effort acting along the connecting-rod;

$$P_0 = P_1 \cos \alpha, \text{ and } \sin \alpha = \frac{\sin \theta}{n}$$

from which α can be obtained, since θ and n are given.

The tangential component—

$$T = P_1 \cos \phi$$

$$\text{and } \phi = 90 - (\theta + \alpha)$$

$$\text{whence } T = \frac{P_0}{\cos \alpha} \cos \{90 - (\theta + \alpha)\} = \frac{P_0 \sin (\theta + \alpha)}{\cos \alpha}$$

Flywheels.—The twisting-moment diagram we have just constructed shows very clearly that the turning effort on the crank-shaft is far from being constant; hence, if the moment of resistance be constant, the angular velocity cannot be constant. In fact, the irregularity is so great in a single-cylinder engine, that if it were not for the flywheel the engine would come to a standstill at the dead centre.

A flywheel is put on a crank-shaft with the object of storing energy while the turning effort is greater than the mean, and giving it back when the effort sinks below the mean, thus making the combined effort, due to both the steam and the flywheel, much more constant than it would otherwise be, and thereby making the velocity of rotation more nearly constant. But, however large a flywheel may be, there must always be some variation in the velocity; but it may be reduced to as small an amount as we please by using a suitable flywheel.

In order to find the dimensions of a flywheel necessary for keeping the cyclical velocity within certain limits, we shall make use of the twisting-moment diagram, plotted for convenience

to a straight instead of a circular base-line, the length of the base being equal to the semicircumference of the crank-pin circle. Such a diagram we give in Fig. 215. The resistance line, which for the present we shall assume to be straight, is shown dotted; the diagonally shaded portions below the mean line are together equal to the horizontally shaded area above.

During the period AC the effort acting on the crank-pin is less than the mean, and the velocity of rotation of the crank-pin is consequently reduced, becoming a minimum at C. During the period CE the effort is greater than the mean, and the velocity of rotation is consequently increased, becoming a maximum at E.

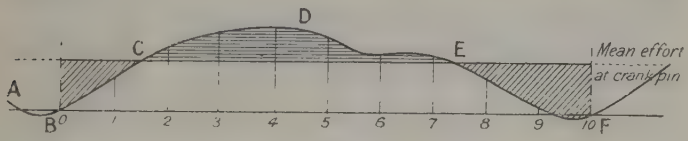


FIG. 215.

Let V = mean velocity of a point on the rim at a radius equal to the radius of gyration of the wheel, in feet per second—usually taken for practical purposes as the velocity of the outside edge of the rim;

V_c = minimum velocity at C (Fig. 215);

V_e = maximum „ E;

W = weight of the flywheel in pounds, usually taken for practical purposes as the weight of the rim;

R_w = radius of gyration in feet of the flywheel rim, usually taken as the external radius for practical purposes.

For most practical purposes it is sheer waste of time calculating the moments of inertia and radii of gyration for all the rotating parts, since the problem is not one that permits of great accuracy of treatment, the form of the indicator diagrams does not remain constant if any of the conditions are altered even to a small extent, then again the coefficient of fluctuation k is not a definitely known quantity, since different authorities give values varying to the extent of two or three hundred per cent. The error involved in using the above approximations is not often greater than five per cent. which is negligible as compared with the other variations, and by adopting them a large amount of time is thereby saved.

[1] Figures 207, 208, 210, 215 are all constructed from the same indicator diagram.

Then the energy stored in the flywheel at C $= \frac{WV_c^2}{2g}$

" " " " $E = \frac{WV_e^2}{2g}$

The increase of energy during CE $= \frac{W}{2g}(V_e^2 - V_c^2)$ (i.)

This increase of energy is derived from the steam or other source of energy; therefore it must be equal to the work represented by the horizontally shaded area CDE = E_n (Fig. 215).

Let $E_n = m \times$ average work done per stroke.

Then the area CDE is m times the work done per stroke, or m times the whole area BCDEF. Or—

$$E_n = \frac{m \times \text{indicated horse-power of engine} \times 33000}{2N} \qquad \text{(ii.)}$$

where N is the number of revolutions of the engine per minute in a double-acting engine.

Whence, from (i.) and (ii.), we have—

$$\frac{W}{2g}(V_e^2 - V_c^2) = \frac{m \times \text{I.H.P.} \times 33000}{2N} \quad . \quad . \quad \text{(iii.)}$$

But $\frac{V_e + V_c}{2} = V$ (approximately)

or $V_e + V_c = 2V$

also $\frac{V_e - V_c}{V} = K$, "the coefficient of speed fluctuation"

and $V_e - V_c = KV$

$V_e^2 - V_c^2 = 2KV^2$

Substituting this value in (iii.)—

$$\frac{W}{2g}(2KV^2) = \frac{m \times \text{I.H.P.} \times 33000}{2N} = E_n$$

and $$W = \frac{48{,}500{,}000m \times \text{I.H.P.}}{KN^3R_w^2} \quad . \quad . \quad . \quad \text{(iv.)}$$

The proportional fluctuation of velocity K is the fluctuation of velocity on either side of the mean; thus, when K = 0·02 it is a fluctuation of 1 per cent. on either side of the mean. The following are suitable values for K:—

K = 0·01 to 0·02 for ordinary electric-lighting engines, but for public lighting and traction stations it often gets as low as 0·0016 to 0·0025 to allow for very sudden and large changes in the load; the weight of all the rotating parts, each multiplied by its own radius of gyration, is to be included in the flywheel;

= 0·02 to 0·04 for factory engines;

= 0·06 to 0·16 for rough engines.

When designing flywheels for public lighting and traction stations where great variations in the load may occur, it is common to allow from 2·4 to 4·5 foot-tons (including rotor) of energy stored per I.H.P.

The calculations necessary for arriving at the value of E_n for any proposed flywheel are somewhat long, and the result when obtained has an element of uncertainty about it, because the indicator diagram must be assumed, since the engine so far only exists on paper. The errors involved in the diagram may not be serious, but the desired result may be arrived at within the same limits of error by the following simpler process. The table of constants given below has been arrived at by constructing such diagrams as that given in Fig. 215 for a large number of cases. They must be taken as fair average values. The length of the connecting-rod, and the amount of pressure required to accelerate and retard the moving parts, affect the result.

The following table gives approximate values of m. In arriving at these figures it was found that if n = number of cranks, then m varies as $\frac{1}{n^2}$ approximately.

APPROXIMATE VALUES OF m FOR DOUBLE-ACTING STEAM-ENGINES.[1]

Cut-off.	Single cylinder.	Two cylinders. Cranks at right angles.	Three cylinders. Cranks at 120°
0·1	0·35	0·088	0·040
0·2	0·33	0·082	0·037
0·4	0·31	0·078	0·034
0·6	0·29	0·072	0·032
0·8	0·28	0·070	0·031
End of stroke	0·27	0·068	0·030

[1] The values of m vary much more in the case of two- and three-cylinder engines than in single-cylinder engines. Sometimes the value of m is twice as great as those given, which are fair averages.

m FOR GAS- AND OIL-ENGINES.

		Otto cycle.			Double acting.	
		Number of cylinders.				
		1	2	4	1	2
Exploding at every cycle	Single	3·7 to 4·5	—	—	2·3 to 2·8	—
	Twin or tandem	—	1·5 to 1·8	0·3 to 0·4	—	0·3 to 0·4
When missing every alternate charge.	Single	8·5 to 9·8	2·5 to 3·0	—	—	—

Gas and oil engines, single acting (Otto) $w = 1{\cdot}5 + \sqrt{d}$

" " double " $w = 2{\cdot}5 + 1{\cdot}6\sqrt{d}$

High speed petrol engines, $w = \frac{d}{8} + \frac{2}{d^2}$

Tandem engines, per line, from 1·8 to 1·9 times the above values.

Where d is the diameter of the cylinder in inches.

Relation between the Work stored in a Flywheel and the Work done per Stroke.—For many purposes it is convenient to express the work stored in the flywheel in terms of the work done per stroke.

$$\text{The energy stored in the wheel} = \frac{WV^2}{2g}$$

Then from equation (iv.), we also have—

$$\left.\begin{array}{l}\text{The energy stored in the wheel}\\ \text{and the average work done per stroke}\end{array}\right\} = \frac{E_n}{2K} = \frac{E_n}{m}$$

$$= \frac{\text{I.H.P.} \times 33000}{2N} \left\{\begin{array}{l}\text{for a double-}\\ \text{acting engine}\end{array}\right.$$

$$\text{the number of average strokes stored in flywheel} = \frac{\frac{E_n}{2K}}{\frac{E_n}{m}} = \frac{m}{2K} \quad \ldots\ldots \text{(v.)}$$

In the following table we give the number of strokes that must be stored in the flywheel in order to allow a total fluctuation of speed of 1 per cent., *i.e.* $\frac{1}{2}$ per cent. on either side of the mean. If a greater variation be permissible in any given case, the number of strokes must be divided by the permissible percentage of fluctuation. Thus, if 4 per cent., *i.e.* K = 0·04, be permitted, the numbers given below must be divided by 4.

NUMBER OF STROKES STORED IN A FLYWHEEL FOR DOUBLE-ACTING STEAM-ENGINES.[1]

Cut-off.	Single cylinder.	Two cylinders. Cranks at right angles.	Three cylinders. Cranks at 120°.
0·1	18	4·4	2·0
0·2	17	4·1	1·9
0·4	16	3·9	1·8
0·6	15	3·6	1·7
0·8	14	3·5	1·6
End of stroke	13	3·4	1·5

GAS-ENGINES (MEAN STROKES).

		Otto cycle.			Double acting.	
		Number of cylinders.				
		1	2	4	1	2
When exploding at every cycle. . .	Single	185 to 225	—	—	115 to 140	—
	Twin or tandem	—	75 to 90	15 to 20	—	15 to 20
When missing every alternate charge .	Single	425 to 490	125 to 150	—	—	—

Shearing, Punching, and Slotting Machines (K not known).—It is usual to store energy in the flywheel equal to the gross work done in two working strokes of the shear, punch, or slotter, amounting to about 15 inch-tons per square inch of metal sheared or punched through.

[1] See note at foot of p. 197.

Gas-Engine Flywheels.—The value of m for a gas-engine can be roughly arrived at by the following method.

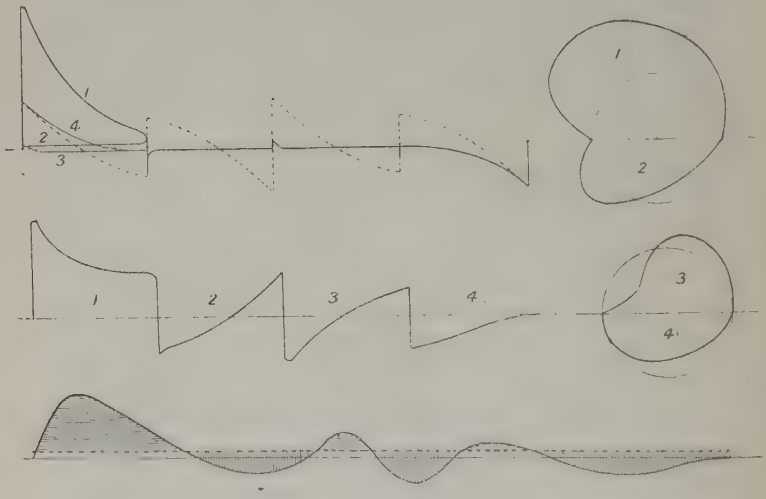


FIG. 216.

The work done in one explosion is spread over four strokes when the mixture explodes at every cycle. Hence the mean effort is only one-fourth of the explosion-stroke effort, and the excess energy is therefore approximately three-fourths of the whole explosion-stroke effort, or three times the mean: hence $m = 3$.

Similarly, when every alternate explosion is missed, $m = 7$.

By referring to the table, it will be seen that both of these values are too low.

The diagram for a 4-stroke case is given in Fig. 216. It has been constructed in precisely the same manner as Figs. 207, 208, and 215. When oil- or gas-engines are used for driving dynamos, a small flywheel is often attached to the dynamo direct, and runs at a very much higher peripheral speed than the engine flywheel. Hence, for a given weight of metal, the small high-speed flywheel stores a much larger amount of energy than the same weight of metal in the engine flywheel.

The peripheral speed of large cast-iron flywheels has to be kept below a mile a minute (see p. 201) on account of their

danger of bursting. The small disc flywheels, such as are used on dynamos, are hooped with a steel ring, shrunk on the rim, which allows them to be safely run at much higher speeds than the flywheel on the engine. The flywheel power of such an arrangement is then the sum of the energy stored in the two wheels.

There is no perceptible flicker in the lights when about forty impulse strokes, or 160 average strokes (when exploding at every cycle, and twice this number when missing alternate cycles), are stored in the flywheels.

Case in which the Resistance varies.—In all the above cases we have assumed that the resistance overcome by the engine is constant. This, however, is not always the case; when the resistance varies, the value of E_n is found thus:

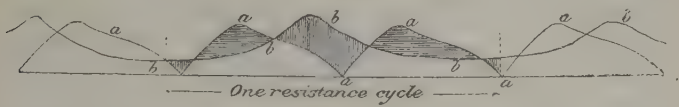


FIG. 217.

The line *aaa* is the engine curve as described above, the line *bbb* the resistance to be overcome, the horizontal shading indicates excess energy, and the vertical deficiency of energy. The excess areas are, of course, equal to the deficiency areas over any complete cycle. The resistance cycle may extend over several engine cycles; an inspection or a measurement will reveal the points of maximum and minimum velocity. The value of m is the ratio of the horizontal shaded areas to the whole area under the line *aaa* described during the complete cycle of operations. See *The Engineer*, January 9, 1885.

Stress in Flywheel Rims.—If we neglect the effects of the arms, the stress in the rim of a flywheel may be treated in the same manner as the stresses in a boiler-shell or, more strictly, a thick cylinder (see p. 421), in which we have the relation—

$$P_r R_w = ft$$

or $P_r R_w = f$, when $t = 1$ inch

The P_r in this instance is the pressure on each unit length of rim due to centifugal force. We

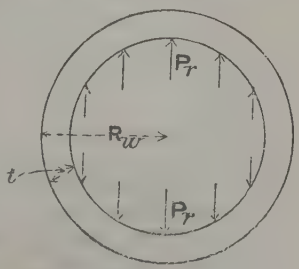


FIG. 218.

shall find it convenient to take the unit of length as 1 foot, because we take the velocity of the rim in *feet* per second. Then—

$$P_r = \frac{W_r V_w^2}{g R_w}, \text{ and } f = \frac{W_r V_w^2}{g}$$

where W_r = the weight of 1 foot length of rim, 1 square inch in section
= 3·1 lbs. for cast iron

We take 1 *sq. inch* in section, because the stress is expressed in pounds per square *inch*. Then substituting the value of W_r in the above equations, we have—

$$\frac{3{\cdot}1 V_w^2}{32{\cdot}2} = f$$

$$f = 0{\cdot}096 V_w^2$$

$$\text{or } f = \frac{V_w^2}{10} \text{ (very nearly)}$$

In English practice V_w is rarely allowed to exceed 100 feet per second, but in American practice much higher speeds are often used, probably due to the fact that American cast iron is much tougher and stronger than the average metal used in England. An old millwright's rule was to limit the speed to a mile a minute, *i.e.* 88 feet per second, corresponding to a stress of about 800 lbs. per square inch.

The above expression gives the tensile stress set up in a thin plain rotating ring, due to centrifugal force; but it is not the only or even the most important stress which occurs in many flywheel and pulley rims. The direct stress in the material causes the rim to stretch and to increase in diameter, but owing to its attachment to the arms, it is unable to do so beyond the amount permitted by the stretch of the arms, with the result that the rim sections bend outwards between the arms, and behave as beams which are constrained in direction at the ends. If the arms stretched sufficiently to allow the rim to remain circular when under centrifugal stress there would be no bending action, and, on the other hand, if the arms were quite rigid the bending stress in the rim sections between the arms could be calculated by treating them as beams built in at each end and supporting an evenly distributed load equal in intensity to the centrifugal force acting on the several portions of the rim; neither of these conditions actually hold and the real state of the beam

is intermediate between that due to the above-mentioned assumptions, the exact amount of bending depending largely upon the stretch of the arms.

A rigid solution of the problem is almost impossible, the results obtained by different authorities are not in agreement, owing to arbitrary assumptions being made. In all cases it is assumed that there are no cooling stresses existing in the arms of the wheel, which every practical man knows is not always correct. However, the results obtained by the more complete reasoning are unquestionably nearer the truth than those obtained by the elementary treatment given above.

For a more complete treatment the reader is referred to Unwin's "Elements of Machine Design," Part II. (Longmans). The following approximate treatment may be of service to those who have not the opportunity of following the more complete theory.

The maximum bending moment in pound-inches on an initially straight beam built in at both ends is $\frac{wl^2}{12}$ (see p. 529), where w is the evenly distributed load per inch run, and l is the length between supports in inches. In the case of the flywheel, w is the centrifugal force acting on the various portions of the rim, and is $\frac{0{\cdot}26V_w^2A}{gR_w}$, where 0·26 is the weight of a cubic inch of cast iron, V_w is the rim velocity in feet per second, A the area of the section of the rim in square inches, g the acceleration of gravity, R_w the radius of the rim in feet, Z the tension modulus of the section (see Chapter XI.).

Then the bending stress in the rim due to centrifugal force is

$$f_b = \frac{0{\cdot}26V_w^2Al^2}{12 \times 32{\cdot}2 \times R_w \times Z}$$

The rim section, however, is not initially straight, hence the ordinary beam formula does not rigidly hold. l is taken as the chord of the arc between the arms. As already explained, the stress due to bending is really less than the above expression gives, but by introducing a constant obtained by comparing this treatment with one more complete, we can bring the results into approximate agreement; this constant is about 2·2. Hence

$$f_b = \frac{0{\cdot}26V_w^2Al^2}{2{\cdot}2 \times 12 \times 32{\cdot}2R_wZ} = \frac{V_w^2Al^2}{3270R_wZ}$$

and the resulting tensile stress in the rim is—

$$f' = \frac{V_w^2}{10}\left\{1 + \frac{Al^2}{327 R_w Z}\right\} \text{ approximately.}$$

All who have had any experience in the foundry will be familiar with the serious nature of the internal cooling stresses in flywheel and pulley arms. The foundry novice not unfrequently finds that one or more of the arms of his pulleys are broken when taken out of the sand, due to unequal cooling; by the exercise of due care the moulder can prevent such catastrophes, but it is a matter of common knowledge that, in spite of the most skilful treatment it is almost impossible to ensure that a wheel is free from cooling stresses. Hence only low-working stresses should be permitted.

When a wheel gets overheated through the use of a friction brake, the risk of bursting is still greater; there are many cases on record in which wheels have burst, in some instances with fatal results, through such overheating. In a case known to the Author of a steam-engine fitted with two flywheels, 5 feet in diameter, and running at 160 revolutions per minute, one of the wheels broke during an engine test. The other wheel was removed with the object of cutting through the boss in order to relieve the cooling stresses; but as soon as the cut was started the wheel broke into several pieces, with a loud report like a cannon, thus proving that it was previously subjected to very serious cooling stresses. In order to reduce the risk of the bursting of large flywheels, when made with solid rims, they should always be provided with split bosses. In some cases the boss is made in several sections, each being attached to a single arm, which effectually prevents the arms from being subjected to initial tension due to cooling. Built-up rims are in general much weaker than solid rims; but when they cannot be avoided, their design should be most carefully considered. The question is discussed in Lanza's "Dynamics of Machinery" (Chapman & Hall).

Large wheels are not infrequently built up entirely of wrought iron or steel sections and plates; in some instances a channel rim has been used into which hard-drawn steel wire is wound under tension. Such wheels are of necessity more costly than cast-iron wheels of the same weight, but since the material is safe under much higher stresses than cast iron, the permissible peripheral speed may be much higher, and consequently the same amount of energy may be stored in a much lighter wheel, with the result that for a given speed fluctuation

the cost of the built-up wheel may be even less than that of a cast-iron wheel.

In the place of arms thin plate webs are often used with great success; such webs support the rim far better than arms, and moreover they have the additional advantage that they materially reduce the air resistance, which is much more important than many are inclined to believe.

Experimental Determination of the Bursting Speed of Flywheels.—Professor C. H. Benjamin, of the Case School, Cleveland, Ohio, has done some excellent research work on the actual bursting speed of flywheels, which well corroborates the general accuracy of the theory. The results he obtained are given below, but the original paper read by him before the American Society of Mechanical Engineers in 1899 should be consulted by those interested in the matter.

Bursting speed in feet per sec. V_w	$f = \frac{V_w^2}{10}$	Thickness of rim.	Remarks.
	lbs. sq. in.	inch.	
430	18,500	0·68	Solid rim, 6 arms, 15 ins. diam.
388	15,000	0·56	,, ,, ,,
192	3,700		Jointed rim, ,, ,,
381	14,500	0·65	Solid rim, 3 arms, ,,
363	13,200	0·38	,, ,, ,,
385	14,800	1·5	,, 6 arms, 24 ins. diam.
190	3,610	0·75	Two internal flanged joints, ,, ,,
305	9,300		Linked joints ,, ,,

From these and other tests, Professor Benjamin concludes that solid rims are by far the safest for wheels of moderate size. The strength is not much affected by bolting the arms to the rim, but joints in the rims are the chief sources of weakness, especially when the joints are near the arms. Thin rims, due to the bending action between the arms, are somewhat weaker than thick rims.

Some interesting work on the bending of rims has been done by Mr. Barraclough (see I C.E. Proceedings, vol. cl.).

For practical details of the construction of flywheels, readers are referred to:—Sharpe on "Flywheels," Manchester Association of Engineers. Haeder and Huskisson's "Handbook on the Gas-Engine." "Flywheels," "Machinery" Reference books.

Arms of Flywheels.—In addition to the unknown tensile stresses in wheels with solid bosses and rims, the arms are under tension due to the centrifugal force acting (i) on the arm itself, (ii) on a portion of the rim, amounting to approximately one-fourth of the length of rim between the arms. In addition to these stresses the arms are subjected to bending due (iii) to a change in the speed of the wheel, (iv) to the power transmitted through the wheel when it is used for driving purposes.

The tension due to (i) is arrived at thus—

Let A_a = Sectional area of the arm in square inches assumed to be the same throughout its length.
r = Radius from centre of wheel to any element of the arm in feet.
ω = Angular velocity of the arm.
r_1 = Radius of inside of the rim of the wheel in feet.
r_2 = Radius of boss of the wheel in feet.
w = Weight of a cubic inch of the material.

The centrifugal force acting on the element of the arm is

$$\frac{12wA_a\omega^2 r dr}{g}$$

and on the whole arm

$$\frac{12wA_a\omega^2}{g}\int_{r_2}^{r_1} r dr = \frac{12wA_a\omega^2}{g}\left(\frac{r_1^2 - r_2^2}{2}\right)$$

or

$$= \frac{12wA_aV_w^2}{gR^2}\left(\frac{r_1^2 - r_2^2}{2}\right)$$

The tension in the arm due to the centrifugal force acting on one-fourth of the rim between the arms is

$$= \frac{wAlV_w^2}{4gR}$$

Hence the tension in the arm at the boss due to both is

$$\frac{wV_w^2}{gR}\left(\frac{12A_a}{R}\frac{(r_1^2 - r_2^2)}{2} + \frac{Al}{4}\right) \text{ approx.}$$

Let the acceleration of the rim, arising from a change of speed of the shaft, be δV feet per sec. per sec. Let W be the weight of the rim in pounds, then, since the arms are built in at both ends, the bending moment on them (see p. 504) is $\frac{6WR_w.\delta V}{g}$ inch-pounds, approximately; the bending stress is

$\frac{6WR_w \cdot \delta V}{gnZ}$, where n is the number of arms and Z the modulus of the section of the arms in bending.

If the flywheel be also used for transmitting power, and P be the effective force acting on the rim of the wheel, the bending stress in the arms is $\frac{12PR_w}{nZ}$, on the assumption that the stress in the most strained arm is twice the mean, which experiments show is a reasonable assumption.

Hence the bending stress in the arms due to both causes is—

$$\frac{6R_w}{nZ}\left(\frac{W \cdot \delta V}{g} + 2P\right)$$

The strength of flywheel and pulley arms should always be checked as regards bending.

Bending Stresses in Locomotive Coupling-rods.—Each point in the rod describes a circle (relatively to the

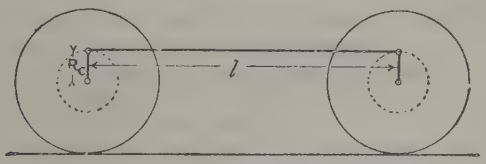


FIG. 219.

engine) as the wheels revolve; hence each particle of the rod is subjected to centrifugal force, which bends the rod upwards when it is at the top of its path and downwards when at the bottom. Since the stress in the rod is given in pounds per square inch, the bending moment on the rod must be expressed in pound-inches; and the length of the rod l in inches. In the expression for centrifugal force we have foot units, hence the radius of the coupling crank must be in feet.

$$\left.\begin{array}{l}\text{The centrifugal force acting}\\ \text{on the rod per } \textit{inch} \text{ run}\end{array}\right\} = C = 0{\cdot}00034wR_cN^2$$

where w is the weight of the rod in pounds per inch run, or $w = 0{\cdot}28A$ pounds, where A is the sectional area of the rod.

The centrifugal force is an evenly distributed load all along the rod if it be parallel.

$$\left.\begin{array}{l}\text{The maximum bending moment}\\ \text{in the middle of the rod}\end{array}\right\} = \frac{Cl^2}{8} = \frac{fA\kappa^2}{y}$$

(see Chapters IX. and XI.)

where κ^2 = the square of the radius of gyration (inch units) about a horizontal axis through the c. of g.;

y = the half-depth of the section (inches).

Then, substituting the value of C, we have—

$$f = \frac{0{\cdot}00034 \times 0{\cdot}28 \times A \times R_c \times N^2 \times l^2 \times y}{8 \times A \times \kappa^2}$$

$$f = \frac{0{\cdot}000012 R_c N^2 l^2 y}{\kappa^2}, \text{ or } = \frac{R_c N^2 l^2 y}{84{,}000\kappa^2}$$

The value of κ^2 can be obtained from Chapter III. For a rectangular section, $\kappa^2 = \frac{h^2}{12}$; and for an I section, $\kappa^2 = \frac{BH^3 - bh^3}{12(BH - bh)}$.

It should be noticed that the stress is independent of the sectional area of the rod, but that it varies inversely as the square of the radius of gyration of the section; hence the importance of making rods of I section, in which the metal is placed as far from the neutral axis as possible. If the stress be calculated for a rectangular rod, and then for the same rod which has been fluted by milling out the sides, it will be found that the fluting very materially reduces the bending stress.

The bending stress can be still further reduced by removing superfluous metal from the ends of the rod, *i.e.* by proportioning each section to the corresponding bending moment, which is a maximum in the middle and diminishes towards the ends. The "bellying" of rods in this manner is a common practice on many railways.

In addition to the bending stress in a vertical plane, there is also a direct stress of nearly uniform intensity acting over the section of the rod, sometimes in tension and sometimes in compression. This stress is due to the driving effort transmitted through the rod from the driving to the coupled wheel, but it is impossible to say what this effort may amount to. It is usual to *assume* that it amounts to one-half of the total pressure on the piston, but a safer method is to calculate it from the maximum adhesion of the coupled wheels. The coefficient of friction between the wheels and rails may be taken at 0·3.

On account of the bending moment on these rods a certain amount of deflection occurs, which reaches its maximum value when the rod is at the top and bottom of its traverse. When the rod is transmitting a compressive stress it becomes a strut loaded out of the centre, and the direct stress is no longer distributed uniformly. The deflection due to the bending moment already considered is

$$\delta = \frac{5Cl^4}{384EI} = \frac{AR_eN^2l^4}{807{,}000EI}$$

if T be the thrust on the rod in pounds.

The bending stress due to the eccentric loading is

$$\frac{T\delta}{Z} = \frac{TR_eN^2l^4}{807{,}000E\kappa^2Z}$$

and the maximum stress due to all causes is

$$\frac{R_eN^2l^2y}{84{,}000\kappa^2}\left(1 + \frac{Tl^2}{9{\cdot}61EI}\right) \pm \frac{T}{A}$$

The + sign refers to the maximum compressive stress and the − sign to the tensile stress. The second term in the brackets is usually very small and negligible.

A more exact treatment will be found in Morley's "Strength of Materials," page 263.

In all cases coupling rods should be checked to see that they are safe against buckling sideways as struts; many breakdowns occur through weakness in this direction.

Bending Stress in Connecting-rods.—In the case of a coupling-rod of uniform section, in which each particle describes a circle of the same radius as the coupling-crank pin, the centrifugal force produces an evenly distributed load; but in the case of a connecting-rod the swing, and therefore the centrifugal force at any section, varies from a maximum at the crank-pin to zero at the gudgeon-pin. The centrifugal force acting on any element distant x from the gudgeon-pin is $\frac{Cx}{l}$, where C is the centrifugal force acting on it if rotating in a circular path of radius R, *i.e.* the radius of the crank, and l is the length of the connecting-rod.

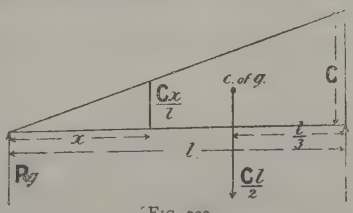


FIG. 220.

Let the rod be in its extreme upper or lower position, and

let the reaction at the gudgeon-pin, due to the centrifugal force acting on the rod, be R_g. Then, since the centrifugal force varies directly as the distance x from the gudgeon-pin, the load distribution diagram is a triangle, and—

$$R_g l = \frac{Cl}{2} \times \frac{l}{3} \quad \text{whence } R_g = \frac{Cl}{6}$$

The shear at a section distant x from the gudgeon-pin $= \frac{Cl}{6} - \left(\frac{Cx}{l} \times \frac{x}{2}\right)$

The shear changes sign when $\frac{Cl}{6} = \frac{Cx^2}{2l}$ or when $x = \frac{l}{\sqrt{3}}$

But the bending moment is a maximum at the section where the shear changes sign (see p. 482). The bending moment at a section y distant x from R_g—

$$M_y = R_g x - \frac{Cx}{l} \times \frac{x}{2} \times \frac{x}{3} = R_g x - \frac{Cx^3}{6l}$$

The position of the maximum bending moment may also be obtained thus—

$$\frac{dM_y}{dx} = \frac{Cl}{6} - \frac{3Cx^2}{6l}$$

for a maximum value

$$\frac{Cl}{6} = \frac{3Cx^2}{6l} \text{ and } x = \frac{l}{\sqrt{3}}$$

By substitution of the values of x and R_g and by reduction, we have—

$$M_{max.} = \frac{Cl^2}{9\sqrt{3}} = \frac{Cl^2}{15{\cdot}6}$$

and the bending stress $f = \frac{RN^2 l^2 y}{1640000\kappa^2}$

which is about one-half as great as the stress in a coupling-rod working under the same conditions.

The rod is also subjected to a direct stress and to a very small bending stress due to the deflection of the rod, which can be treated by the method given for coupling rods.

Readers who wish to go very thoroughly into this question should refer to a series of articles in the *Engineer*, March, 1903.

Balancing Revolving Axles.

Case I. "*Standing Balance.*"—If an unbalanced pulley or wheel be mounted on a shaft and the shaft be laid across two levelled straight-edges, the shaft will roll until the heavy side of the wheel comes to the bottom.

If the same shaft and wheel are mounted in bearings and rotated rapidly, the centrifugal force acting on the unbalanced portion would cause a pressure on the bearings acting always in the direction of the unbalanced portion; if the bearings were very slack and the shaft light, it would lift bodily at every revolution. In order to prevent this action, a balance weight or weights must be attached to the wheel *in its own plane of rotation*, with the centre of gravity diametrically opposite to the unbalanced portion.

Let W = the weight of the unbalanced portion;
W_1 = „ „ balance weight;
R = the radius of the c. of g. of the unbalanced portion;
R_1 = the radius of the c. of g. of the balance weight.

Then, in order that the centrifugal force acting on the balance weight may exactly counteract the centrifugal force acting on the unbalanced portion, we must have—

$$0{\cdot}00034WRN^2 = 0{\cdot}00034W_1R_1N^2$$
$$\text{or } WR = W_1R_1$$
$$\text{or } WR - W_1R_1 = 0$$

that is to say, the algebraic sum of the moments of the rotating weights about the axis of rotation must be zero, which is equivalent to saying that the centre of gravity of all the rotating weights must coincide with the axis of rotation. When this is the case, the shaft will not tend to roll on levelled straight-edges, and therefore the shaft is said to have "standing balance."

When a shaft has standing balance, it will also be perfectly balanced at all speeds, *provided that all the weights rotate in the same plane.*

We must now consider the case in which all the weights do not rotate in the same plane.

Case II. *Running Balance.*—If we have two or more weights attached to a shaft which fulfil the conditions for standing balance, but yet do not rotate in the same plane, the shaft will no longer tend to lift bodily at each revolution; but it will tend to wobble, that is, it will tend to turn about an axis perpendicular to its own when it rotates rapidly. If the bearings were very slack, it would trace out the surface of a double cone in space as indicated by the dotted lines, and the axis would be constantly shifting its position, *i.e.* it would not be permanent. The reason for this is, that the two centrifugal forces c and c_1 form a couple, tending to turn the shaft about some point A between them. In order to

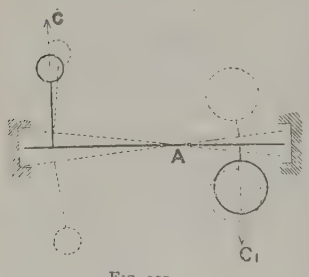


Fig. 221.

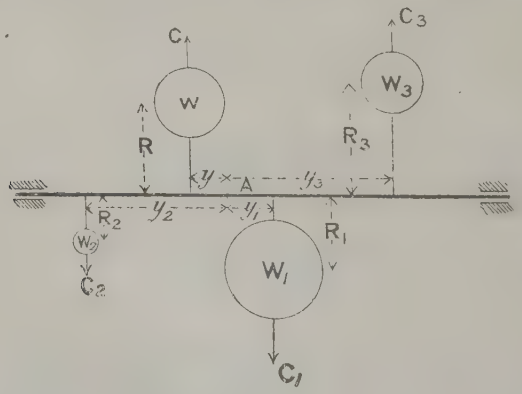


Fig. 222.

counteract this turning action, an equal and opposite couple must be introduced by placing balance weights diametrically opposite, which fulfil the conditions for "standing balance," and

moreover their centrifugal moments about any point in the axis of rotation must be equal and opposite in effect to those of the original weights. Then, of course, the algebraic sum of all the centrifugal moments is zero, and the shaft will have no tendency to wobble, and the axis of rotation will be permanent.

In the figure, let the weights W and W_1 be the original weights, balanced as regards "standing balance," but when rotating they exert a centrifugal couple tending to alter the direction of the axis of rotation. Let the balance weights W_2 and W_3 be attached to the shaft in the same plane as W_1 and W, *i.e.* diametrically opposite to them, also having "standing balance." Then, in order that the axis may be permanent, the following condition must be fulfilled :—

$$cy + c_1y_1 = c_2y_2 + c_3y_3$$
$$0{\cdot}00034N^2(WRy + W_1R_1y_1) = 0{\cdot}00034N^2(W_2R_2y_2 + W_3R_3y_3)$$
$$\text{or } WRy + W_1R_1y_1 - W_2R_2y_2 - W_3R_3y_3 = 0$$

The point A, about which the moments are taken, may be chosen anywhere along the axis of the shaft without affecting the results in the slightest degree. Great care must be taken with the signs, viz. a + sign for a clockwise moment, and a − sign for a contra-clockwise moment.

The condition for standing balance in this case is—

$$WR - W_1R_1 - W_2R_2 + W_3R_3 = 0$$

So far we have only dealt with the case in which the balance weights are placed diametrically opposite to the weight to be balanced. In some cases this may lead to more than one balance weight in a plane of rotation; the reduction to one equivalent weight is a simple matter, and will be dealt with shortly. Then, remembering this condition, the only other conditions for securing a permanent axis of rotation, or a "running balance," are—

$$\Sigma WR = 0$$
$$\text{and } \Sigma WRy = 0$$

where ΣWR is the algebraic sum of the moments of all the rotating weights about the axis of rotation, and y is the distance, measured parallel to the shaft, of the plane of rotation of each weight from some given point in the axis of rotation. Thus the c. of g. of all the weights must lie in the axis of rotation.

Graphic Treatment of Balance Weights.—Such a problem as the one just dealt with can be very readily treated graphically. For the sake, however, of giving a more general application of the method, we will take a case in which the

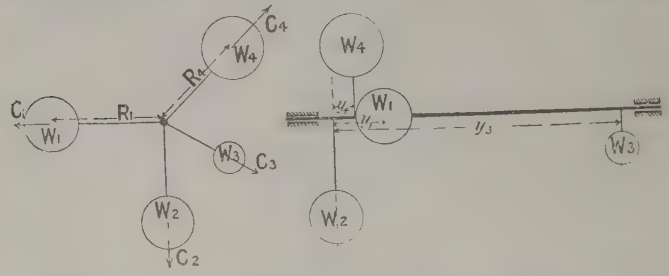


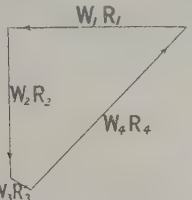
FIG. 223.

weights are not placed diametrically opposite, but are as shown in the figure.

Let all the quantities be given except the position and weight of W_4, and the arm y_4, which we shall proceed to find by construction.

Standing balance.

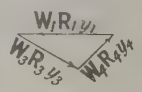
There must be no tendency for the axis to lift bodily; hence the vector sum of the forces C_1, C_2, C_3, C_4, must be zero, *i.e.* they must form a closed polygon. Since C is proportional to WR, set off W_1R_1,



W_2R_2, W_3R_3, to some suitable scale and in their respective directions; then the closing line of the force polygon gives us W_4R_4 in direction, magnitude, and sense. The radius R_4 is given, whence W_4 is found by dividing by R_4.

Running balance.

There must be no tendency for the axis to wobble; hence the vector sum of the moments C_1y_1, etc., about a given plane must be zero, *i.e.* they, like the forces, must form a closed polygon. We adopt Professor Dalby's method of taking the plane of one of the rotating masses, viz. W_2 for our plane of reference; then the force C_2 has no moment about the plane. Constructing the triangle of moments, we get the value of $W_4R_4y_4$ from the closing line of the triangle. Then dividing by W_4R_4, we get the value of y_4.



Provided the above-mentioned conditions are fulfilled, the axle will be perfectly balanced at all speeds. It should be noted that the second condition cannot be fulfilled if the number of rotating masses be less than four.

Balancing of Stationary Steam-Engines.—Let the sketch represent the scheme of a two-cylinder vertical steam-engine with cranks at right angles. Consider the moments of the unbalanced forces p_a and p_b about the point O. When the piston A is at the bottom of its stroke, there is a contra-clockwise moment, $p_a y_a$, due to the acceleration pressure p_a tending to turn the whole engine round in a contra-clockwise direction about the point O. The force p_b is zero in this position (neglecting the effect of the obliquity of the rod). When, however, A gets to the top of its stroke, there is a moment, $p_a y_a$, tending to turn the whole engine in a contrary direction about the point O. Likewise with B; hence there is a constant tendency for the engine to lift first at O, then at P, which has to be counteracted by the holding-down bolts, and may give rise to very serious vibrations unless the foundations be very massive. It must be clearly understood that the cushioning of the steam mentioned on p. 190 in no way tends to reduce this effect; balance weights on the cranks will partially remedy the evil, but it is quite possible to entirely eliminate it in such an engine as this.

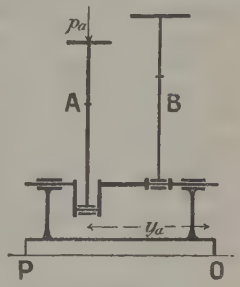


FIG. 224.

A two-cylinder engine can, however, be arranged so that the balance is perfect in every respect. Such a one is found in the Barker engine. In this engine the two cylinders are in line, and the cranks are immediately opposite and of equal throw. The connecting-rod of the A piston is forked, while that of the B piston is coupled to a central crank; thus any forces that may act on either of the two rods are equally distributed between the two main bearings of the bed-plate, and consequently no disturbing moments are set up. Then if the mass of A and its attachments is equal to that of B, also if the moments of inertia of the two connecting-rods about the gudgeon-pins are the same, the

FIG. 225.

disturbing effect of the obliquity of the rods will be entirely eliminated.

A single cylinder engine can be balanced by a similar device. Let B represent the piston, cross-head, and connecting rod of a single-cylinder engine, and let a "bob-weight" be substituted for the piston and cross-head of the cylinder A. Then, provided the "bob-weight" slides to and fro in a similar manner and fulfils the conditions mentioned above, a perfect balance will be established. In certain cases it may be more convenient to use two "bob-weights," A_1 and A_2, each attached to a separate connecting rod. Let the distances of the planes of the connecting rods from a plane taken through B be x_1 and x_2, and the "bob-weights" be W_1 and W_2, and the radii of the cranks r_1 and r_2 respectively. Let the weight of the reciprocating parts of B be W and the radius of the crank r. Then using connecting rods of equal moment of inertia about the gudgeon pin for A_1 and A_2, and whose combined moments of inertia are equal to that of B, we must have—

$$W_1 r_1(x_1 + x_2) = W r x_2, \text{ also } W_2 r_2(x_1 + x_2) = W r x_1.$$

The same arrangement can be used for three-cylinder engines, the "bob-weights" then become the weights of the reciprocating parts of the two cylinders A_1 and A_2.

Any three-cylinder engine can be balanced in a similar manner by the addition of two extra cranks or eccentrics to drive suitable "bob-weights," the calculations for arriving at the necessary weights, radii of cranks and their positions along the shaft can be readily made by the methods shortly to be discussed.

If slotted cross-heads are used in order to secure simple harmonic motion for the reciprocating parts the matter is simplified in that no connecting rods are used and consequently there are no moments of inertia to be considered.

A three-cylinder vertical engine having cranks at 120°, and having equal reciprocating and rotating masses for each cylinder, can be entirely balanced along the centre line of the engine. The truth of this statement can be readily demonstrated by inserting the angles θ, $\theta + 120°$, and $\theta + 240°$ in the equation on p. 186; the sum of the inertia forces will be found to be zero. The proof was first given by M. Normand of Havre, and will be found in "Ripper's Steam Engine Theory and Practice."

There will, however, be small unbalanced forces acting at right angles to the centre line, tending to make the engine rock about an axis parallel to the centre line of the crank-shaft.

A four-cylinder engine, apart from a small error due to the obliquity of the rods, can be perfectly balanced; thus—

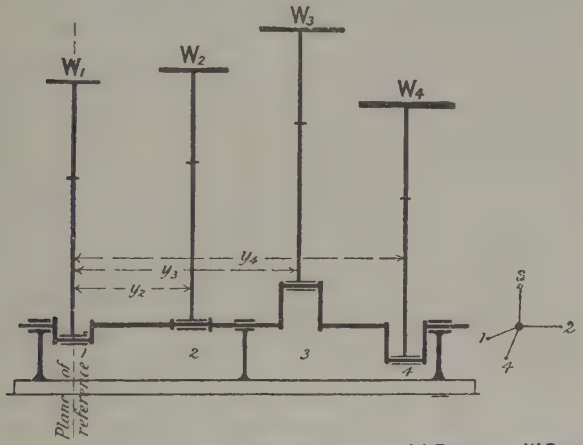
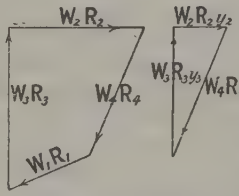


FIG. 226.



Let the reciprocating masses be W_1, W_2, etc.;
the radii of the cranks be R_1, R_2, etc.;
the distance from the plane of reference taken through the first crank be y_2, y_3, etc.

Then the acceleration pressure, neglecting the obliquity of the rods at each end of the stroke, will be $0{\cdot}00034WRN^2$, with the corresponding suffixes for each cylinder. Since the speed of all of them is the same, the acceleration pressure will be proportional to WR. It will be convenient to tabulate the various quantities, thus—

Cylinder.	Weight of reciprocating parts. lbs.	Radius of crank.	Proportional acceleration force.	Distance of centre line from plane of reference.	Proportional acceleration force moment.
1	$W_1 = 750$	$R_1 = 12''$	$W_1R_1 = 9{,}000$	0	0
2	$W_2 = 1000$	$R_2 = 14''$	$W_2R_2 = 14{,}000$	$y_2 = 40''$	$W_2R_2y_2 = 560{,}000$
3	$W_3 = 1200$	$R_3 = 14''$	$W_3R_3 = 16{,}800$	$y_3 = 80''$	$W_3R_3y_3 = 1{,}344{,}000$
4	$W_4 = 1230$	$R_4 = 12''$	$W_4R_4 = 14{,}800$	$y_4 = 112''$	$W_4R_4y_4 = 1{,}653{,}000$

The vector sum of both the forces and the moments of the forces must be zero to secure perfect balance, *i.e.* they must form closed polygons; such polygons are drawn to show how the cranks must be arranged and the weights distributed.

The method is due to Professor Dalby, who treats the whole question of balancing very thoroughly in his "Balancing of Engines." The reader is recommended to consult this book for further details.

Balancing Locomotives.—In order that a locomotive may run steadily at high speeds, the rotating and reciprocating parts must be very carefully balanced. If the rotating parts be left unbalanced, there will be a serious blow on the rails every time the unbalanced portion gets to the bottom; this is known as the "hammer blow." If the reciprocating parts be left unbalanced, the engine will oscillate to and fro at every revolution about a vertical axis situated near the middle of the crank-shaft; this is known as the "elbowing action."

By balancing the rotating parts, the hammer blow may be overcome, but then the engine will elbow; if, in addition, the reciprocating parts be entirely balanced, the engine will be overbalanced vertically; hence we have to compromise matters by only partially balancing the reciprocating parts. Then, again, the obliquity of the connecting-rod causes the pressure due to the inertia of the reciprocating parts to be greater at one end of the stroke than at the other, a variation which cannot be compensated for by balance weights rotating at a constant radius.

Thus we see that it is absolutely impossible to perfectly balance a locomotive of ordinary design, and the compromise we adopt must be based on experience.

The following symbols will be used in the paragraphs on locomotive balancing:—

W_ρ, for rotating weights (pounds) to be balanced.

W_ρ, for reciprocating weights (pounds) to be balanced.

W_B, for balance weights; if with a suffix ρ, as $W_{B\rho}$, it will indicate the balance weight for the reciprocating parts, and so on with other suffixes.

R, for radius of crank (feet).

$R_{c\rho}$,, ,, coupling-crank.

R_B, ,, ,, balance weights.

Rotating Parts of Locomotive.—The balancing of the rotating parts is effected in the manner described in the paragraph on standing balance, p. 211, which gives us—

$$W_r R = W_{Br} R_B$$

$$\text{and } W_{Br} = \frac{W_r R}{R_B}$$

The weights included in the W_r vary in different types of engines; we shall consider each case as it arises.

Reciprocating Parts of Locomotive.— We have already shown (p. 181) that the acceleration pressure at the

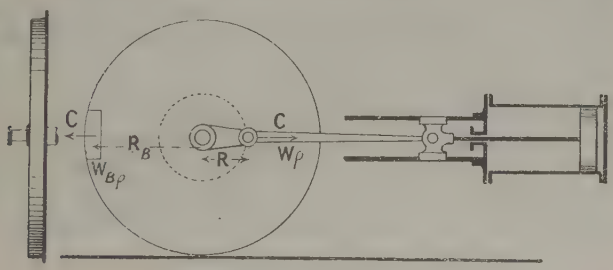


FIG. 227.

end of the stroke due to the reciprocating parts is equal to the centrifugal force, assuming them to be concentrated at the crank-pin, and neglecting the obliquity of the connecting-rod.

Then, for the present, assuming the balance weight to rotate in the plane of the crank-pin, in order that the reciprocating parts may be balanced, we must have—

$$C_B = C$$

$$0{\cdot}00034 W_{B\rho} \cdot R_B \cdot N^2 = 0{\cdot}00034 W_\rho \cdot R \cdot N^2$$

$$W_{B\rho} \cdot R_B = W_\rho \cdot R$$

$$\text{and } W_{B\rho} = \frac{W_\rho \cdot R}{R_B} \quad \ldots\ldots \quad \text{(i.)}$$

On comparing this with the result obtained for rotating parts, we see that reciprocating parts, when the obliquity of the connecting-rod is neglected, may for every purpose be regarded as though their weight were concentrated in a heavy ring round the crank-pin.

Now we come to a much-discussed point. We showed above that with a short connecting-rod of n cranks long, the acceleration pressure was $\frac{1}{n}$ greater at one end and $\frac{1}{n}$ less at the other end of the stroke than the pressure with an infinitely long rod: hence if we make $W_{B\rho}$ $\frac{1}{n}$ greater to allow for the

obliquity of the rod at one end, it will be $\frac{2}{n}$ too great at the other end of the stroke. Thus we really do mischief by attempting to compensate for the obliquity of the rod at either end; we shall therefore proceed as though the rod were of infinite length.

If the reader wishes to follow the effect of the obliquity of the rod at all parts of the stroke, he should consult a paper by Mr. Hill, in the *Proceedings of the Institute of Civil Engineers*, vol. civ.; or Barker's "Graphic Methods of Engine Design;" also Dalby's "Balancing of Engines."

The portion of the connecting-rod which may be regarded as rotating with the crank-pin and the portion as reciprocating with the cross-head may be most readily obtained by finding the centre of gravity of the whole rod—let its distance from the crank-pin centre be x, the length of the rod centres l, then the portion to be included in the reciprocating parts is $\frac{Wx}{l}$, where W is the weight of the whole rod. The remainder $W\left(1 - \frac{x}{l}\right)$ is to be included in the rotating parts. If the rod be placed horizontally with the small end on a weighing machine or be suspended from a spring balance the reading will give the weight to be included in the reciprocating parts. For most purposes it is sufficiently near to take the weight of the small end together with one-half the plain part as reciprocating, and the big end with one-half the plain part as rotating.

Inside-cylinder Engine (uncoupled).—In this case we have—

W_p = weight of (piston + piston-rod + cross-head + small end of connecting-rod + $\frac{1}{2}$ plain part of rod);

W_r = weight of (crank-pin + crank-webs[1] + big end of connecting-rod + $\frac{1}{2}$ plain part of rod).

If we arrange balance weights so that their c. of g. rotates in the same plane as the crank-pins, their combined weight would be $W_{Br} + W_{Bp}$, placed at the radius R_B, and if we only counterbalance two-thirds of the reciprocating parts, we get—

$$W_{B0} = \frac{R(\frac{2}{3}W_p + W_r)}{R_B} \quad \ldots\ldots \quad \text{(ii.)}$$

[1] See p. 229.

Balance weights are not usually placed opposite the crank-webs as shown, but are distributed over the wheels in such a manner that their centrifugal moments about the plane of rotation of the crank-pin is zero. If W be the balance weight on one wheel, and W_1 the other, distant y' and y_1' from the plane of the crank, then—

$$WR_By' = W_1R_By_1'$$
$$\text{or } Wy' = W_1y_1'$$

which is equivalent to saying that the centre of gravity of the two weights lies in the plane of rotation of the crank. The object of this particular arrangement is to keep the axis of

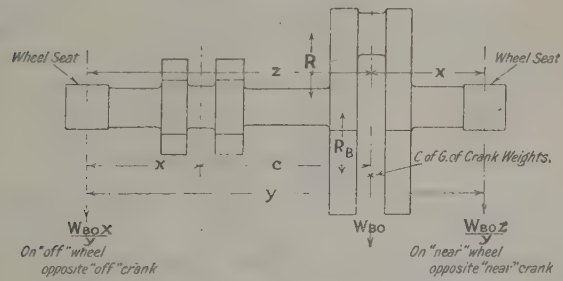


FIG. 228.

rotation permanent. Then, considering the vertical crank shown in Fig. 228, by taking moments, we get the equivalent weights at the wheel centres as given in the figure.

We have, from the figure—

$$x = \frac{y}{2} - \frac{c}{2} \qquad x = \frac{y - c}{2}$$

$$z = \frac{y}{2} + \frac{c}{2} \qquad z = \frac{y + c}{2}$$

Substituting these values, we get—

$\frac{W_{B0}}{2y}(y - c) = W_{B1}$, as the proportion of the balance weight on the "off" wheel opposite the far crank

and $\frac{W_{B0}}{2y}(y + c) = W_{B2}$, as the proportion of the balance weight on the "near" wheel opposite near crank

Exactly similar balance weights are required for the other crank. Thus on each wheel we get one large balance weight W_{B2} at N (Fig. 229), opposite the near crank, and one small one W_{B1} at F, opposite the far crank. Such an arrangement would, however, be very clumsy, so we shall combine the two balance weights by the parallelogram of forces as shown, and for them substitute the large weight W_B at M.

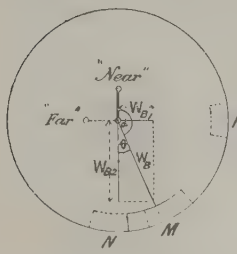


FIG. 229.

Then $W_B = \sqrt{W_{B1}^2 + W_{B2}^2}$

On substituting the values given above for W_{B1} and W_{B2} we have, when simplified—

$$W_B = \frac{\sqrt{2}W_{B0}}{2y}\sqrt{y^2 + c^2}$$

In English practice $y = 2{\cdot}5c$ (approximately)

On substitution, we get—

$$W_B = 0{\cdot}76W_{B0}$$

Substituting from ii., we have—

$$W_B = \frac{0{\cdot}76R(\frac{2}{3}W_\rho + W_r)}{R_B}$$

Let the angle between the final balance weight and the near crank be α, and the far crank $\theta + 90$.

Then $\alpha = 180 - \theta$

and $\tan\theta = \frac{W_{B1}}{W_{B2}} = \frac{y - c}{y + c}$

Substituting the value of y for English practice, we get—

$$\tan\theta = \frac{1{\cdot}5}{3{\cdot}5} = 0{\cdot}429$$
$$\theta = 23^{\circ}$$

Now, $\theta = \frac{90}{4}$ very nearly; hence, for English practice, if the quadrant opposite the crank quadrant be divided into

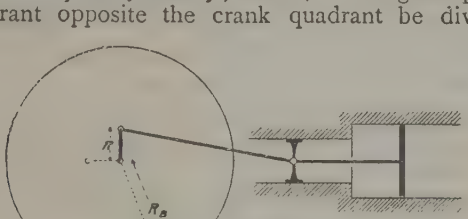


FIG. 230.

four equal parts, the balance weight must be placed on the first of these, counting from the line opposite the near crank.

Outside-cylinder Engine (uncoupled).—W_p and W_r are the same as in the last paragraph. If the plane of rotation of the crank-pin nearly coincides, as it frequently does, with the plane of rotation of the balance weight, we have—

$$W_B = W_{B0} = \frac{R(\frac{2}{3}W_p + W_r)}{R_B} \text{ nearly}$$

and the balance weight is placed diametrically opposite the crank.

When the planes do not approximately coincide—

Let y = the distance between the wheel centres;
c = " " " cylinder centres;
x = " " " cylinder centre line and the "near" wheel;
z = " " " cylinder centre line and the "off" wheel.

$$x = \frac{c-y}{2} \qquad z = \frac{c+y}{2}$$

The balance weight required on the "off" wheel opposite the "far" crank $= \frac{W_{B0}x}{y} = \frac{W_{B0}}{2y}(c-y) = W_{B1}$

The balance weight required on the "near" wheel opposite the "near" crank $= \frac{W_{B0}z}{y} = \frac{W_{B0}}{2y}(c+y) = W_{B2}$

$$\text{Then } W_B = \frac{\sqrt{2}W_{B0}\sqrt{y^2+c^2}}{2y}$$

which is precisely the same expression as we obtained for inside-cylinder engines, but in this case $y = 0{\cdot}8c$ to $0{\cdot}9c$. On substitution, we get $W_B = 1{\cdot}13W_{B0}$ to $1{\cdot}05W_{B0}$, and $\theta = 6°$ to $3°$.

The same reasoning applies to the coupling-rod balance weights W_{BC} in the next paragraphs.

Inside-cylinder Engine (coupled).—In this case we have W_p the same as in the previous cases.

W_C = the weight of coupling crank-web and pin [1] + coupling rod from *a* to *b*, or *c* to *d*, or *b* to *c* (Fig. 195), as the case may be;

W_{BC} = the weight of the balance weight required to counterbalance the coupling attachments;

R_C = the radius of the coupling crank.

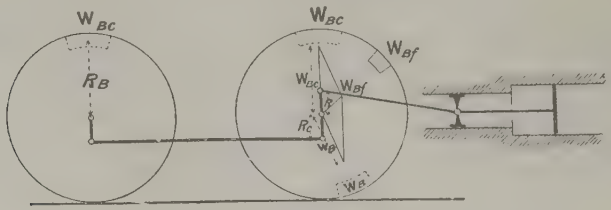


FIG. 231.

In the case of the driving-wheel of the four-wheel coupled engine, we have W_B arrived at in precisely the same manner as in the case of the inside-cylinder uncoupled engine, and $W_{BC} = \frac{R_C W_C}{R_B}$.

The portion of the coupling rod included in the W_C is, in this case, one-half the whole rod. The balance weight W_{BC} is placed diametrically opposite the coupling crank-pin. After finding W_B and W_{BC}, they are combined in one weight W_{BF} by the parallelogram of forces, as already described.

With this type of engine the balance weight is usually small. Sometimes the weights of the rods are so adjusted that a balance weight may be dispensed with on the driving-wheel.

[1] See p. 229.

It frequently happens, however, that W_B is larger than W_{BC}; in that case W_{BF} is placed much nearer W_B than is shown in the figure.

On the coupled wheel the balance weight W_{BC} is of the same value as that given above, and is placed diametrically opposite the coupling crank-pin.

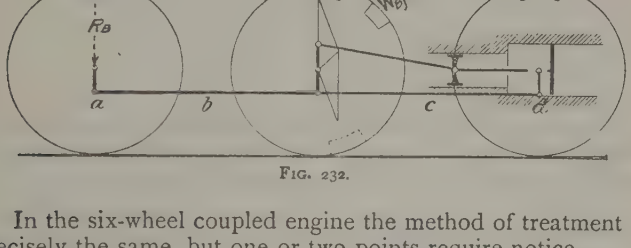


FIG. 232.

In the six-wheel coupled engine the method of treatment is precisely the same, but one or two points require notice.

$$W'_{BC} = \frac{R_C W_C}{R_B}$$

The portion of the coupling rod included in the W_C is from b to c; whereas in the W_{BC} the portion is from a to b or c to d.

Coupling cranks [1] have been placed *with* the crank-pins; the balance weights then become very much greater. They are treated in precisely the same way.

Some locomotive-builders evenly distribute the balance weights on coupled engines over all the wheels: most authorities strongly condemn this practice. Space will not allow of this point being discussed here.

Outside-cylinder Engine (coupled).

W is the same as before;

W_r is the weight of crank-web [2] and pin + coupling rod from a to b + big end of connecting-rod + half plain part of rod;

W_C is the same as in the last paragraph;

$$R_C = R$$

$$W_B = W_{B0} = \frac{R(\frac{2}{3}W_p + W_r)}{R_B}$$

$$W_{BC} = \frac{RW_C}{R_B}$$

[1] See *Proc. Inst. C.E.*, vol. lxxxi. p. 122.

[2] See p. 229.

The six-wheel coupled engine is treated in a similar way; the remarks in the last paragraph also apply here.

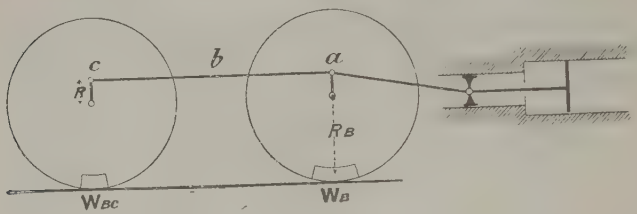


FIG. 233.

The above treatment only holds when the planes of the crank-pins and wheels nearly coincide, as already explained when dealing with the uncoupled outside-cylinder engine.

On some narrow-gauge railways, in which the wheels are placed inside the frames, the crank and coupling pins are often at a considerable distance from the plane of the wheels. Let the coupling rods be on the outer pins. It will be convenient, when dealing with this case, to find the distance between the planes containing the centres of gravity of the coupling and connecting rods, viz. C_g.

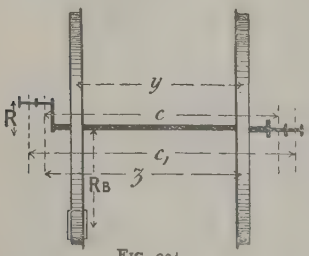


FIG. 234.

$$C_g = \frac{W_c C_1 + (W_r + \frac{2}{3}W_p)C}{W_c + W_r + \frac{2}{3}W_p}$$

The W_r must include the weight of the crank-webs and pins all reduced to the radius of the pin and to the distance C. For all practical purposes, C_g may be taken as the distance between the insides of the collars on the crank-pin. Then, by precisely similar reasoning to that given above—

$$W_B = \frac{\sqrt{2}W_{B0}}{2y}\sqrt{y^2 + C_g^2}$$

$$\text{where } W_{B0} = \frac{R(\frac{2}{3}W_p + W_c + W_r)}{R_B}$$

In some cases y is only $0{\cdot}5C_g$; then—

$$W_B = 1{\cdot}58W_{B0}, \text{ and } \theta = 18°$$

Hammer Blow.—If the rotating parts only of a locomotive are fully balanced there is no variation of load on the rail due to their centrifugal force, but when, in addition, the reciprocating parts are partially or fully balanced the vertical component of the centrifugal force of the excess balance weight over and above that required to balance the rotating parts causes a considerable variation of the load on the rail, tending to lift the wheel off the rail when the balance weight is on top and causing a very rapid increase of rail load —almost amounting to a blow—when the balance weight is at the bottom. The hammer blow is rather severe on the cross girders of bridges and on the permanent way generally. At very high speeds the upward force will actually lift the wheel off the rail if it exceeds the dead weight on the wheel. On some American railroads the proportion of the reciprocating parts to be balanced is settled by the maximum speed the engine is likely to reach.

In arriving at this speed, the balance weight required to completely balance the rotating parts is first determined, then the balance weight for both rotating and reciprocating parts is found, the difference between the two is the unbalanced portion which is responsible for the hammer blow and the tendency to lift the wheel off the rail.

Let the difference be W_d, then the centrifugal force tending to lift the wheel is

$$0{\cdot}00034 W_d R_B N^2$$

and when this exceeds the dead weight W_w on the rail the wheel will lift, hence the speed of lifting is

$$N = \sqrt{\frac{W_w}{0{\cdot}00034 W_d R_B}}$$

When calculating the speed of the train from N it must not be forgotten that the radius of the wheel is greater than R_B.

Centre of Gravity of Balance Weights and Crank-webs.—The usual methods adopted for finding the position and weight of balance weights are long and tedious; the following method will be found more convenient. The effective balance weight is the whole weight *minus* the weight of the spokes embedded.

Let Figs. 235, 236, 237 represent sections through a part of the balance weight and a spoke; then, instead of dealing first with the balance weight as a whole, and afterwards deducting

the spokes, we shall deduct the spokes first. Draw the centre lines of the spokes x, x, and from them set off a width w on

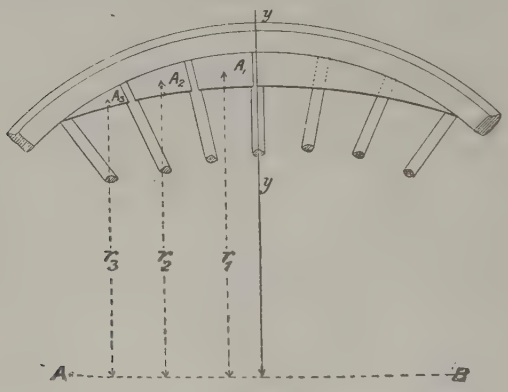


FIG. 235.

each side as shown in Fig. 236, where wt = half the area of the spoke section; in the case

of the elliptical spoke, $wt = \frac{0{\cdot}785Dd}{2}$

$$w = \frac{0{\cdot}392Dd}{t}$$

of the rectangular spoke, $wt = \frac{Dd}{2}$

$$w = \frac{0{\cdot}5Dd}{t}$$

By doing this we have not altered either the weight or the position of the centre of gravity of the section of the balance weight, but we have reduced it to a much simpler form to deal with. If a centre line yy (Fig. 235) be drawn through the balance weight, it is only necessary to dealt with the segments on one side of it.

Measure the area of the segments when thus treated.

Let them be A_1, A_2, A_3; then the weight of the whole balance weight is the sum of these segments—

$$W_B = 2tw_m(A_1 + A_2 + A_3)$$

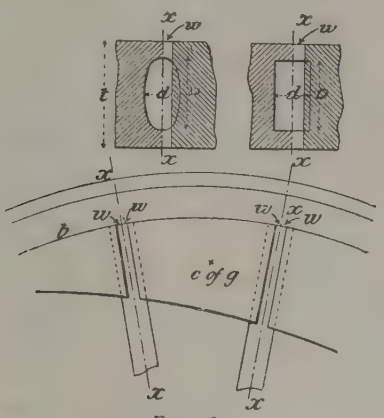


FIG. 236.

where w_m = the weight per cubic inch of the metal.

For a cast-iron weight—

$$W_B = 0{\cdot}52t(A_1 + A_2 + A_3)$$

For a wrought-iron or cast-steel weight—

$$W_B = 0{\cdot}56t(A_1 + A_2 + A_3)$$

all dimensions being in inches.

The centre of gravity of each section can be calculated, but it is far less trouble to cut out pieces of cardboard to the shape of each segment, and then find the position of the centre of gravity by balancing, as described on p. 75. Measure the distance of each centre of gravity from the line AB drawn through the centre of the wheel.

Let them be r_1, r_2, r_3 respectively; then the radius of the centre of gravity of the whole weight (see Fig. 235)—

$$R_B = \frac{A_1r_1 + A_2r_2 + A_3r_3}{A_1 + A_2 + A_3} \text{ (see p. 58)}$$

$$\text{and } W_BR_B = \begin{Bmatrix} 0{\cdot}52 \\ \text{or} \\ 0{\cdot}56 \end{Bmatrix} t(A_1r_1 + A_2r_2 + A_3r_3)$$

If there were more segments than those shown, we should get further similar terms in the brackets.

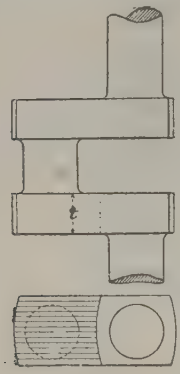


FIG. 237.

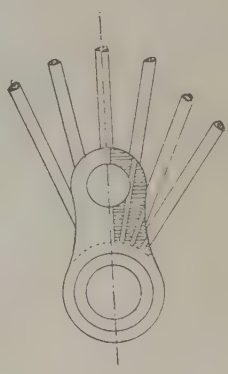


FIG. 238.

When dealing with cranks, precisely the same method may be adopted for finding their weight and the position of the centre of gravity.

In the figures, the weight of the crank $= 2tw_m \times$ shaded areas. The position of the centre of gravity is found as before, but no material error will be introduced by assuming it to be at the crank-pin.

Governors.—The function of a flywheel is to keep the speed of an engine approximately constant during one revolution or one cycle of its operations, but the function of a governor is to regulate the number of revolutions or cycles that the engine makes per minute. In order to regulate the speed, the supply of energy must be varied proportionately to the resistance overcome; this is usually achieved automatically by a governor consisting essentially of a rotating weight suspended in such a manner that its position relatively to the axis of rotation varies as the centrifugal force acting upon it, and therefore as the speed. As the position of the weight varies, it either directly or indirectly opens and closes the valve through which the energy is supplied, closing it when the speed rises, opening it when it falls.

The governor weight shifts its position on account of a change in speed; hence some variation of speed must always take place when the resistance is varied, but the change in

speed can be reduced to a very small amount by suitably arranging the governor.

Simple Watt Governor. — Let the ball shown in the figure be suspended by an arm pivoted at O, and let it rotate round the axis OO_1 at a constant rate. The ball is kept in equilibrium by the three forces W, the weight of the ball acting vertically downwards (we shall for the present neglect the weight of the arm and its attachments, also friction on the joints); C, the centrifugal force acting horizontally; T, the tension in the supporting arm.

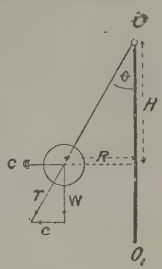


FIG. 239.

Let H = height of the governor in feet;
h = " " " " inches;
R = radius of the ball path in feet;
N_s = number of revolutions made by the governor per second;
N = number of revolutions made by the governor per minute;
V = velocity (linear) in feet per second of the balls.

By taking moments about the pin O, we have—

$$CH = WR, \qquad \frac{WV^2H}{gR} = WR$$

$$\text{hence } H = \frac{gR^2}{4\pi^2R^2N_s^2} = \frac{0{\cdot}816}{N_s^2}$$

Expressing the height in inches, and the speed in revolutions per minute, we get—

$$h = \frac{35230}{N^2}$$

Thus we see that the height at which a simple Watt governor will run is entirely dependent upon the number of revolutions per minute at which it runs. The size of the balls and length of arms make no difference whatever as regards the height when the balls are "floating."

The following table gives the height of a simple Watt governor for various speeds :—

Revolutions per minute (n).	Height of governor in inches (h).	Change of height corresponding to a change of speed of 10 revolutions per minute.
		Inches.
50	14·09	—
(54·2)	(12·00)	—
60	9·79	4·30
70	7·19	2·60
80	5·51	1·68
90	4·35	1·16
100	3·52	0·83
110	2·91	0·61
120	2·45	0·46

These figures show very clearly that the change of height corresponding to a given change of speed falls off very rapidly as the height of the governor decreases or as the apex angle θ increases; but as the governing is done entirely by a change in the height of the governor in opening or closing a throttle or other valve, it will be seen that the regulating of the motor is much more rapid when the height of the governor is great than when it is small; hence, if we desire to keep the speed within narrow limits, we must keep the height of the governor as great as possible, or the apex angle θ as small as possible, within reasonable limits.

Suppose, for instance, that a change of height of 2 inches were required to fully open or close the throttle or other valve; then, if the governor were running at 60 revolutions per minute, the 2-inch movement would correspond to about 7 per cent. change of speed; at 80, 15 per cent.; at 100, 24 per cent.; at 120, 36 per cent.

The greater the change of height corresponding to a given change of speed, the greater is said to be the *sensitiveness* of the governor.

A simple Watt governor can be made as sensitive as we please by running it with a very small apex angle, but it then becomes very cumbersome, and, moreover, it then possesses very little "power" to overcome external resistances.

Loaded Governor.—In order to illustrate the principle of the loaded governor, suppose a simple Watt governor to be loaded as shown. The broken lines show the position of the governor when unloaded.

When the load W_w is placed on the balls, the "equivalent

height of the simple Watt governor" is increased from H to H_e. Then, constructing the triangle of forces, or, by taking moments about the pin, and remembering that $\frac{W_w}{2}$ acts vertically downwards through the centre of the ball, we have—

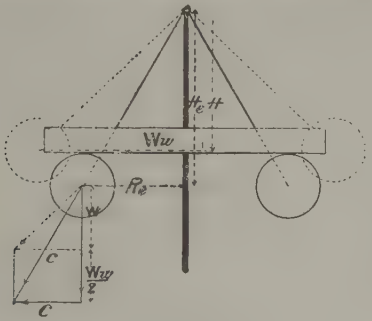


FIG. 240.

$$\frac{H_e}{R_e} = \frac{W + \frac{W_w}{2}}{C}$$

Then, by precisely the same reasoning as in the case given above, we have—

$$H_e = \frac{0{\cdot}816}{N_e^2}\left(\frac{W + \frac{W_w}{2}}{W}\right)$$

$$\text{or } h_e = \frac{35230}{N^2}\left(\frac{W + \frac{W_w}{2}}{W}\right)$$

If W_w be m times the weight of one ball, we have—

$$h_e = \frac{35230}{N^2}\left(1 + \frac{m}{2}\right)$$

the value of m usually varies from 10 to 50.

This expression must, however, be used with caution. Consider the case of a simple Watt governor both when unloaded and when loaded as shown in Figs. 239 and 240. If the *same* governor be taken in both instances, it is evident that its maximum height, *i.e.* when it just begins to lift, also its

minimum apex angle, will be the same whether loaded or unloaded, and cannot in any case be greater than the length of the suspension arm measured to the centre of the ball. The speed of the loaded governor corresponding to any given height will, however, be greater than that of the unloaded governor in the ratio $\sqrt{1 + \frac{m}{2}}$ to 1, and if the *engine* runs at the same speed in both cases, the governor must be geared up in this ratio, but the alteration in height for any given alteration in the speed of the engine will be the same in both cases, or, in other words, the proportional sensitiveness will be the same whether loaded or unloaded. We shall later on show, however, that the loaded governor is better on account of its greater power.

In the author's opinion most writers on this subject are in error; they compare the sensitiveness of a loaded governor at heights which are physically impossible (because greater even than the length of the suspension arms), with the much smaller, but possible, heights of an unloaded governor. If the reader wishes to appeal to experiment he can easily do so, and will find that the sensitiveness actually is the same in both cases.

The following table may help to make this point clear. m has been chosen as 16; then $\sqrt{1 + \frac{m}{2}} = 3$.

On comparing the last column of this table with that for the unloaded governor, it will be seen that they are identical, or the sensitiveness is the same in the two cases.

Revolutions per minute of governor.	Height of loaded governor in inches.	Change of height corresponding to a change of speed of 30 revolutions per minute of governor, or 10 revolutions per minute of engine.
		Inches.
150	14·09	
180	9·79	4·30
210	7·19	2·60
240	5·51	1·68
270	4·35	1·16
300	3·52	0·83
330	2·91	0·61
360	2·45	0·46

If, by any system of leverage, the weight W_w moves up and down x times as fast as the balls, the above expression becomes—

$$h_e = \frac{35230}{N^2}\left(1 + \frac{xm}{2}\right)$$

Porter and other Loaded Governors.—The method of loading shown in Fig. 240 is not convenient, and is rarely adopted in practice. The usual method is that shown in Fig. 241, viz. the Porter governor, in which the links are usually of equal length, thus making $x = 2$; but this proportion is not always adhered to. Then—

$$h_e = \frac{35230}{N^2}\left(\frac{W + W_w}{W}\right)$$

$$h_e = \frac{35230}{N^2}(1 + m)$$

Occasionally, governors of this type are loaded by means of a spring, as shown in Fig. 243, instead of a central weight. The arrangement is, however, bad, since the central load increases as the governor rises, and consequently makes it far too sluggish in its action.

Let the length of the spring be such that it is free from load when the balls are right in, *i.e.* when their centres coincide with the axis of rotation, or when the apex angle is zero. The reason for making this stipulation will be apparent when we have dealt with other forms of spring governors.

Let the pressure on the spring = Px_e lbs. when the spring is compressed x_e feet;

l = the length of each link in feet;
H_e = the equivalent height of the governor in feet;
W = the weight of each ball in lbs.;
$x_e = 2(l - H_e)$;
N = revolutions of governor per minute;

Then, as in the Porter governor, or, by taking moments about the apex pin, and remembering that the force Px_e acts parallel to the spindle, we have—

$$\frac{H_e}{R_e} = \frac{W + Px_e}{C} = \frac{W + 2Pl - 2PH_e}{0{\cdot}00034WR_eN^2}$$

$$H_e(0{\cdot}00034WN^2 + 2P) = W + 2Pl$$

$$H_e = \frac{W + 2Pl}{0{\cdot}00034WN^2 + 2P} \text{ for a vertical governor}$$

$$H_e = \frac{2Pl}{0{\cdot}00034WN^2 + 2P} \quad \text{,, horizontal ,,}$$

In the type of governor shown in Fig. 244, which is frequently met with, springs are often used instead of a dead weight. The value of x is usually a small fraction, consequently a huge weight would be required to give the same results as a Porter or similar type of governor. But it has other inherent defects which will shortly be apparent.

Isochronous Governors. — A perfectly isochronous governor will go through its whole range with the slightest

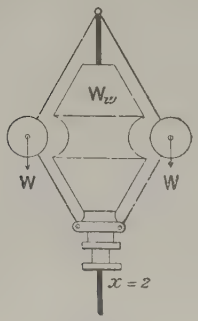


FIG. 241.

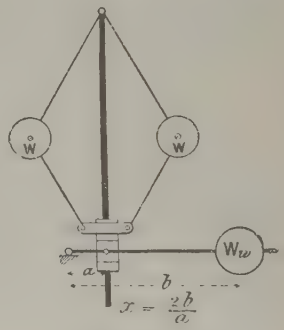


FIG. 242.

variation in speed; but such a governor is practically useless for governing an engine, for reasons shortly to be discussed.

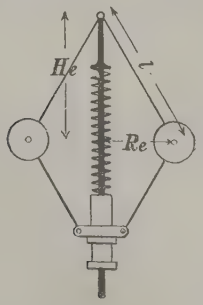


FIG. 243.

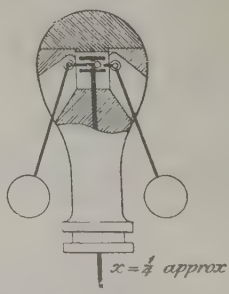


FIG. 244.

But when designing a governor which is required to be very sensitive, we sail as near the wind as we dare, and make it very nearly isochronous. In the governors we have considered,

the height of the governor has to be altered in order to alter the throttle or other valve opening. If this could be accomplished without altering the height of the governor, it could also be accomplished without altering the speed, and we should have an isochronous governor. Such a governor can be constructed by causing the balls to move in the arc of a parabola, the axis being the axis of rotation. Then, from the properties of the parabola, we know that the height of the governor, *i.e.* the subnormal to the path of the balls, is constant for all positions of the balls; therefore the sleeve which actuates the governing valve moves through its entire range for the smallest increase in speed. We shall only consider an approximate form which is very commonly used, viz. the crossed-arm governor.

The curve *abc* is a parabolic arc; the axis of the parabola is O*b*; then, if normals be drawn to the curve at the highest and lowest positions of the ball, they intersect at some point *d* on the other side of the axis. Then, if the balls be suspended from this point, they will move in an approximately parabolic arc, and the governor will therefore be approximately isochronous—and probably useless because too sensitive. If it be desired to make the governor more stable, the points *d*, *d* are brought in nearer the axis. The virtual centre of the arms is at their intersection; hence the height of the governor is H, which is approximately constant. The equivalent height can be raised by adding a central weight as in a Porter governor. It, of course, does not affect the sensitiveness, but it increases the power of the governor to overcome resistances. The speed at which a crossed-arm governor lifts depends upon the height in precisely the same manner as in the simple Watt governor.

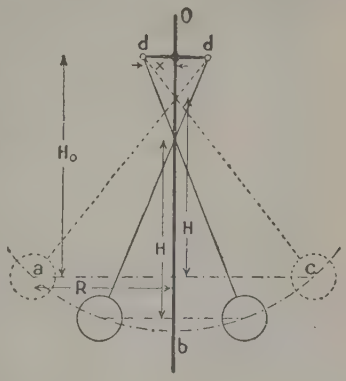


FIG. 245.

It can also be arrived at thus. By taking moments about the pin *d*, W is the weight of the ball, *l* the length of the arm *ad*.

$$W(R + x) = CH_0 = 0{\cdot}00034WRN^2\sqrt{l^2 - (R + x)^2}$$

$$N = \sqrt{\frac{R + x}{0{\cdot}00034R\sqrt{l^2 - (R + x)^2}}}$$

or thus: $\frac{x}{R} = \frac{H_0 - H}{H}$ and $H = \frac{RH_0}{x + R} = \frac{0{\cdot}816 \times 60^2}{N^2}$

$$N = \sqrt{\frac{2937(x + R)}{RH_0}}$$

when the dimensions of the governor are taken in feet and the speed in revolutions per minute.

In some instances Watt governors are made with the arms suspended at some distance, say x, from the axis of rotation, as shown in Fig. 249, but without the central weight. Then the above expression becomes—

$$N = \sqrt{\frac{R - x}{0{\cdot}00034R\sqrt{l^2 - (R - x)^2}}}$$

and when the expression for H is used (p. 231), the height H is measured from the level of the ball centres to the point where the two arms cross (see Fig. 249).

Astronomical Clock Governor.—A beautiful application of the crossed-arm principle as applied to isochronous governors is found in the governors used on the astronomical clocks made by Messrs. Warner and Swazey of Cleveland, Ohio. Such a clock is used for turning an equatorial telescope on its axis at such a speed that the telescope shall keep exactly focussed on a star for many hours together, usually for the purpose of taking a photograph of that portion of the heavens immediately surrounding the star. If the telescope failed to move in the desired direction, and at the exact apparent speed of the star, the relative motion of the telescope and star would not be zero, and a blurred image would be produced; hence an extreme degree of accuracy in driving is required. The results obtained with this governor are so perfect that no ordinary means of measuring time are sufficiently accurate to detect any error.

The spindle A and cradle are driven by the clock, whose

speed has to be controlled. A short link, *c*, is pivoted to arms on the driving spindle at *b*; the governor weights are suspended by links from the point *d*; a brake shoe, *e*, covered with some soft material, is attached to a lug on the link *c*, and, as the governor rotates, presses on the fixed drum *f*. The point of suspension, *d*, is so chosen that the governor is practically isochronous. The weights rest in their cradle until the speed of the governor is sufficiently high to cause them to lift; when in their lowest position, the centre line of the weight arm passes through *b*, and consequently the pull along the arm has no moment about this point, but, as soon as the speed rises sufficiently to lift the weights, the centre line of the weight arm no longer coincides with *b*, and the pull acting along the weight arm now has a moment about *b*, and thus sets up a pressure between the rotating brake shoe *e* and the fixed drum *f*. The friction between the two acts as a brake, and thus checks the speed of the clock.

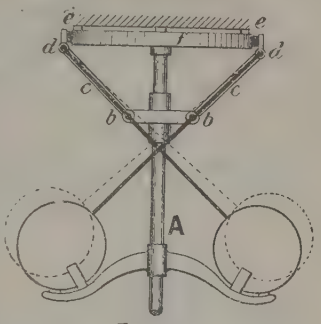


FIG. 246.

It will be seen that this is an extremely sensitive arrangement, since the moment of the force acting along the ball link, and with it the pressure on the brake shoe, varies rapidly as the ball rises; but since the governor is practically isochronous, only an extremely small variation in speed is possible. It should be noted that the driving effort should be slightly in excess of that required to drive the clock and telescope, apart from the friction on the governor drum, in order to ensure that there is always some pressure between the brake shoe and the drum.

Wilson Hartnell Governor.—Another well-known and highly successful isochronous governor is the "Wilson Hartnell" governor.

In the diagram, c is the centrifugal force acting on the ball, and p the pressure due to the spring, *i.e.* one-half the total pressure. As the balls fly out the spring is compressed, and since the pressure increases directly as the compression, the pressure p increases directly (or very nearly so) as the radius r of the balls; hence we may write $p = Kr$, where K is a constant depending on the stiffness of the spring.

Let $r_0 = nr$, frequently $n = 1$.

$$\text{Then } cr_0 = pr$$

$$\text{and } 0{\cdot}00034\text{W}r^2n\text{N}^2 = \text{K}r^2$$

$$\text{and } \text{N}^2 = \frac{\text{K}}{0{\cdot}00034\text{W}n}$$

For any given governor the weight W of the ball is constant; hence the denominator of the fraction is constant, whence N^2, and therefore N, is constant; *i.e.* there is only one speed at which the governor will float, and any increase or decrease in the speed will cause the balls to fly right out or in, or, in other words, will close or fully open the governing valve; therefore the governor is isochronous.

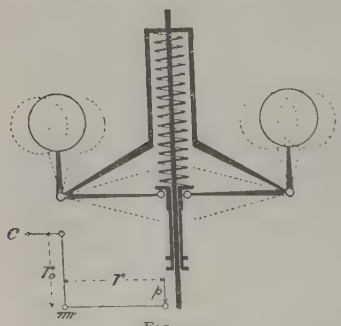


FIG. 247.

There are one or two small points that slightly affect the isochronous character of the governor. For example, the weight of the ball, except when its arm is vertical, has a moment about the pivot. Then, except when the spring arm is horizontal, the centrifugal force acting on the spring arm tends to make the ball fly in or out according as the arm is above or below the horizontal.

We shall shortly show how the sensitiveness can be varied by altering the compression on the spring.

Weight of Governor Arms.—Up to the present we have neglected the weight of the governor arms and links, and have simply dealt with the weight of the balls themselves; but with some forms of governors such an approximate treatment would give results very far from the truth.

Dealing first with the case of the arm, and afterwards with the ball—

Let the vertical section of the arm be a square inches, and all other dimensions be in inches. The centrifugal moment acting on the element is—

$$\frac{wadr \cdot r\omega^2 h}{12g} = \frac{wa\omega^2 r^2 dr}{12g \tan \theta}$$

and on the whole arm

$$\frac{wa\omega^2}{12g\tan\theta}\int_0^R r^2dr$$

$$\frac{wa\omega^2R^3}{12\times 3g\tan\theta}$$

which may be written

$$waR\times\frac{R^2}{3}\times\frac{\omega^2}{12g\tan\theta}$$

The quantity waR is the weight of the whole arm W_a, and $\frac{R^2}{3}$ is the square of the radius of gyration of the arm about the axis of rotation, and the product is the moment of inertia of the arm in pounds weight and inch ^{3} units.

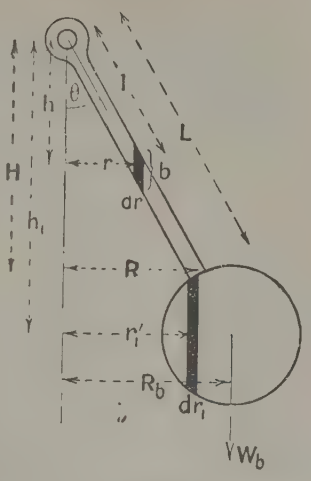


FIG. 248.

$$\left.\begin{matrix}\text{The centrifugal moment}\\ \text{acting on the arm}\end{matrix}\right\}=\frac{I\omega^2}{12g\tan\theta}=\frac{W_aR^2\omega^2}{36g\tan\theta}$$

In the case of the ball we have the centrifugal moment for the elemental slice.

$$\frac{wa_1dr_1r'_1\omega^2h_1}{12g}=wa_1r'_1dr_1\left(\frac{\omega^2h_1}{12g}\right)$$

and for the whole ball—

$$\left(\frac{\omega^2h_1}{12g}\right)w\int a_1r'_1dr_1$$

The quantity outside the brackets is the sum of the moments of the weight of each vertical slice of the ball about the axis of rotation, which is the product of the weight of the ball and the distance of its centre of gravity from the axis. The centrifugal moment of the ball is—

$$\frac{W_B\times R_B\times\omega^2\times h_1}{12g}$$

This is the value usually taken for the centrifugal moment

of the governor, but it of course neglects the arms. A common way of taking the arms into account is to assume that the centrifugal force acts at the centre of gravity of the arm; but it is incorrect. By this assumption we get for the centrifugal moment—

$$\frac{wa\mathrm{R}}{g} \times \frac{\mathrm{R}}{2 \times 12} \times \frac{\mathrm{H}}{2} \times \omega^2 = \frac{wa\mathrm{R}\omega^2\mathrm{R}^2}{48g \tan \theta}$$

$$= \frac{\mathrm{W}_a\mathrm{R}^2\omega^2}{48g \tan \theta}$$

Hence the ratio of the true centrifugal moment of the arm to the approximation commonly used is $\frac{4}{3}$, thus the error involved in the assumption is 33 per cent. of the centrifugal moment of the arm. In cases in which the weight of the arm is small compared with the weight of the ball the error is not serious, but in the case of some governors in which thick stumpy arms are used, commonly found where the balls and arms are of cast or malleable iron, the error may amount to as much as 20 per cent., which represents about 11 per cent. error in the speed. The centre of gravity assumption is a convenient one, and may be adhered to without error by assuming that the weight of the arm is $\frac{4}{3}$ of its real weight. When the arm is pivoted at a point which is not on the axis of rotation, the corresponding moment of inertia of the arm should be taken. The error involved in assuming it to be on the axis is quite small in nearly all cases.

In all cases the centripetal moment in a gravity governor is found by taking the moment of the arm and ball about the point of suspension. The weight of the arm and ball is considered as concentrated at the centre of gravity; but in the case of the lower links, the bottom joint rises approximately twice as fast as the upper joint, hence its centre of gravity rises 1·5 times as fast as the top joint, and its weight must be taken as 1·5 times its real weight; this remark also applies to the centrifugal moment, in that case the lower link is taken as twice its real weight.

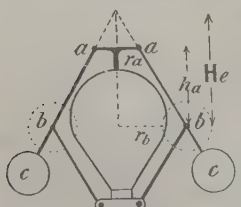


FIG. 249.

The height of this governor is H_e, not h_a; *i.e.* the height is measured from the virtual centre at the apex.

$$H_e = \frac{h_a r_b}{r_b - r_a}$$

A governor having arms suspended in this manner is very much more stable and sluggish than when the arms are suspended from a central pin, and still more so than when the arms are crossed.

In Fig. 250 we show the governor used on the De Laval steam turbine. The ball weights in this case consist of two halves of a hollow cylinder mounted on knife-edges to reduce the friction. The speed of these governors is usually calculated by assuming that the mass of each arm is concentrated at its centre of gravity, or that it is a governor having weightless arms carrying equivalent balls, as shown in broken lines in the figure. It can be readily shown by such reasoning as that given above that such an assumption is correct provided the arms are parallel with the spindle, and that the error is small provided that the arms only move through a small angle. These governors work exceedingly well, and keep the speed within very narrow limits. The figure is not drawn to scale.

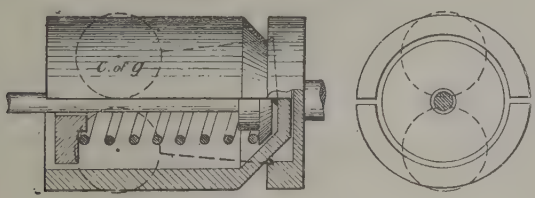


FIG. 250.

Crank-shaft Governors.—The governing of steam-engines is often effected by varying the point at which the steam is cut off in the cylinder. Any of the forms of governor that we have considered can be adapted to this method, but the one which lends itself most readily to it is the crank-shaft governor, which alters the cut-off by altering the throw of the eccentric. We will consider one typical instance only, the Hartnell-McLaren governor, chosen because it contains many good points, and, moreover, has a great reputation for governing within extremely fine limits (Figs. 251 and 252).

The eccentric E is attached to a plate pivoted at A, and suspended by spherical-ended rods at B and C. A curved cam,

DD, attached to this plate, fits in a groove in the governor weight W in such a manner that, as the weight flies outwards due to centrifugal force, it causes the eccentric plate to tilt, and so bring the centre of the eccentric nearer to the centre of the shaft, or, in other words, to reduce its eccentricity, and consequently the travel of the valve, thus causing the steam to be cut off earlier in the stroke. The cam DD is so arranged that when the weight W is right in, the cut-off is as late as the slide-valve will allow it to be. Then, when the weight is right out, the travel of the valve is so reduced that no steam is admitted to the cylinder. A spring, SS, is attached to the weight arm to supply the necessary centripetal force. The speed of the engine is regulated by the tension on this spring. In order to alter the speed while the engine is running, the lower end of the spring is attached to a screwed hook, F. The nut G is in the form of a worm wheel; the worm spindle is provided with a small milled wheel, H. If it be desired to alter the speed when running, a leather-covered lever is pushed into gear, so that the rim of the wheel H comes in contact with it at each revolution, and is thereby turned through a small amount, thus tightening or loosening the spring as the case may be. If the lever bears on the one edge of the wheel H, the spring is tightened and the speed of the engine increased, and if on the other edge the reverse. The spring S is attached to the weight arm as near its centre of gravity as possible, in order to eliminate friction on the pin J when the engine is running.

The governor is designed to be extremely sensitive, and, in order to prevent hunting, a dashpot K is attached to the weight arm.

In the actual governor two weights are used, coupled together by rods running across the wheel. The figure must be regarded as purely diagrammatic.

It will be seen that this governor is practically isochronous, for the load on the spring increases as the radius of the weight, and therefore, as explained in the Hartnell governor, as the centrifugal force.

The sensitiveness can be varied by altering the position of suspension, J. In order to be isochronous, the path of the weight must as nearly as possible coincide with a radial line drawn from O, and the direction of S must be parallel to this radial line.

A later form of the same governor is shown in Fig. 252, an inertia weight I is attached to the eccentric. The speeding

up is accomplished by the differential bevil gear shown. The outer pulley A is attached to the inner bevil wheel A, and the inner pulley B to the adjoining wheel; by applying a brake to one pulley the bevil wheels turn in one direction, and when the brake is applied to the other pulley the wheels turn in the opposite direction, and so tighten or slacken the springs S S.

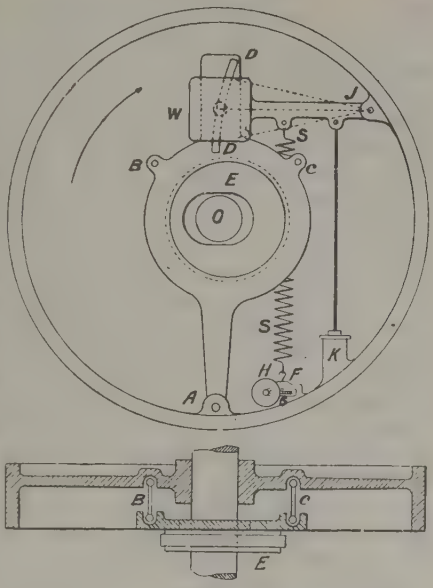


FIG. 251.

Inertia Effects on Governors. — Many governors rely

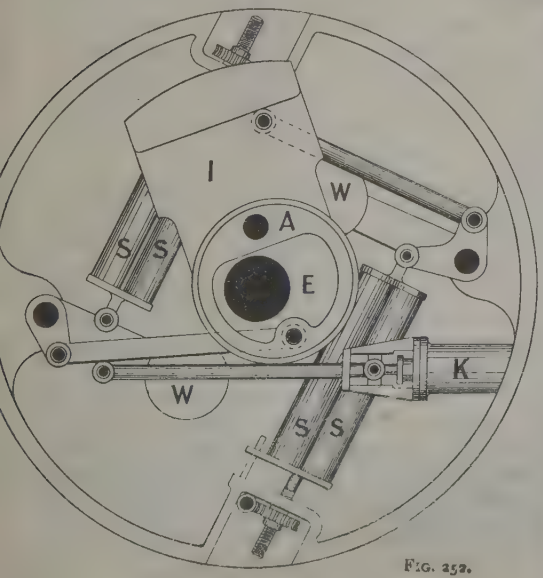
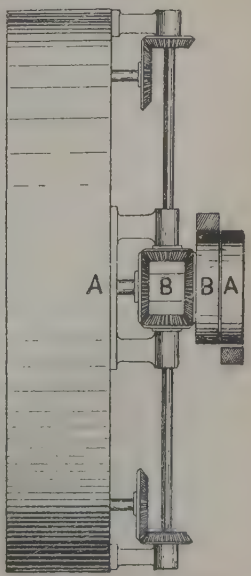


FIG. 252.



entirely on the inertia of their weights or balls for regulating the supply of steam to the engine when a change of speed occurs, while in other cases the inertia effect on the weights is so small that it is often neglected; it is, however, well when designing a governor to arrange the mechanism in such a manner that the inertia effects shall act *with* rather than against the centrifugal effects.

In all cases of governors the weights or balls tend to fly out radially under the action of the centrifugal force, but in the case of crank-shaft governors, in which the balls rotate in one plane, they are subjected to another force, acting at right angles to the centrifugal, whenever a change of speed takes place; the latter force, therefore, acts tangentially, and is due to the tangential acceleration of the weights. For convenience of expression we shall term the latter the "inertia force."

The precise effect of this inertia force on the governor entirely depends upon the sign of its moment about the point of suspension of the ball arm: if the moment of the inertia force be of the same sign as that of the centrifugal force about the pivot, the inertia effects will assist the governor in causing it to act more promptly; but if the two be of opposite sign, they will tend to neutralize one another, and will make the governor sluggish in its action.

Since the inertia of a body is the resistance it offers to having its velocity increased, it will be evident that the inertia force acts in an opposite sense to that of the rotation. In the figures and table given below we have only stated the case in which the speed of rotation is increased; when it is decreased the effect on the governor is the same as before, since the moments act together or against one another.

In the case of a governor in which the inertia moment assists the centrifugal, if the speed be suddenly increased, both the centrifugal and the inertia moments tend to make the balls fly out, and thereby to partially or wholly shut off the supply of steam,—the resulting moment is therefore the *sum* of the two, and a prompt action is secured; but if, on the other hand, the inertia moment acts against the centrifugal, the resulting moment is the *difference* of the two, and a sluggish action results. If, as is quite possible, the inertia moment were greater than the centrifugal, and of opposite sign, a sudden increase of speed would cause the governor balls to close in and to admit more steam, thus producing serious disturbances. The table given below will serve to show the effect of the two moments on the governor shown in Fig. 253.

In every case c is the centrifugal force, and T the inertia force. $T = M\frac{dv}{dt}$, or $= \frac{W}{g}\cdot\frac{dv}{dt}$, where W is the weight of the ball, and $\frac{dv}{dt}$ the acceleration in feet per second per second. For example, let the centre of gravity of a crank-shaft governor arm and weight be at a radius of 4 inches when the governor is running at 300 revolutions per minute, and let it be at a radius of 7 inches when the governor is running at 312 revolutions per minute, and let the change take place in 0·2 second. The weight of the arm and weight is 25 pounds.

The change of velocity is—

$$\frac{2 \times 3{\cdot}14}{12 \times 60}(7 \times 312 - 4 \times 300) = 8{\cdot}59 \text{ feet per second,}$$

and the acceleration $\frac{\delta V}{\delta t} = \frac{8{\cdot}59}{0{\cdot}2} = 42{\cdot}95$ feet per sec. per sec.,

and the force $T = \frac{42{\cdot}95 \times 25}{32{\cdot}2} = 33{\cdot}4$ pounds.

Sense of rotation.	Position of ball.	Centrifugal moment. A	Centrifugal moment. B	Inertia moment for an increase of speed. A	Inertia moment for an increase of speed. B	Effect of inertia on governor.
+	1	$-c_1x_1$	c_1x_1	T_1y_1	$-T_1'y_1$	Retards its action
+	2	$-c_2x_2$	c_2x_2	0	0	No effect
+	3	$-c_3x_3$	c_3x_3	$-T_3y_3$	$T_3'y_3$	Assists its action
−	1	$-c_1x_1$	c_1x_1	$-T_1y_1$	$T_1'y_1$	,, ,,
−	2	$-c_2x_2$	c_2x_2	0	0	No effect
−	3	$-c_3x_3$	c_3x_3	T_3y_3	$-T_3'y_3$	Retards its action

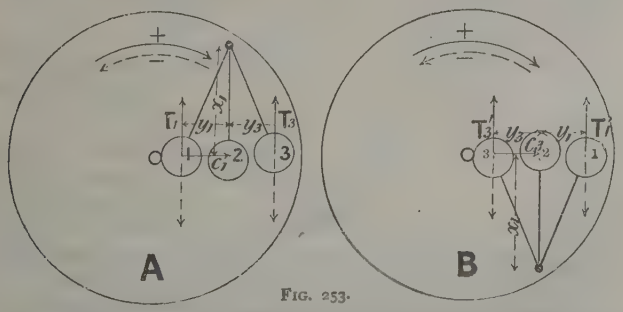


FIG. 253.

Sensitiveness of Governors.—The sensitiveness and behaviour of a governor when running can be very conveniently studied by means of a diagram showing the rate of increase of

the centrifugal and centripetal moments as the governor balls fly outwards. These diagrams are the invention of Mr. Wilson Hartnell, who first described them in a paper read before the Institute of Mechanical Engineers in 1882.

In Fig. 254 we give such a diagram for a simple Watt governor, neglecting the weight of the arms. The axis OO_1 is the axis of rotation. The ball is shown in its two extreme positions. The ball is under the action of two moments—the centrifugal moment CH and the centripetal moment W_cR, which are equal for all positions of the ball, unless the ball is being accelerated or retarded. The centrifugal moment tends to carry the ball outwards, the centripetal to bring it back. The four numbered curves show the relation between the moment tending to make the balls fly out (ordinates) and the position of the balls. The centripetal moment line shows the relation between the moment tending to bring the balls back and the position of the balls, which is independent of the speed.

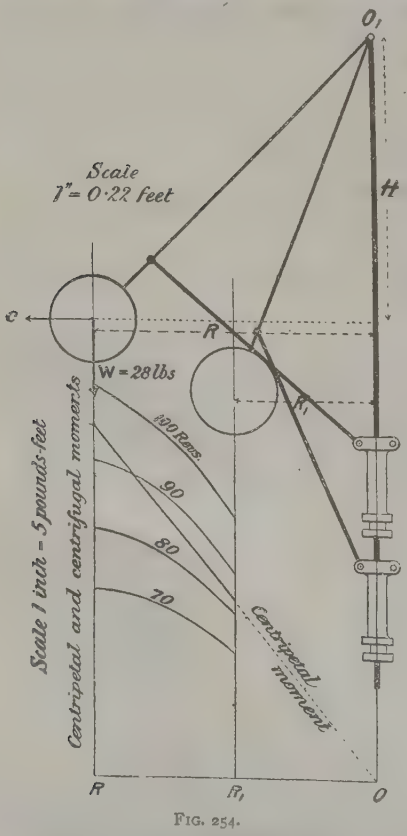


FIG. 254.

We have—

$$CH = 0{\cdot}00034WRN^2H$$
$$= 0{\cdot}00034WN^2(RH)$$
$$= KRH$$

The quantity $0{\cdot}00034WN^2$ is constant for any given ball running at any given speed. Values of KRH have been calculated for various positions and speeds, and the curves plotted.

The value of W_cR varies directly as the radius;

hence the centripetal line is straight, and passes through the origin O. From this we see that the governor begins to lift at a speed of about 82 revolutions per minute, but gets to a speed of about 94 before the governor lifts to its extreme position. Hence, if it were intended to run at a mean speed of 88 revolutions per minute, it would, if free from friction, vary about 9 per cent. on either side of the mean, and when retarded by friction it will vary to a greater extent.

For the centrifugal moment, W = weight of (ball + $\frac{4}{3}$ arm + $\frac{4}{3}$ link). The resultant acts at the centre of gravity of W. For the centripetal moment, W_c = weight of (ball + sleeve + $\frac{1}{2}$arm + $\frac{3}{2}$ link). The resultant acts at the centre of gravity of W_c, the weight of the sleeve being regarded as concentrated at the top joint of the link. It is here assumed that the sleeve rises twice as fast as the top pin of the link.

If the centrifugal and centripetal curves coincided, the governor would be isochronous. If the slope of the centrifugal curve be less than that of the centripetal, the governor is too stable; but if, on the other hand, the slope of the centrifugal curve be greater than that of the centripetal, the governor is too sensitive, for as soon as the governor begins to lift, the centrifugal moment, tending to make the balls fly out, increases more rapidly than the centripetal moment, tending to keep the balls in—consequently the balls are accelerated, and fly out to their extreme position, completely closing the governing valve, which immediately causes the engine to slow down. But as soon as this occurs, the balls close right in and fully open the governing valve, thus causing the engine to race and the balls to fly out again, and so on. This alternate racing and slowing down is known as *hunting*, and is the most common defect of governors intended to be sensitive.

It will be seen that this action cannot possibly occur with a simple Watt governor unless there is some disturbing action.

When designing a governor which is intended to regulate the speed within narrow limits, it is important to so arrange it that any given change in the speed of the engine shall be constant for any given change in the height throughout its range. Thus if a rise of 1 inch in the sleeve corresponds to a difference of 5 revolutions per minute in the speed, then each $\frac{1}{5}$th of an inch rise should produce a difference of 1 revolution per minute of the engine in whatever position the governor

may be. This condition can be much more readily realized in automatic expansion governors than in throttling governors.

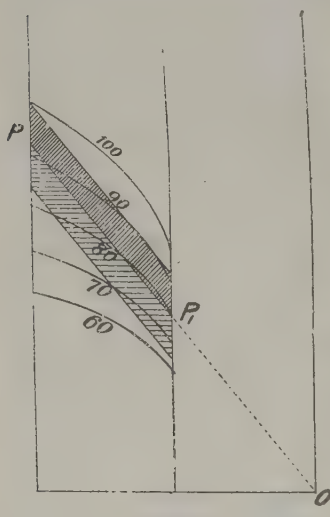


FIG. 255.

Friction of Governors.—So far, we have neglected the effect of friction on the sensitiveness, but it is in reality one of the most important factors to be considered in connection with sensitive governors. Many a governor is practically perfect on paper—friction neglected—but is to all intents and purposes useless in the material form on an engine, on account of retardation due to friction. The friction is not merely due to the pins, etc., of the governor itself, but to the moving of the governing valve or its equivalent and its connections.

In Fig. 255 we show how friction affects the sensitiveness of a governor. The vertical height of the shaded portion represents the friction moment that the governor has to overcome. Instead of the governor lifting at 80 revolutions per minute, the speed at which it should lift if there were no friction, it does not lift till the speed gets to about 92 revolutions per minute; likewise on falling, the speed falls to 64 revolutions per minute. Thus with friction the speed varies about 22 per cent. above and below the mean. Unfortunately, very little experimental data exists on the friction of governors and their attachments;[1] but a designer cannot err by doing his utmost to reduce it even to the extent of fitting all joints, etc., with ball-bearings or with knife-edges (see Fig. 250).

The effect of friction is to increase the height of the governor when it is rising, and to reduce it when falling. The exact difference in height can be calculated if the frictional

[1] See Paper by Ransome, *Proc. Inst. C.E.*, vol. cxiii.; the question was investigated some years ago by one of the author's students, Mr. Eurich, who found that when oiled the Watt governor tested lagged behind to the extent of 7·5 per cent., and when unoiled 17·5 per cent.

resistance referred to the sleeve is known; it is equivalent to increasing the weight on the sleeve when rising and reducing it when falling.

In the well-known Pickering governor, the friction of the governor itself is reduced to a minimum by mounting the balls on a number of thin band springs instead of arms moving on pins. The attachment of the spring at the c. of g. of the weight and arm, as in the McLaren governor, is a point also worthy of attention. We will now examine in detail several types of governor by the method just described.

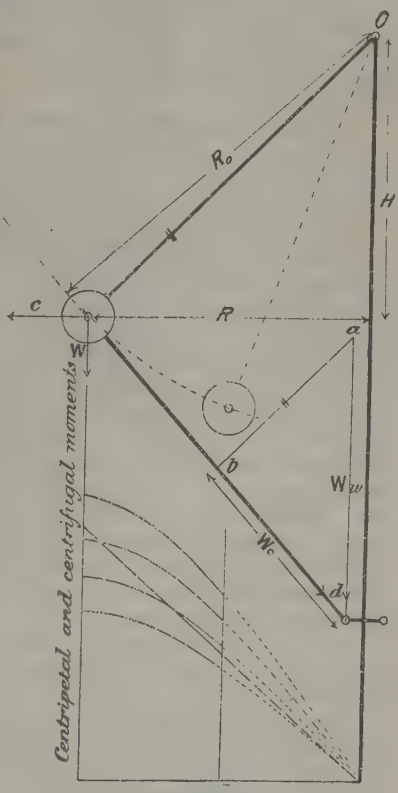


FIG. 256.

Porter Governor Diagram.—In this case the centripetal force is greatly increased while the centrifugal is unaffected by the central weight W_w, which rises twice as fast as the balls (Fig. 256) when the links are of equal length. Resolve W_w in the directions of the two arms as shown: it is evident that the component *ab*, acting along the upper arm, has no moment about O, but $bd = W_0$ has a centripetal moment, W_0R_0; then we have—

$$CH = WR + W_0R_0$$

Values of each have been calculated and plotted as in Fig. 254. In the central spring governor W_w varies as the balls lift; in other respects the construction is the same.

It should be noticed that the centripetal and centrifugal

moment curves coincide much more closely as the height of the governor increases; thus the sensitiveness increases with the height—a conclusion we have already come to by another process of reasoning.

Crossed-arm Governor Diagram.—In this governor H is constant, and as C varies directly as the radius for any given speed, it is evident that the centripetal and centrifugal lines are both straight and coincident, hence the governor is isochronous.

Wilson Hartnell Governor Diagram (Fig. 257).—When constructing the curves *a*, *b*, *c*, *d*, *e*, the moment of the weight of the ball on either side of the suspension pin, also the other disturbing causes, have been neglected.

We have shown that $cr_0 = pr$, also that c and p vary as R, hence the centrifugal moment lines (shown in full) and the centripetal moment line O*a* both pass through the origin, under these conditions the governor is isochronous. A common method of varying the speed of such governors is to alter the load on the spring by the lock nuts at the top; this has the effect of bodily shifting the centripetal moment line up or down, but it does not alter the slope, such as *db*, *ec*, both of which are parallel to O*a*. But such an alteration also affects the sensitiveness; if the centripetal line was *db*, the governor would hunt, and if *ec*, it would be too stable. These defects can, however, be remedied by altering the stiffness of the spring, by throwing more or less coils out of action by the corkscrew nut shown in section, by means of which *db* can be altered to *dc* and *ec* to *eb*. For fine governing both of these adjustments are necessary.

When the moment of the weight of the ball and other disturbing causes are taken into account, the curves *f* and *g* are obtained.

Instead of altering the spring for adjusting the speed, some makers leave a hollow space in the balls for the insertion of lead until the exact weight and speed are obtained. It is usually accomplished by making the hollow spaces on the inside edge of the ball, then the centrifugal force tends to keep the lead in position.

The sensitiveness of the Wilson Hartnell governor may also be varied at will by a simple method devised by the author some years ago, which has been successfully applied to several forms of governor. In general, if a governor tends to hunt, it can be corrected by making the centripetal moment increase more rapidly, or, if it be too sluggish, by making it increase less

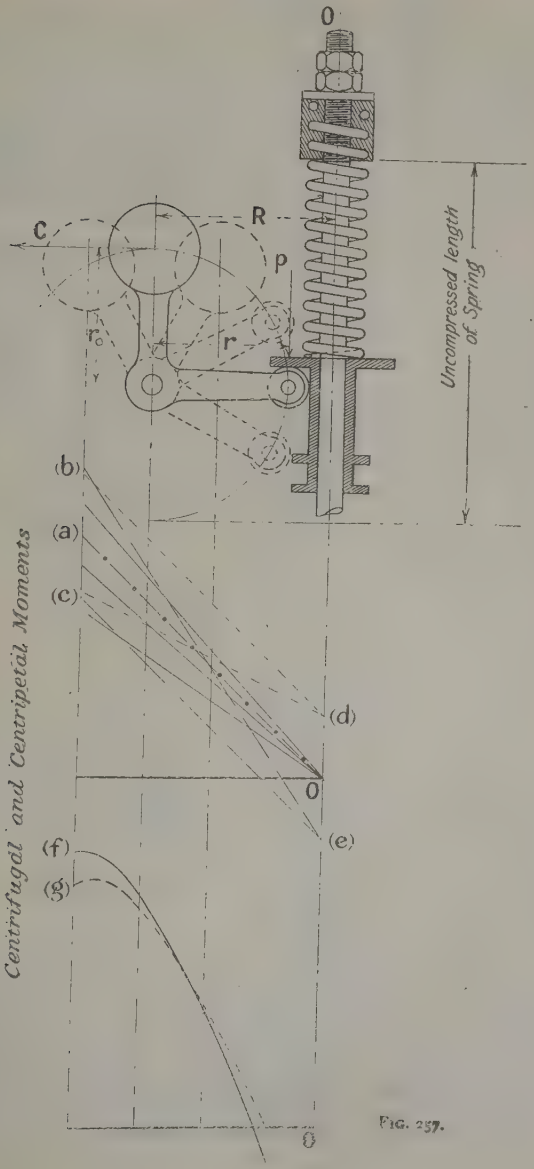

0
C
R
P
r_0
r
Uncompressed length of Spring
(b)
(a)
(c)
(d)
0
(e)
(f)
(g)
Centrifugal and Centripetal Moments
0

FIG. 257.

rapidly as the centrifugal moment of the balls increases. The governor, which is shown in Fig. 258, is of the four-ball horizontal type; it originally hunted very badly, and in order to correct it the conical washer A was fitted to the ball path, which was previously flat. It will be seen that as the balls fly out the inclined ball path causes the spring to be compressed more rapidly than if the path were flat, and consequently the rate of increase of the centripetal moment is increased, and with it the stability of the governor.

In constructing the diagram it was found convenient to make use of the virtual centre of the ball arm in each position; after finding it, the method of procedure is similar to that already given for other cases. In order to show the effect of the conical washer, a second centripetal curve is shown by a broken line for a flat plate. With the conical washer, neglecting friction, the diagram shows that the governor lifts at 430 revolutions, and reaches 490 revolutions at its extreme range; by experiment it was found that it began to lift at 440, and rose to 500, when the balls were lifting, and it began to fall at 480, getting down to 415 before the balls finally closed in. The conical washer A in this case is rather too steep for accurate governing.

The centrifugal moment at any instant is—

$$4 \times 0{\cdot}00034 W R N^2 H$$

where W is the weight of one ball.

And the centripetal moment is—

$$\text{Load on spring} \times R_s$$

See Fig. 258 for the meaning of R_s, viz. the distance of the virtual centre from the point of suspension of the arm.

Taking position 4, we have for the centrifugal moment at 450 revolutions per minute—

$$4 \times 0{\cdot}00034 \times 2{\cdot}5 \times \tfrac{2{\cdot}88}{12} \times 450^2 \times 1{\cdot}58 = 260 \text{ pound-inches}$$

and for the centripetal moment—

The load on the spring = 111 lbs.; and $R_s = 2{\cdot}78$

centripetal moment = $111 \times 2{\cdot}78 = 310$ pound-inches

McLaren's Crank-shaft Governor.—In this governor we have $CR_0 = SR_s$; but C varies as R, hence if there be no tension on the spring when R is zero, it will be evident that S will vary directly as R; but C also varies in the same manner, hence the centrifugal and centripetal moment lines are nearly

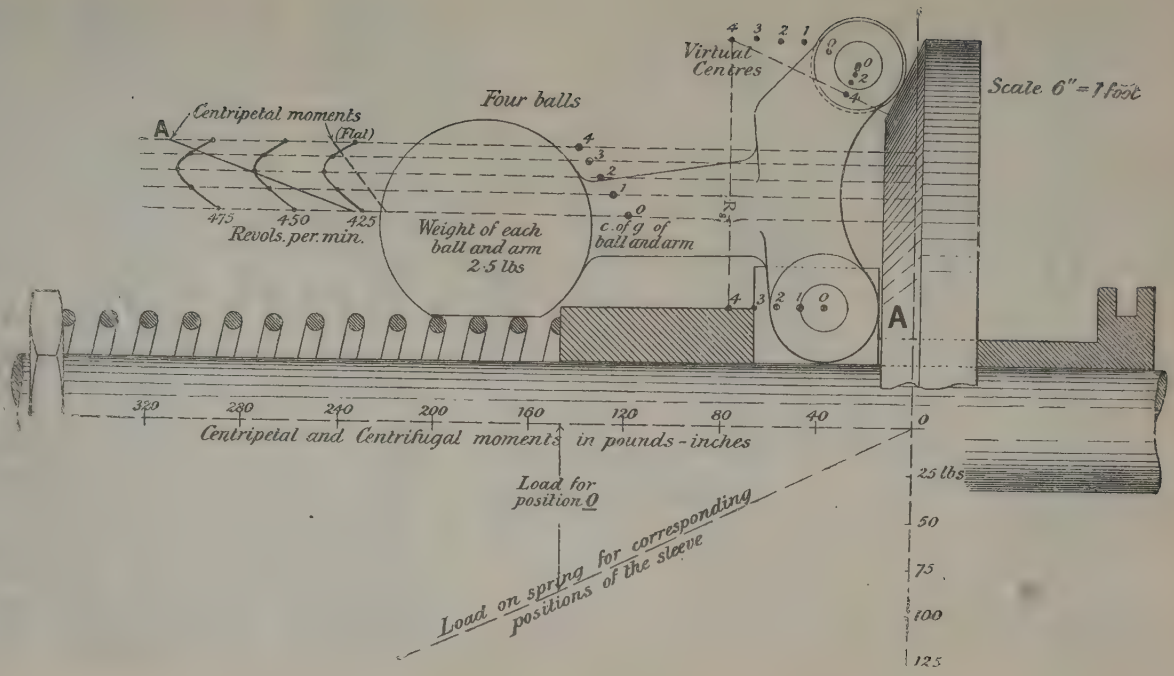

Four balls
Centripetal moments
(Flat)
A
475
450
425
Revols. per min.
Weight of each ball and arm 2·5 lbs
c. of g of ball and arm
R
Virtual Centres
Scale 6" = 1 foot
A
320
280
240
200
160
120
80
40
0
Centripetal and Centrifugal moments in pounds-inches
Load for position 0
25 lbs
50
75
100
125
Load on spring for corresponding positions of the sleeve

FIG. 258.

straight and coincident. The centrifugal lines are not absolutely straight, because the weight does not move exactly on a radial line from the centre of the crank-shaft.

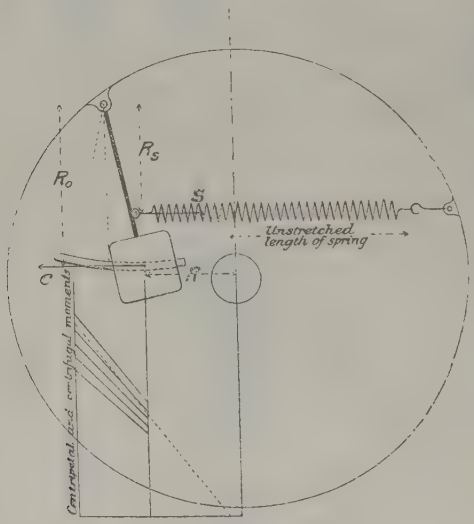


FIG. 259.

Governor Dashpots.—A dashpot consists essentially of a cylinder with a leaky piston, around which oil, air, or other fluid has to leak. An extremely small force will move the piston slowly, but very great resistance is offered by the fluid if a rapid movement be attempted.

Very sensitive governors are therefore generally fitted with dashpots, to prevent them from suddenly flying in or out, and thus causing the engine to hunt.

If a governor be required to work over a very wide range of power, such as all the load suddenly thrown off, a sensitive, almost isochronous governor with dashpot gives the best result; but if very fine governing be required over small variations of load, a slightly less sensitive governor without a dashpot will be the best.

However good a governor may be, it cannot possibly govern well unless the engine be provided with sufficient fly-wheel power. If an engine have, say, a 2-per-cent. cyclical

variation and a very sensitive governor, the balls will be constantly fluctuating in and out during every stroke.

Power of Governors.—The "power" of a governor is its capacity for overcoming external resistances. The greater the power, the greater the external resistance it will overcome with a given alteration in speed.

Nearly all governor failures are due to their lack of power.

The useful energy stored in a governor is readily found thus, approximately:—

Simple Watt governor, crossed-arm and others of a similar type—

$$\text{Energy} = \text{weight of both balls} \times \text{vertical rise of balls}$$

Porter and other loaded governors—

$$\text{Energy} = \text{weight of both balls} \times \text{vertical rise of balls} + \text{weight of central weight} \times \text{its vertical rise}$$

Spring governors—

$$\text{Energy} = \text{weight of both balls} \times \text{vertical rise (if any) of balls} + n\left(\frac{\text{max. load on spring} + \text{min. load on spring}}{2}\right) \times \text{the stretch or compression of spring}$$

where n = the number of springs employed; express weights in pounds, and distances in feet.

The following may be taken as a rough guide as to the energy that should be stored in a governor to get good results: it is always better to store too much rather than too little energy in a governor:—

Type of governor.	Foot-pounds of energy stored per inch diameter of cylinder.
For trip gears and where small resistances have to be overcome	0·5–0·75
For fairly well balanced throttle-valves	0·75–1

In the earlier editions of this book values were given for automatic expansion gears, which were based on the only data available to the author at the time; but since collecting a considerable amount of information, he fears that no definite values can be given in this form. For example, in the case of governors acting through reversible mechanisms on well-balanced slide-valves, about 100 foot-pounds of energy per inch diameter of the high-pressure cylinder is found to give good results; but in other cases, with unbalanced slide-valves, five

times that amount of energy stored is found to be insufficient. If the driving mechanism of the governor be non-reversible, only about one-half of this amount of energy will be required.

A better method of dealing with this question is to calculate, by such diagrams as those given in the "Mechanisms" chapter, the actual effort that the governor is capable of exerting on the valve rod, and ensuring that this effort shall be greatly in excess of that required to drive the slide-valve. Experiments show that the latter amounts to about one-fifth to one-sixth of the total pressure on the back of a slide-valve (*i.e.* the whole area of the back × the steam pressure) in the case of unbalanced valves. The effort a governor is capable of exerting can also be arrived at approximately by finding the energy stored in the springs, and dividing it by the distance the slide-valve moves while the springs move through their extreme range.

Generally speaking, it is better to so design the governor that the valve-gear cannot react upon it, then no amount of pressure on the valve-gear will alter the height of the governor; that is to say, the reversed efficiency of the mechanism which alters the cut-off must be negative, or the efficiency of the mechanism must be less than 50 per cent. On referring to the McLaren governor, it will be seen that no amount of pressure on the eccentric will cause the main weight W to move in or out.

Readers who wish to go more fully into the question of governors will find detailed information in "Governors and Governing Mechanism," by H. R. Hall, The Technical Publishing Co., Manchester; "Dynamics of Machinery," by Lanza, Chapman and Hall; "Shaft Governors," Trinks and Housum, Van Nostrand & Co., New York.

CHAPTER VII.

VIBRATION.

Simple Harmonic Motion (S.H.M.).—When a crank rotates at a uniform velocity, a slotted cross-head, such as is shown in Fig. 160, moves to and fro with simple harmonic motion. In Chapter VI. it is shown that the force required to accelerate the cross-head is

$$P_1 = \frac{WV^2}{gR} \times \frac{x}{R} \quad \ldots \ldots \quad (i)$$

where W = weight of the cross-head in pounds.
V = velocity of the crank-pin in feet per sec.
R = radius of the crank in feet.
x = displacement of cross-head in feet from the central position.
g = acceleration of gravity.
K = radius of gyration in feet.

Then the acceleration of the crosshead in this position

$$= \frac{P_1 g}{W} = \frac{V^2 x}{R^2} = \frac{V^2}{R^2} \times \left\{ \begin{array}{l} \text{its displacement from the middle} \\ \text{of the stroke,} \end{array} \right.$$

or $$\frac{V}{R} = \sqrt{\frac{\text{acceleration}}{\text{displacement}}} = \sqrt{\frac{P_1 g}{Wx}} \quad \ldots \ldots \quad (ii)$$

These expressions show that the acceleration of the cross-head is proportional to the displacement x, and since the force tending to make it slide is zero at the middle of the stroke and varies directly as x, it is clear that the direction of the acceleration is always towards the centre O.

The time t taken by the cross-head in making one complete journey *to and fro* is the same as that taken by the crank in making one complete revolution.

$$\text{Hence } t = \frac{2\pi R}{V} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

$$t = 2\pi\sqrt{\frac{Wx}{P_1 g}} \quad \ldots\ldots\ldots \quad \text{(iii)}$$

On referring to the argument leading up to expression (i) in Chapter VI. it will be seen that P_1 is the force acting along the centre line of the cross-head when it is displaced an amount x from its middle or zero-force position. When dealing with the vibration of elastic bodies W is the weight of the vibrating body in pounds, and P_1 is the force in pounds weight required to strain the body through a distance x feet.

When dealing with angular oscillations we substitute thus—

Angular.	Linear.
The moment of inertia of the body $I = \frac{WK^2}{g}$ in pound-foot units.	Mass of the body $\left(\frac{W}{g}\right)$, where W is in pounds.
The couple acting on the body $C = IA$ in pound-foot units.	Force acting on the body $P_1 = \frac{W}{g}f$ in pounds.
The angular displacement (θ) in radians.	Linear displacement (x) in feet.
The angular velocity (ω) in radians per second.	The linear velocity (v) in feet per second.
The angular acceleration (A) in radians per second per second.	The linear acceleration (f) in feet per second per second.
Kinetic energy $\frac{1}{2} I\omega^2$	Kinetic energy $\frac{1}{2} \frac{W}{g} v^2$

Hence, when a body is making angular oscillations under the influence of a couple which varies as the angular displacement, the time of a complete oscillation is—

$$t = 2\pi\sqrt{\frac{I\theta}{C}} \quad \ldots\ldots\ldots \quad \text{(iv)}$$

These expressions enable a large number of vibration problems to be readily solved.

Simple Pendulum.—In the case of a simple pendulum the weight of the suspension wire is regarded as negligible when compared with the weight of the bob, and the displacement x is small compared to l. The tension in the wire is normal to the path, hence the only accelerating force is the component of W, the weight of the bob, tangential to the path,

hence $P_1 = W \sin\theta = \frac{Wx}{l}$

$$t = 2\pi\sqrt{\frac{\overline{Wxl}}{\overline{Wxg}}} = 2\pi\sqrt{\frac{\overline{l}}{g}} \quad . \quad . \quad . \quad (v)$$

FIG. 260.

where l is the length of the pendulum in feet measured from the point of suspension to the c. of g. of the bob; t is the time in seconds taken by the pendulum in making one complete oscillation through a small arc.

The above expression may be obtained direct from (iv) by substituting $\frac{Wl^2}{g}$ for I, and Wx or $Wl\theta$ for C. Then

$$t = 2\pi\sqrt{\frac{\overline{Wl^2\theta}}{\overline{Wl\theta g}}} = 2\pi\sqrt{\frac{\overline{l}}{g}}$$

Compound Pendulum.—When the weight of the arm is not a negligible quantity, the pendulum is termed compound.

Let W be the weight of the bob and arm, and l_g be the distance of their c. of g. from the point of suspension. Then if x be a small horizontal displacement of the c. of g.

$C = Wx = Wl_g\theta$ (nearly), and

$$t = 2\pi\sqrt{\frac{\overline{I\theta}}{\overline{Wl_g\theta}}} = 2\pi\sqrt{\frac{\overline{I}}{\overline{Wl_g}}} \quad . \quad (vi)$$

$$= 2\pi\sqrt{\frac{\overline{K_0^2}}{l_g g}}$$

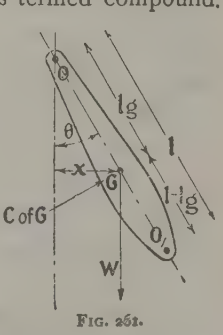


FIG. 261.

where K_0 is the radius of gyration of the body about the point of suspension. If K be the radius of gyration of the body about an axis through the centre of gravity, then (see p. 76)

$$I = \frac{WK^2}{g} + \frac{Wl_g^2}{g} = \frac{W}{g}(K^2 + l_g^2)$$

and $$t = 2\pi\sqrt{\frac{K^2 + l_g^2}{l_g g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii)}$$

If l be the length of a simple pendulum which has the same period of oscillation, then

$$2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{K^2 + l_g^2}{l_g g}}$$

$$l = \frac{K^2 + l_g^2}{l_g} \text{ and } l_g(l - l_g) = K^2$$

or $OG.GO_1 = K^2$, and since the position of G will be the same whether the point of suspension be O or O_1 the time of oscillation will be the same for each.

Oscillation of Springs.—Let a weight W be suspended from a helical spring as shown, and let it stretch an amount δ (inches) when supporting the load W, where $\delta = \frac{8D^3Wn}{Gd^4}$ (see page 587).

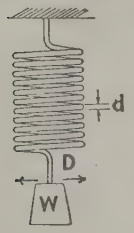


FIG. 262.

D is the mean diameter of the coils in inches
d ,, diameter of the wire in inches
n ,, number of free coils
G ,, modulus of rigidity in pounds per square inch
W ,, weight in pounds,

Then, neglecting the weight of the spring itself, we have, for the time of one complete oscillation of the spring (from iii)

$$t = 2\pi\sqrt{\frac{Wx}{P_1 g}} = 2\pi\sqrt{\frac{W\delta}{12Wg}} = 2\pi\sqrt{\frac{D^3Wn}{1{\cdot}5gGd^4}}$$

$$t = 0{\cdot}904\sqrt{\frac{D^3Wn}{Gd^4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(viii)}$$

$t = 2\pi\sqrt{\frac{\Delta}{g}}$, where Δ is the deflection in feet. For a simple pendulum $t = 2\pi\sqrt{\frac{l}{g}}$, hence the time of one complete oscillation of a spring is the same as that of a simple pendulum

whose length is equal to the static deflection of the spring due to the weight W.

The weight of the spring itself does not usually affect the problem to any material extent, but it can be taken into account thus: the coil at the free end of the spring oscillates at the same velocity as the weight, but the remainder of the coils move at a velocity proportional to their distance from the free end.

Let the weight per cubic inch of the spring $= w$, the area of the section $= a$. Consider a short length of spring dl, distant l from the fixed end of the spring, L being the total length of the wire in the spring.

$$\text{Weight of short length} = wa \,.\, dl$$

$$\text{Velocity of element} = \frac{\mathrm{V}l}{\mathrm{L}}$$

$$\text{Kinetic energy of element} = \frac{w \,.\, a \,.\, \mathrm{V}^2}{2g\mathrm{L}^2} l^2 dl$$

$$\text{Kinetic energy of spring} = \frac{wa\mathrm{V}^2}{2g\mathrm{L}^2}\int_{l=0}^{l=\mathrm{L}} l^2 dl = \frac{wa\mathrm{V}^2\mathrm{L}^3}{3 \times 2g\mathrm{L}^2}$$

$$= \frac{wa\mathrm{V}^2\mathrm{L}}{3 \times 2g} = \frac{1}{3}\left(\frac{\mathrm{W}_s\mathrm{V}^2}{2g}\right)$$

where W_s is the weight of the spring. Thus the kinetic energy stored in the spring at any instant is equal to that stored in an oscillating body of one-third the weight of the spring. Thus the W in the expression for the time of vibration should include one-third the weight of the spring in addition to that of the weight itself.

Expression (viii.) now becomes—

$$t = 2\pi\sqrt{\frac{\left(\mathrm{W} + \frac{\mathrm{W}_s}{3}\right)x}{\mathrm{P}_1 g}} = 0{\cdot}904\sqrt{\frac{\mathrm{D}^3\left(\mathrm{W} + \frac{\mathrm{W}_s}{3}\right)n}{\mathrm{G}d^4}}$$

Example.—D = 2 inches $\quad d = 0{\cdot}2$ inch
W = 80 pounds $\quad n = 8$
G = 12,000,000 pounds per sq. inch.

Find the time of one complete oscillation, (*a*) neglecting the weight of the spring, (*b*) taking it into consideration.

$$t = 0{\cdot}904\sqrt{\frac{2^3 \times 80 \times 8}{12{,}000{,}000 \times 0{\cdot}0016}} = 0{\cdot}904\sqrt{0{\cdot}267}$$
$$= 0{\cdot}4668 \text{ sec., say } 0{\cdot}47 \text{ second.}$$

In some instances the deflection x is given in terms of the force P_1; if the deflection δ be worked out by the expression given on page 587, it will be found to be $1{\cdot}33''$ or $0{\cdot}11$ foot for every 50 lbs. Then—

$$t = 2\pi\sqrt{\frac{80 \times 0{\cdot}11}{50 \times 32{\cdot}2}} = 0{\cdot}47 \text{ sec.}$$

The weight of the spring $= 0{\cdot}69nd^2D$
$= 0{\cdot}69 \times 8 \times 0{\cdot}04 \times 2$
$Ws = 0{\cdot}44$ lb.

Then allowing for the weight of the spring—

$$t = 0{\cdot}904\sqrt{\frac{8 \times 80{\cdot}15 \times 8}{12{,}000{,}000 \times 0{\cdot}0016}} = 0{\cdot}4674 \text{ second}$$

say $0{\cdot}47$ second.

Thus it is only when extreme accuracy is required that the weight of the spring need be taken into account.

In the case of a weight W pounds on the end of a cantilever L inches long, the time of one complete oscillation is found by inserting the value for δ (see p. 587) in equation (iii), which gives—

$$t = 2\pi\sqrt{\frac{WL^3}{36EIg}} = 0{\cdot}185\sqrt{\frac{WL^3}{EI}}$$

The value of δ can be inserted in the general equation for any cases that may arise.

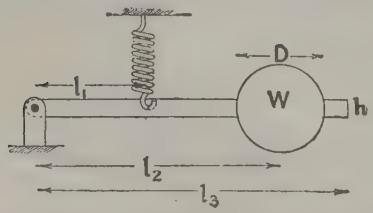


FIG. 263.

Oscillation of Spring-controlled Governor Arms.—In the design of governors it is often of importance to know the period of oscillation of the governor arms. This is a case of angular oscillation, the time of which has already been given in equation (iv), viz.

$$t = 2\pi\sqrt{\frac{I\theta}{C}}$$

The I is the moment of inertia of the weight and arm about the pivot in foot and pound units, and is obtained thus (see page 76)

$$I = \frac{W}{g}(K^2 + l_2^2) + \frac{w}{g}\left\{\frac{(l_3^2 + h^2)}{12} + \frac{l_2^2}{4}\right\}$$

where K is the radius of gyration of the weight about an axis parallel to the pivot and passing through the centre of gravity. For a cylindrical weight $K^2 = \frac{D^2}{8}$, and for a spherical weight $K^2 = \frac{D^2}{10}$. The weight of the arm, which is assumed to be of uniform section is *w*. More complex forms of arms and weights must be dealt with by the graphic methods given on page 101.

Example.—The cylindrical weight W is 48 pounds, and is 5 inches diameter. The weight of the arm = 12 pounds, $l_3 = 18$ inches, $l_2 = 12$ inches, $l_1 = 8$ inches, $h = 2$ inches. The spring stretches 0·3 inch per 100 pounds. If the dimensions be kept in inch units, the value for the moment of inertia must be divided by 144 to bring it to foot units.

$$I = \frac{48}{32{\cdot}2 \times 144}\left(\frac{25}{8} + 144\right) + \frac{12}{32{\cdot}2 \times 144}\left\{\left(\frac{324 + 4}{12}\right) + \frac{324}{4}\right\}$$

$= 1{\cdot}8$ pounds-feet units.

$$\theta = \frac{0{\cdot}3}{8} \text{ when } C = \frac{100 \times 8}{12}$$

$$\text{hence } t = 2 \times 3{\cdot}14\sqrt{\frac{1\,8 \times 0{\cdot}3 \times 12}{8 \times 100 \times 8}} = 0{\cdot}2 \text{ sec.}$$

Vertical Tension Rod.—For a rod of length *l* supported at top with a weight W at the lower end. Inserting the values $x = \frac{fl}{E}$ and $P_1 = Af$ we get $t = 2\pi\sqrt{\frac{Wl}{EAg}}$

Torsional Oscillations of Shafts.—If an elastic shaft be held at one end, and a body be attached at the other as shown, it will oscillate with simple harmonic motion if it be turned through a small arc, and then released. The time of one complete oscillation has already been shown (iv) to be

$$t = 2\pi\sqrt{\frac{I\theta}{C}}$$

on page 579 it is shown that

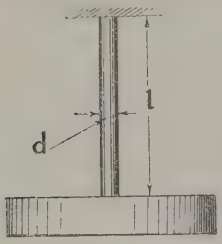


FIG. 264.

$$\theta = \frac{M_t l}{GI_p}$$

radians, where M_t is the twisting moment in pounds inches, and I_p is the polar moment of inertia of the shaft section in inch^3 units. G is the modulus of rigidity in pounds per square inch, l is the length in inches.

Substituting this value, we get—

$$t = 2\pi\sqrt{\frac{IM_t l}{CGI_p}} = 2\pi\sqrt{\frac{12IM_t l}{M_t GI_p}} = 2\pi\sqrt{\frac{12Il}{GI_p}}$$

NOTE.—C is in pounds-feet, and M_t in pounds-inches units. Writing L for the length in feet, we get—

$$t = 2\pi\sqrt{\frac{144IL}{GI_p}} = 2\pi\sqrt{\frac{I_t L}{GI_p}}$$

where I_t is the moment of inertia of the oscillating body in pound and inch units. If all the quantities be expressed in inch units, and the constants be reduced, we get

$$t = 1{\cdot}81\sqrt{\frac{I_t l}{GI_p}}$$

If greater accuracy is required, one-third of the polar moment of inertia of the rod should be added to I_t, but in general it is quite a negligible quantity.

If there are two rotating bodies or wheels on a shaft, there will be a node somewhere between them, and the time of a complete oscillation will be

$$2\pi\sqrt{\frac{I_1 l_1}{GI_p}} = 2\pi\sqrt{\frac{I_2 l_2}{GI_p}}$$

$$\sqrt{I_1 l_1} = \sqrt{I_2 l_2}$$

and

$$\frac{l_1}{l_2} = \frac{I_2}{I_1}$$

It not infrequently happens in practice that engine crank

shafts fracture although the torsional stresses calculated from static conditions are well within safe limits. On looking more closely into the matter, it may often be found that the frequency of the crank effort fluctuations agrees very nearly with the frequency of torsional vibration of the shaft, and when this is the case the strain energy of the system may be so great as to cause fracture. An investigation of this character shows why some engine crank shafts are much more liable to fracture when running at certain speeds if fitted with two flywheels, one at each end of the shaft, than if fitted with one wheel of approximately twice the weight and moment of inertia. In both cases the effective length l is roughly one half the distance between the two wheels, or it may be the distance from the centre of the crank pin to the flywheel; but the moment of inertia I in the one case is only one half as great as in the other, and consequently the natural period of torsional vibration with the two flywheels is only $\frac{1}{\sqrt{2}} = 0{\cdot}7$ of the period with the one flywheel. If the lower speed happens to synchronise, or nearly so, with the crank effort fluctuations, fracture is liable to occur, which would not happen with the higher period of torsional vibration.

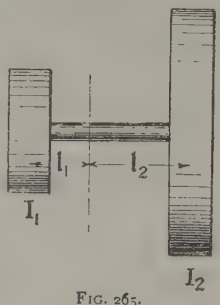


FIG. 265.

Vibration and Whirling of Shafts.—If the experiment be made of gradually increasing the speed of rotation of a long, thin horizontal shaft, freely supported at each end, it will run true up to a certain speed, apart from a little wobbling at first due to the shaft being not quite true, and will then quite suddenly start to vibrate violently, and will whirl into a single bow with a node at each bearing. As the speed increases the shaft straightens and becomes quite rigid till a much higher speed is reached, when it suddenly whirls into a double bow with a node in the middle. It afterwards straightens and whirls into three bows, and so on. High speed shafts in practice are liable to behave in this way, and may cause serious disasters, hence it is of great importance to avoid running at or near the speed at which whirling is likely to occur. An exact solution of many of the cases which occur in practice is a very complex matter (see a paper by Dunkerley, *Phil. Trans.*, Vol. 185), but the following approximate

treatment for certain simple cases gives results sufficiently accurate for practical purposes.

The energy stored at any instant in a vibrating body consists partly of kinetic energy and partly of potential or strain energy, the total energy at any instant being constant, but the relation between the two kinds of energy changes at every instant. In the case of a simple pendulum the whole energy stored in the bob is potential when it is at its extreme position, but it is wholly kinetic in its central position, and in intermediate positions it is partly potential and partly kinetic.

If a periodic force of the same frequency act on the pendulum bob it will increase the energy at each application, and unless there are other disturbing causes the energy will continue to increase until some disaster occurs. In an elastic vibrating structure the energy increases until the elastic limit is passed, or possibly even until rupture takes place. When a horizontal shaft rests on its bearings the weight of the shaft sets up bending stresses, consequently some of the fibres are subjected to tension and some to compression, depending upon whether they are above or below the neutral axis, hence when the shaft rotates each fibre is alternately brought into tension and compression at every revolution. The same distribution and intensity of stress can be produced in a stationary shaft by causing it to vibrate laterally. Hence if a shaft revolve at such a speed that the period of rotation of the shaft agrees with the natural period of vibration, the energy will be increased at each revolution, and if this particular speed be persistent the bending of the shaft will continue to increase until the elastic limit is reached, or the shaft collapses. The speed at which this occurs is known as the whirling speed of the shaft. The problem of calculating this speed thus resolves itself into finding the natural period of lateral vibration of the shaft, if this be expressed in vibrations per minute, the same number gives the revolutions per minute at which whirling occurs.

The fundamental equation is the one already arrived at, viz.—

$$t = 2\pi\sqrt{\frac{\overline{W}x}{gP_1}}$$

For the present purpose it is more convenient to use revolutions per minute N rather than the time in seconds of one vibration, also to express the deflection of the shaft δ in inches.

$$N = \frac{60}{t} = \frac{60}{2\pi}\sqrt{\frac{12gP_1}{W\delta}}$$

In this case the displacing force P_1 is the load W which tends to bend the shaft, and x is the deflection. Reducing the constants we get—

$$N = \frac{187}{\sqrt{\delta}}$$

In the expression for the deflection of a beam it is usual to take the length (l) in inches, but in Professor Dunkerley's classical papers on the whirling of shafts he takes the length L in feet, and for the sake of comparing our approximations with his exact values, the length in the following expressions is taken in feet. The diameter of the shaft (d) is in inches. The weight per cubic inch = 0·28 lb. E is taken at 30,000,000 lbs. per square inch. The value of I is $\frac{\pi d^4}{64}$. For the values of δ see Chapter XIII.

Parallel Shaft Supported freely at each end.—

$$\delta = \frac{5wl^4}{384EI} = \frac{270wL^4}{EI} = \frac{L^4}{25000d^2}$$

where $w = \frac{0{\cdot}28\pi d^2}{4}$ pounds

$$N = \frac{187}{\sqrt{\frac{L^4}{25000d^2}}} = \frac{29590d}{L^2} \quad \left\{\begin{array}{l}\text{First critical speed} \\ \text{—single bow}\end{array}\right.$$

(Dunkerley gets $\frac{32864d}{L^2}$)

At the second critical speed the virtual length is $\frac{L}{2}$, hence

$$N_2 = \frac{29590d}{\left(\frac{L}{2}\right)^2} = \frac{118400d}{L^2} \text{ double bow}$$

and $$N_3 = \frac{29590d}{\left(\frac{L}{3}\right)^2} = \frac{266300d}{L^2} \text{ treble bow.}$$

The following experimental results by the author will show to what extent the calculations may be trusted. Steel shaft—short bearings, d = 0·372 inch, L = 5·44 feet.

Condition.	Whirling speed R. p.m.	
	Calculated.	Experiment.
Single bow	370	350 to 400
Double bow . . .	1480	1460 to 1550
Treble bow. . . .	3340	3200 to 3500

It is not always easy to determine the exact speed at which whirling occurs, the experimental results show to what degree of accuracy the speed can be determined with an ordinary speed indicator.

Parallel Shaft, supported freely at each end, central load W.—

$$\delta = \frac{Wl^3}{48EI} = \frac{36WL^3}{EI}$$

$$N = \frac{377000d^2}{\sqrt{WL^3}} \qquad \left(\text{Dunkerley gets } \frac{373500d^2}{\sqrt{WL^3}}\right)$$

Experimental results—$d = 0{\cdot}995$ inch, $L = 3{\cdot}17$ feet

Condition.	Whirling speed R. p.m.	
	Calculated.	Experiment.
Unloaded	2940	3100
W = 33 pounds . .	1080	1100

The weight of the shaft itself may be taken into account thus—the deflection coefficient for the weight of the shaft is $\frac{5}{384}$ = say $\frac{1}{77}$, and for the added load, $\frac{1}{48}$, hence $\frac{48}{77}$ of the weight of the shaft should be added to W, in general the weight of the shaft is a negligible quantity.

Tapered Shaft, supported freely at each end, central load.—If the taper varies in such a manner that $\frac{M}{I}$ is constant except close to the ends where it must be enlarged on account of the shear, but such a departure from the

assumed conditions does not appreciably affect the deflection, then since $\rho = \frac{EI}{M} = \text{constant}$, the beam bends to an arc of a circle, and—

$$\delta = \frac{Wl^3}{32EI} = \frac{54WL^3}{EI}$$

$$N = \frac{31050d^2}{\sqrt{WL^3}}$$

Note.—The d is the diameter at the largest section.

If the taper varies so that the stress is constant, then $\frac{M}{Z}$ is constant, and we have—

$$\delta = \frac{Wl^3}{26{\cdot}5EI} = \frac{65WL^3}{EI}$$

$$N = \frac{28100d^2}{\sqrt{WL^3}}$$

Shafts usually conform roughly to one of the above conditions but it is improbable that the taper will be such as to give a lower whirling speed than the latter value. For calculating a shaft of any profile the following method is convenient.

In Chapter XIII., we show that

$$\delta = \frac{1}{EI}\int_0^{\frac{l}{2}} m \, x \, . \, dx \text{ for a parallel shaft.}$$

In the case of a tapered shaft this becomes

$$\delta_t = \frac{1}{EI}\sum_0^{\frac{l}{2}}\left(\frac{m_1x_1I}{I_1} + \frac{m_2x_2I}{I_2} + \text{etc.}\right)\delta x$$

$$\frac{\delta}{\delta^t} = \frac{\int mx\delta x}{\sum\left(\frac{m_1x_1I}{I_1} + \text{etc.}\right)\delta x}$$

$$= \frac{\text{Moment of the area of half the bending moment diagram}}{\text{Moment of the area of half the b.m. diagram after altering its depth at each section in the ratio of the moment of inertia in the middle to the moment of inertia at that section.}}$$

$$\delta_t = \frac{Wl^3}{48EI} \times \frac{\text{area } efgh \times l_c}{\text{area } chg \times \frac{l}{3}} \quad \text{(Fig. 266)}$$

where $\frac{ac}{ab} = \frac{I}{I_1} = \frac{D^4}{d^4}$ a similar correction being made at every section.

The I in the above expressions is for the largest section, viz. $\frac{\pi D}{64}$.

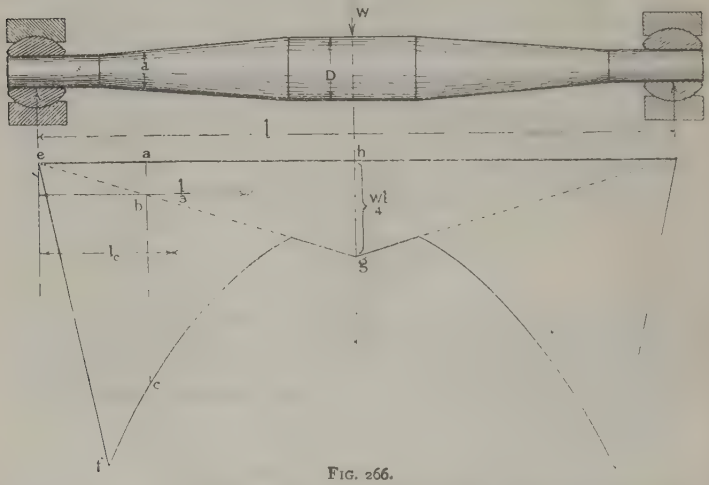


FIG. 266.

For some purposes the following method is more convenient. Let the bending moment diagram *cgh* be divided into a number of vertical strips of equal width, it will be convenient to take 10 strips, each $\frac{l}{20}$ wide. Start numbering from the support (see page 508).

Mean bending moment at the middle of the	The distance of each strip from the support.
1st strip $= \frac{W}{2} \times \frac{l}{40}$	$\frac{l}{40}$
2nd ,, $= \frac{W}{2} \times \frac{3l}{40}$	$\frac{3l}{40}$
3rd ,, $= \frac{W}{2} \times \frac{5l}{40}$	$\frac{5l}{40}$
and so on.	and so on.

The respective moments of inertia are I_1, I_2, I_3, etc., that at the middle of the shaft being I.
Then

$$\delta = \frac{1}{EI}\left(\frac{WlI}{80I_1} \times \frac{l}{20} \times \frac{l}{40} + \frac{3WlI}{80I_2} \times \frac{l}{20} \times \frac{3l}{40} + \frac{5WlI}{80I_3} \times \frac{l}{20} \times \frac{5l}{40} + \text{etc.}\right)$$

$$\delta = \frac{Wl^3}{64000E}\left(\frac{1}{I_1} + \frac{9}{I_2} + \frac{25}{I_3} + \frac{49}{I_4} + \frac{81}{I_5} + \frac{121}{I_6} + \frac{169}{I_7} + \frac{225}{I_8} + \frac{289}{I_9} + \frac{361}{I_{10}}\right)$$

$$\delta = \frac{Wl^3}{3142E}\left(\frac{1}{d_1^4} + \frac{9}{d_2^4} + \frac{25}{d_3^4} + \frac{49}{d_4^4} + \text{etc.} + \frac{361}{d_{10}^4}\right)$$

$$\delta = \frac{WL^3}{1{\cdot}82E}\left(\frac{1}{d_1^4} + \text{etc.} + \frac{361}{d_{10}^4}\right)$$

Experimental Results.—$d = 0{\cdot}995$ inch, parallel in middle, tapering to 0·5 inch diameter at ends. $L = 3{\cdot}17$ feet.

	Whirling speed R. p.m.	
	Calculated.	Experiment.
Parallel for 4 ins. in middle, unloaded .	2606	2750
Ditto for 1 in. . . .	2320	2425 to 2500
Ditto for 4 ins. Loaded, W = 33 lbs.	900	900 to 910
Ditto for 1 in. Loaded, W = 33 lbs.	800	750 to 770

When the middle of a shaft is *rigidly* gripped by a wheel boss, or its equivalent, of length l_b, the virtual length of shaft for deflection and whirling purposes is $l - l_b$ instead of l.

If the framing on which the shaft bearings are mounted is

not stiff, the natural period of vibration of the frame must be considered. An insufficiently rigid frame may cause the shaft to whirl at speeds far below those calculated.

Vertical shafts, if perfectly balanced, do not whirl, but if thrown even a small amount out of balance they will whirl at the same speeds as horizontal shafts of similar dimensions.

The following table gives a brief summary of the results obtained by Dunkerley. See "The Whirling and Vibration of Shafts" read before the Liverpool Engineering Society, Session 1894–5, also by the same author in "The Transactions of the Royal Society," Vol. 185A, p. 299.

N = the number of revolutions per minute of the shaft when whirling takes place
d = the diameter of the shaft in inches
L = the length of the shaft in feet

Description of shaft.	Remarks on *l*.	N.	L.
Unloaded: overhanging from a *long* bearing which fixes its direction	Length of overhang in feet	$11645\frac{d}{L^2}$	$108\sqrt{\frac{d}{N}}$
Unloaded: resting in short or swivelled bearings at each end	Distance between centres of bearings in feet	$32864\frac{d}{L^2}$	$181\sqrt{\frac{d}{N}}$
Unloaded: supported as in last case, but one end overhanging c feet	Ditto $r = \frac{c}{L}$	$a\frac{d}{L^2}$ For values of a see Table I.	$\sqrt{a}\sqrt{\frac{d}{N}}$
Unloaded: supported in a *long* bearing at one end, and a short or swivelled bearing at the other	Distance between inner edge of long bearing and centre of short bearing	$51340\frac{d}{L^2}$	$227\sqrt{\frac{d}{N}}$
Unloaded: shaft supported in three short or swivelled bearings, one at each end $r = \frac{L_1}{L_2}$	L_1 = shorter span L_2 = longer span in feet	$a\frac{d}{L_2^2}$ For values of a see Table II.	$L_2 = \sqrt{a}\sqrt{\frac{d}{N}}$

Description of shaft.	Remarks on *l*.	N.	L.
Unloaded : shaft supported in *long* bearings which fix its direction at each end	Clear span between inner edges of bearings	$749711\frac{d}{L^2}$	$274\sqrt{\frac{d}{N}}$
Unloaded : long continuous shaft supported on short or swivelled bearings equidistant	Distance between centres of bearings in feet	$32864\frac{d}{L^2}$	$181\sqrt{\frac{d}{N}}$
Loaded : shaft supported in short or swivelled bearings, single pulley of weight W pounds centrally placed between the bearings	Distance between the centres of the bearings in feet	$N = 37{,}350\frac{d^2}{\sqrt{WL^3}}$	
Loaded : long bearings which fix its direction at each end	Clear span between inner edges of bearings	$N = 74{,}700\frac{d^2}{\sqrt{WL^3}}$	

TABLE I.

Value of $r = \frac{c}{L}$.	a.	$\sqrt{a}$.
1·00	7,554	87
0·75	12,044	110
0·50	20,931	145
0·33	28,095	168
0·25 to 0·10	31,000	176
Very small	31,590	178
Zero	32,864	181

TABLE II.

Value of $r = \frac{d_1}{d_2}$	a.	$\sqrt{a}$.
1·00 to 0·75	32,864	181
0·5 to 0·75	36,884	192
0·33	41,026	203
0·25	43,289	208
0·20 to 0·14	44,312	211
0·125 to 0·10	47,125	217
Very small	50,654	225

For hollow shafts in which—

d_1 = the external diameter in inches
d_2 = „ internal „ „

substitute for d in the expressions given above the value—

$$\sqrt{d_1^2 + d_2^2}$$ for unloaded shafts,

and substitute for d^2 the value—

$$\sqrt{d_1^4 - d_2^4}$$ for loaded shafts.

CHAPTER VIII.

GYROSCOPIC ACTION.

When a wheel or other body is rotating at a high speed a considerable resistance is experienced if an attempt be made to rapidly change the direction of its plane of rotation. This statement can be verified by a simple experiment on the front wheel of a bicycle. Take the front wheel and axle out of the fork, suspend the axle from one end only by means of a piece of string. Hold the axle horizontal and spin the wheel rapidly in a vertical plane, on releasing the free end of the axle it will be found that the wheel retains its vertical position so long as it continues to spin at a high speed. The external couple required to keep the wheel and axle in this position is known as the gyroscopic couple.

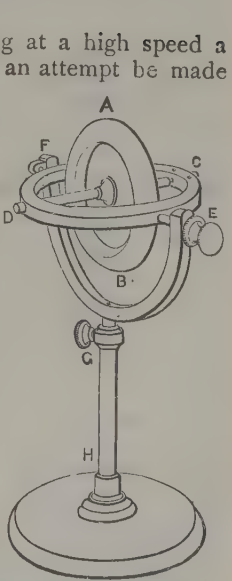


Fig. 267.
From Worthington's "Dynamics of Rotation."

The gyroscopic top is also a familiar and striking instance of gyroscopic action, and moreover affords an excellent illustration for demonstrating the principle involved, which is simply the rotational analogue of Newton's second law of motion, and may be stated thus—

When a $\left\{\begin{matrix}\text{force}\\\text{couple}\end{matrix}\right\}$ acts on a body, or self-contained system the change of $\left\{\begin{matrix}\text{momentum}\\\text{angular momentum}\end{matrix}\right\}$ generated in a given time is proportional to the impressed $\left\{\begin{matrix}\text{force}\\\text{couple}\end{matrix}\right\}$.

Hence, if an external couple act for a given time on the wheel of a gyroscope the angular momentum about the axis of the couple will be increased in proportion to the couple.

If a second external couple be impressed on the system for the same length of time about the same or another axis the angular momentum will be proportionately increased about that axis. Since angular momenta are vector quantities they can be combined by the method used in the parallelogram of forces.

Precession of Gyroscope.—A general view of a gyroscopic top is shown in Fig. 267, a plan and part sectional elevation in Figs. 268 and 269.

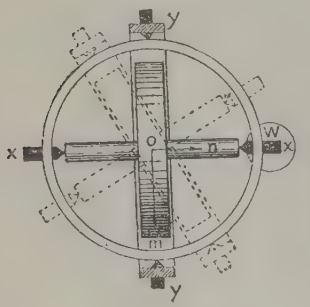


FIG. 268.

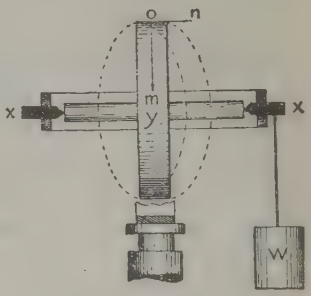


FIG. 269.

If the wheel were not rotating and a weight W were suspended from the pivot x as shown, the wheel and gimbal ring would naturally rotate about the axis yy, but when the wheel is rotating at a high speed the action of the weight at first sight appears to be entirely different.

Let the axis OO be vertical, and let the gimbal ring be placed horizontal to start with. Looking at the top of the wheel (Fig. 268), let the angular momentum of a particle on the rim be represented by Om in a horizontal plane, and let On represent the angular momentum in a direction at right angles to Om generated in the same time by the moment of W about yy. The resultant angular momentum will be represented by Or, that is to say, the plane of rotation of the wheel will alter or precess into the direction shown by Or; thus, to the observer, it precesses in a contra-clockwise sense at a rate shortly to be determined. The precession will be reversed in sense if the wheel rotates in the opposite direction or if the external couple tends to lift the right-hand pivot x.

If any other particle on the rim be considered, similar

results, as regards the precession of the system will be obtained.

Consider next the case of a horizontal force P applied to the right-hand pivot.

Let *ym* (Fig. 270) represent the angular momentum of a particle on the rim of the wheel, which at the instant is moving in a vertical plane. Let *yn* represent the angular momentum generated by the external couple of P about the axis OO. The resultant angular momentum is represented by *yr*, thus indicating that the plane of rotation which was vertical before the application of the external moment has now tilted over or precessed to the inclined position indicated by *yr*. Thus an observer at y_1 looking horizontally at the edge of the wheel sees it precessing about y_1y in a contra-clockwise sense. If the sense of rotation of the wheel be reversed, or if P act in the opposite direction, the sense of the precession will also be reversed.

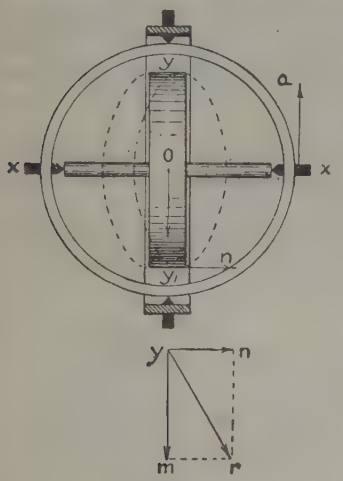


FIG. 270.

We thus get the very curious result with a spinning gyroscope that when it is acted on by an external couple or twisting moment the resulting rotation does not occur about the axis of the couple but about an axis at right angles to it.

An easily remembered method of determining the sense in which the gyroscope tends to precess is to consider the motion of a point on the rim of the wheel just as it is penetrating the plane containing the axis of the wheel and the external couple. In Fig. 268 this point is on the axis OO on either the top or bottom of the rim. In Fig. 270 it is on the axis *yy*. The diagram of velocity[1] is constructed on a plane tangential

[1] Angular momentum $= I\omega = \frac{IV}{K}$

Since I and K are constant, the angular momentum generated is proportional to V, hence velocity diagrams may be used instead of angular momentum diagrams.

to the rim of the wheel at the point in question and parallel to the axis of rotation, by drawing lines to represent

(i) the velocity of the point owing to the rotation of the wheel;

(ii) the velocity of the point due to the external couple;

(iii) the resultant velocity, the direction of which indicates the plane towards which the wheel tends to precess.

In the case of rotating bodies in which the precession is forced the sense and direction of lines (i) and (iii) are known; hence the sense and direction of (ii) are readily obtained, thus supplying all the data for finding the sense of the external couple.

When only the sense of the precession is required, it is not necessary to regard the magnitude of the velocities or the corresponding lengths of the lines.

Rate of Precession.—When the wheel of a gyroscope is rotating at a given speed, the rate at which precession occurs

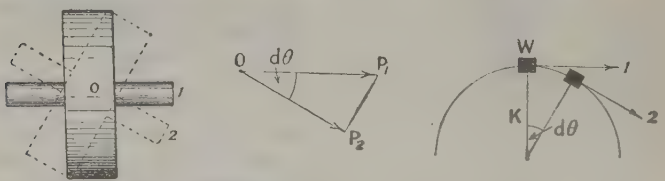


FIG. 271.

is entirely dependent upon the applied twisting moment, the magnitude of which can be readily found by a process of reasoning similar to that used for finding centrifugal force. See the paragraph on the Hodograph on p. 17.

The close connection between the two phenomena is shown by the following statements:—

Centrifugal force:— / (Gyroscopic couple):— } Let a { body / (wheel) } be kept { moving / (spinning) } at a constant { linear / (angular) } speed and be constrained in such a manner that it { moves / (precesses) } about { a centre O in the plane of / (an axis OO at right angles motion / to the axis of rotation) }.

Let OP_1 represent the { linear / (angular) } velocity of the { body / (wheel) }

when in position 1 and let a $\left\{\begin{matrix}\text{force}\\(\text{couple})\end{matrix}\right\}$ act on the $\left\{\begin{matrix}\text{body}\\(\text{wheel})\end{matrix}\right\}$ in such a manner as to bring it into position 2.

Then OP_2 represents its corresponding $\left\{\begin{matrix}\text{linear}\\(\text{angular})\end{matrix}\right\}$ velocity, and P_1P_2 represents, to the same scale, the change of $\left\{\begin{matrix}\text{velocity}\\(\text{angular velocity})\end{matrix}\right\}$.

The $\left\{\begin{matrix}\text{force}\\(\text{couple})\end{matrix}\right\}$ required to effect this change of $\left\{\begin{matrix}\text{velocity}\\(\text{angular velocity})\end{matrix}\right\}$ is given by

$$\left\{\begin{matrix}C = \dfrac{WV^2}{gK} = \dfrac{WV\Omega}{g}\\ \left(CK = \dfrac{WVK\Omega}{g} = \dfrac{WK^2}{g}\Omega\omega = I\Omega\omega\right)\end{matrix}\right\}$$

or Centrifugal force

= mass of body × velocity × angular velocity.

Gyroscopic couple

$$= \left\{\begin{matrix}\text{moment of inertia of}\\\text{wheel about the axis}\\\text{of rotation}\end{matrix}\right\} \times \left\{\begin{matrix}\text{angular}\\\text{velocity}\\\text{of wheel}\end{matrix}\right\} \times \left\{\begin{matrix}\text{angular velocity}\\\text{of precession.}\end{matrix}\right.$$

In the above expressions—

W = Weight of body or the wheel in pounds.
V = Velocity in feet per second of the c. of g. of the body or of a point on the wheel at a radius equal to the radius of gyration K (in feet).
Ω = The angular velocity of the body or the wheel in radians per second.
ω = The rate of precession in radians per second.
N = Revolutions per minute of the wheel.
n = Revolutions per minute of the precession.

For engineering purposes it is generally more convenient to express the speeds in revolutions per minute. Inserting the value of g and reducing we get—

Gyroscopic couple $= \dfrac{WK^2Nn}{2937} = 0{\cdot}00034WK^2Nn$ in pounds-feet.

The following examples of gyroscopic effects may serve to show the magnitude of the forces to be dealt with.

The wheels of a locomotive weigh 2000 pounds each, the diameter on the tread is 6 feet 6 inches, and the radius of gyration is 2·8 feet. Find the gyroscopic couple acting on the wheels when running round a curve at 50 miles per hour. Radius of curve 400 feet. Find the vertical load on the outer and inner rails when there is no super-elevation, and when the dead load on each wheel is 10,000 pounds. Rail centres 5 feet.

$N = 215{\cdot}5 \qquad n = 1{\cdot}75$

$$\text{Gyroscopic couple} = \frac{2000 \times 2{\cdot}8^2 \times 215{\cdot}5 \times 1{\cdot}75}{2937}$$

$$= 2015 \text{ pounds-feet.}$$

Arm of couple, 5 feet. Force, 403 pounds.

The figure will assist in determining the sense in which the external couple acts. Since the wheels precess about a

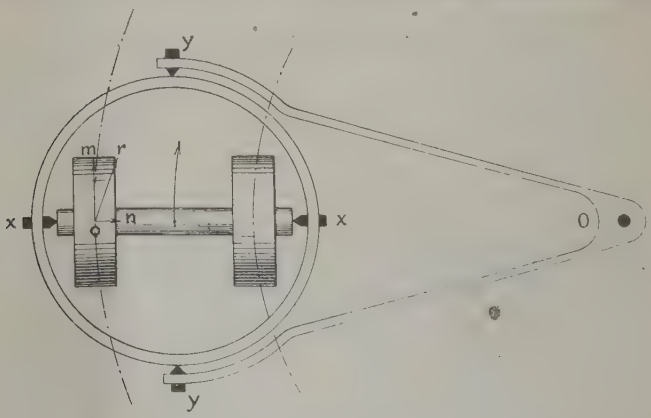


Fig. 272.

vertical axis passing through the centre of the curve, we know that the external forces act in a vertical direction. As the wheels traverse the curve their direction is changed from *om* to *or*, hence the external couple tends to move the top of the wheel in the direction *on*. The force, which is the reaction of the outer rail, therefore acts upwards, which is equivalent to saying that the vertical pressure on the rail is greater by the

amount of the gyroscopic force than the dead weight on the wheel. Since the total pressure due to both wheels is equal to the dead weight upon them, it follows that the vertical pressure on the inner rail is less by the amount of the gyroscopic force than the dead weight on the wheel. The vertical pressure on the outer rail is 10,009 + 403 = 10,403 pounds, and on the inner rail 10,000 − 403 = 9597 pounds. (No super-elevation.)

Thus it will be seen that the gyroscopic effect intensifies the centrifugal effect as far as the overturning moment is concerned.

If the figure represented a pair of spinning gyroscopic wheels which are not resting upon rails, but with the frame supported at the pivot O, the gyroscope would precess in the direction of the arrow if an upward force were applied to the left-hand pivot x, or if a weight were suspended from the right-hand pivot x.

In the case of a motor-car, in which the plane of the flywheel is parallel to that of the driving wheels, the angular momentum of the flywheel must be added to that of the road wheels, when the sense of rotation of the flywheel is the same as that of the road wheels, and subtracted when rotating in the opposite sense.

When the flywheel rotates in a plane at right angles to that of the road wheels, and its sense of rotation is clockwise when viewed from the front of the car, the weight on the steering wheels will be increased and that on the driving wheels will be diminished when the car steers to the chauffeur's left hand, and *vice versâ* when steering to the right; the actual amount, however, is very small. When a car turns very suddenly, as when it skids in turning a corner on a greasy road, the gyroscopic effect may be so serious as to bend the crank shaft. Lanchester strengthens the crank web and the neck of the crank-shaft next the flywheel, in order to provide against such an accident.

CHAPTER IX.

FRICTION.

When one body, whether solid, liquid, or gaseous, is caused to slide over the surface of another, a resistance to sliding is experienced, which is termed the "friction" between the two bodies.

Many theories have been advanced to account for the friction between sliding bodies, but none are quite satisfactory. To attribute it merely to the roughness between the surfaces is but a very crude and incomplete statement; the theory that the surfaces somewhat resemble a short-bristled brush or velvet pile much more nearly accounts for known phenomena, but still is far from being satisfactory.

However, our province is not to *account* for what happens, but simply to observe and classify, and, if possible, to sum up our whole experience in a brief statement or formula.

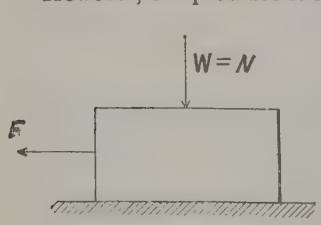


Fig. 273.

Frictional Resistance. (F).—If a block of weight W be placed on a horizontal plane, as shown, and the force F applied parallel to the surface be required to make it slide, the force F is termed the frictional resistance of the block. The normal pressure between the surfaces is N.

Coefficient of Friction (μ).—Referring to the figure above, the ratio $\frac{F}{W}$ or $\frac{F}{N} = \mu$, and is termed the coefficient of friction. It is, in more popular terms, the ratio the friction bears to the normal pressure between the surfaces. It may be found by dragging a block along a plane surface and measuring F and N, or it may be found by tilting the surface as in Fig. 274. The plane is tilted till the block just begins to slide. The vertical

weight W may be resolved normal N and parallel to the plane F. The friction is due to the normal pressure N, and the

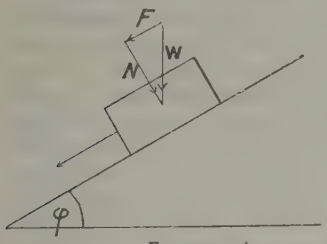


FIG. 274.

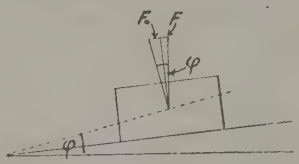


FIG. 275.

force required to make the body slide is F; then the coefficient of friction $\mu = \frac{F}{N}$ as before. But $\frac{F}{N} = \tan \phi$, where ϕ is the angle the plane makes to the horizontal when sliding just commences.

The angle ϕ is termed the "friction angle," or "angle of repose." The body will not slide if the plane be tilted at an angle less than the friction angle, a force F_0 (Fig. 275) will

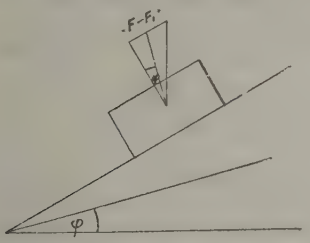


FIG. 276.

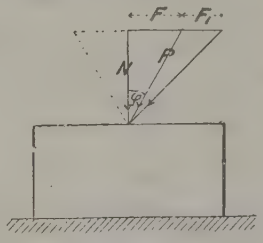


FIG. 277.

then have to be applied parallel to the plane in order to make it slide. Whereas, if the angle be greater than ϕ, the body will be accelerated due to the force F_1 (Fig. 276).

There is yet another way of looking at this problem. Let the body rest on a horizontal plane, and let a force P be applied at an angle to the normal; the body will not begin to slide until the angle becomes equal to the angle ϕ, the angle of friction. If the line representing P be revolved round the normal, it will describe the surface of a cone in space, the apex angle being 2ϕ; this cone is known as the "friction cone."

If the angle with the normal be less than ϕ, the block will not slide, and if greater the block will be accelerated, due to the force F_1. In this case the weight of the block is neglected.

If P be very great compared with the area of the surfaces in contact, the surfaces will seize or cling to one another, and if continued the surfaces will be torn or abraded.

Friction of Dry Surfaces.—The experiments usually quoted on the friction of dry surfaces are those made by Morin and Coulomb; they were made under very small pressures and speeds, hence the laws deduced from them only hold very imperfectly for the pressures and speeds usually met with in practice. They are as follows:—

1. The friction is directly proportional to the normal pressure between the two surfaces.

2. The friction is independent of the area of the surfaces in contact for any given normal pressure, *i.e.* it is independent of the *intensity* of the normal pressure.

3. The friction is independent of the velocity of rubbing.

4. The friction between two surfaces at rest is greater than when they are in motion, or the friction of rest[1] is greater than the friction of motion.

5. The friction depends upon the nature of the surfaces in contact.

We will now see how the above laws agree with experiments made on a larger scale.

The first two laws are based on the assumption that the coefficient of friction is constant for all pressures; this, however, is not the case.

The curves in Fig. 278 show approximately the difference between Coulomb's law and actual experiments carried to high pressures. At the low pressure at which the early workers worked, the two curves practically agree, but at higher pressures the friction falls off, and then rises until seizing takes place.

Instead of the frictional resistance being

$$F = \mu N$$

it is more nearly given by $F = \mu N^{0.97}$, or $F = \mu^{1.03}\sqrt{N}$

The variation is really in μ and not in N, but the expression, which is empirical, assumes its simplest form as given above.

For dry surfaces μ has the following values:—

[1] The friction of rest has been very aptly termed the "sticktion."

Wood on wood	...	0·25 to 0·50	Leather on wood	...	0·25 to 0·50
Metal ,,	...	0·20 to 0·60	,, metal	...	0·30 to 0·60
Metal on metal	...	0·15 to 0·30	Stone on stone	...	0·40 to 0·65

These coefficients must always be taken with caution; they vary very greatly indeed with the state of the surfaces in contact.

The third law given above is far from representing facts; in the limit the fourth law becomes a special case of the third. If the surfaces were perfectly clean, and there were no film of air between, this law would probably be strictly accurate, but all experiments show that the friction decreases with velocity of rubbing.

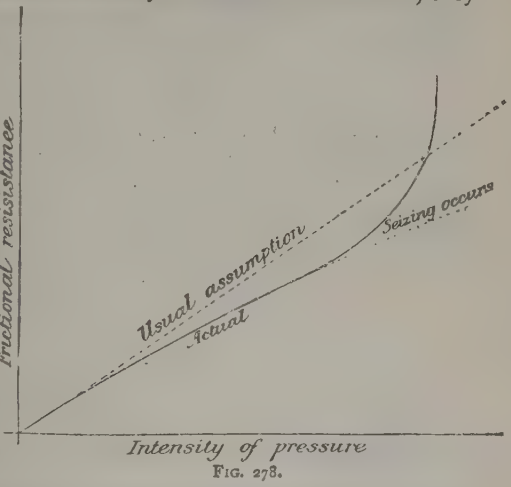


FIG. 278.

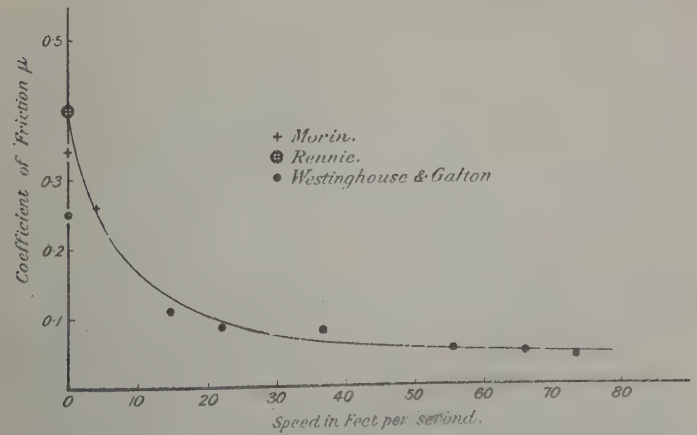


FIG. 279.

The following empirical formula fairly well agrees with experiments :—

Let μ = coefficient of friction ;
K = a constant to be determined by experiment ;
V = the velocity of sliding.

$$\text{Then } \mu = \frac{K}{2{\cdot}4\sqrt{V}}$$

The results obtained with dry surfaces by various experimenters are shown in Fig. 279.

The fourth law has been observed by nearly every experimenter on friction. The following figures by Morin and others will suffice to make this clear :—

Materials.	Coefficient of friction.	
	Rest.	Velocity 3 to 5 ft.-sec.
Wood on wood	0·54	0·46
„ „	0·69	0·43
Metal on metal	0·34	0·26
Stone on stone	0·74	0·63
Leather on iron	0·59	0·52

The figures already quoted quite clearly demonstrate the truth of the fifth law given above.

Special Cases of Sliding Bodies.—In the cases we are about to consider, we shall for sake of simplicity, assume that Coulomb's laws hold good.

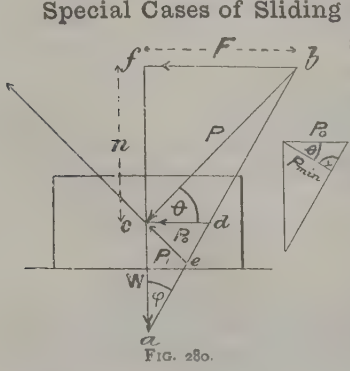


FIG. 280.

Oblique Force required to make a Body slide on a Horizontal Plane.—If an oblique force P act upon a block of weight W, making an angle θ with the direction of sliding, we can find the magnitude of P required to make the block slide ; the total normal pressure on the plane is the normal component of P, viz. n, together with W. From a draw a line making an angle ϕ (the friction angle) with W, cutting P in

the point b; then bc, measured to the same scale as W, is the magnitude of the force P required to make the body slide. The frictional resistance is F, and the total normal pressure $n + W$; hence $F = \mu(n + W)$. When $\theta = 0$, $P_0 = cd = \mu W$.

When θ is negative, it simply indicates that P_1 is pulling away from the plane: the magnitude is given by ce. From the figure it is clear that the least value of P is when its direction is normal to ab, *i.e.* when $\theta = \phi$; then—

$$\begin{aligned} P_{min.} &= P_0 \cos \phi = \mu W \cos \phi \\ &= \tan \phi \ W \cos \phi \\ &= W \sin \phi \end{aligned}$$

It will be seen from the figure that P is infinitely great when ab is parallel to bc—that is, when P is just on the edge of the friction cone, or when $90 - \theta = \phi$. When $\theta = -90°$, P acts vertically upwards and is equal to W.

A general expression for P is found thus—

$$\begin{aligned} n &= P \sin \theta \\ F &= P \cos \theta = \mu(W \mp n) \\ F &= \mu(W \mp P \sin \theta) \end{aligned}$$

and $P(\cos \theta \mp \mu \sin \theta) = \mu W$

$$\text{hence } P = \frac{\mu W}{\cos \theta \mp \mu \sin \theta} = \frac{\tan \phi \ W}{\cos \theta \mp \mu \sin \theta}$$

$$P = \frac{\sin \phi \ W}{\cos \phi \cos \theta \mp \sin \phi \sin \theta}$$

$$P = \frac{W \sin \phi}{\cos (\phi \mp \theta)}$$

When P acts upwards away from the plane, the $-$ sign is to be used in the denominator; and for the minimum value of P, $\phi = -\theta$; then the denominator is unity, and $P = W \sin \phi$, the result given above, but arrived at by a different process.

Thus, in order to drag a load, whether sliding or on wheels, along a plane, the line of pull should be upwards, making an angle with the plane equal to the friction angle.

Force required to make a Body slide on an Inclined Plane.—A special case of the above is that in which the plane is inclined to the horizontal at an angle α. Let the block of weight W rest on the inclined plane as shown. In order to make it slide up the plane, work must be done in lifting the block as well as overcoming the friction. The pull required to raise the block is readily obtained thus: Set down a line bc to represent the weight W, and from c draw a line cd, making

an angle a with it; then, if from b a line be drawn parallel to the direction of pull P_1, the line bd_1 represents to the same scale as W the required pull if there were no friction. An examination of the diagram will at once show that bd_1e is simply the triangle of forces acting on the block; the line cd_1 is, of course, normal to the plane.

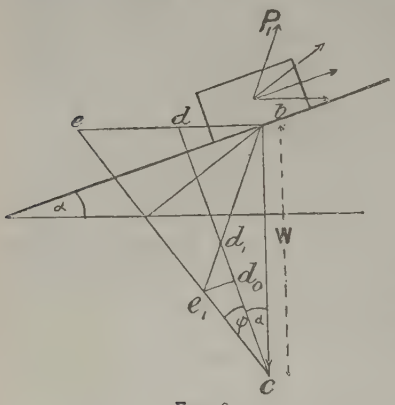


FIG. 281.

When friction is taken into account, draw the line ce, making an angle ϕ (the friction angle) with cd; then be_1 gives the pull P_1 required to drag the block up the plane including friction.

For it will be seen that the normal pressure on the plane is cd_0, and that the friction parallel to the direction of sliding, viz. normal to cd, is—

$$\mu \cdot \overline{cd_0} = \tan \phi \cdot \overline{cd_0}$$
$$= d_0e_1$$

Then resolving d_0e_1 in the direction of the pull, we get the pull required to overcome the friction d_1e_1; hence the total pull required to both raise the block and overcome the friction is be_1.

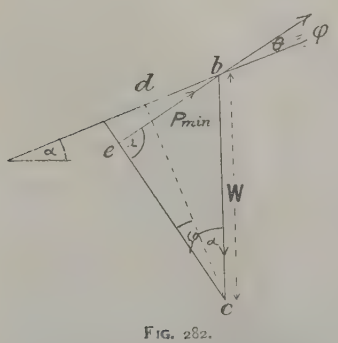


FIG. 282.

Least Pull.—The least pull required to pull the block up the plane is when be has its least value, *i.e.* when be is normal to ce; the direction of pull then makes an angle ϕ with the plane, or $\theta = \phi$, for cd is normal to the plane, and ce makes an angle ϕ with cd.

Then $P_{min} = W \sin (\phi + \alpha)$ . . . . (i.)

Horizontal Pull.—When the body is raised by a horizontal pull, we have (Fig. 283)—

$$P_b = W \tan (\phi + \alpha) \quad . \quad . \quad . \quad . \quad . \quad (ii.)$$

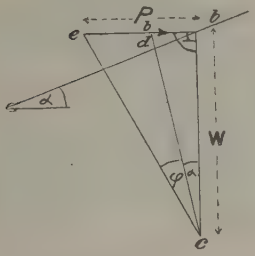


FIG. 283.

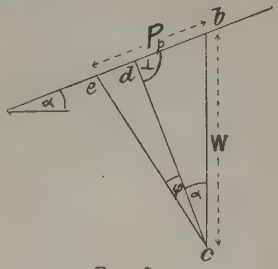


FIG. 284.

Thus, in all cases, the effect of friction is equivalent to making the slope steeper by an amount equal to the friction angle.

Parallel Pull.—When the body is raised by a pull parallel to the plane, we have (Fig. 284)—

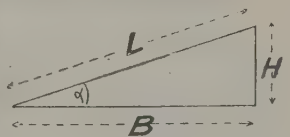


FIG. 285.

$$P_p = ed + db$$
$$\text{But } \overline{ed} = \overline{dc} \tan \phi = \mu \overline{dc}$$
$$\text{and } \overline{dc} = W \cdot \cos \alpha$$
$$\text{therefore } \overline{ed} = \mu \cdot W \cdot \cos \alpha$$
$$\text{and } \overline{db} = W \sin \alpha$$
$$\text{hence } P_p = W(\mu \cdot \cos \alpha + \sin \alpha) \quad . \quad . \quad . \quad (iii.)$$

This may be expressed thus (see Fig. 285)—

$$P_p = W\left(\mu \frac{B}{L} + \frac{H}{L}\right)$$
$$\text{or } P_p L = W\mu B + WH$$

or, Work done in dragging a body of weight W up a plane, by a force acting parallel to the plane = { work done in dragging the body through the same distance on a horizontal plane against friction. } + { work done in lifting the body

General Case.—When the body is raised by a pull making an angle θ with the plane—

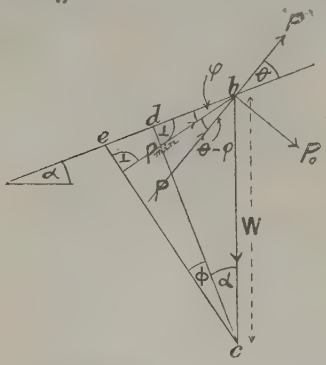


FIG. 286.

$$P = \frac{P_{min.}}{\cos(\theta - \phi)}$$

Substituting the value of $P_{min.}$ from equation (i.)—

$$P = \frac{W \sin(\phi + a)}{\cos(\theta - \phi)} \quad . \quad \text{(iv.)}$$

When the line of action of P is towards the plane, as in P_0, the θ becomes *minus*, and we get—

$$P_0 = \frac{W \sin(\phi + a)}{\cos(-\theta - \phi)}$$

All the above expressions may be obtained from this.

When the direction of pull, P_0, is parallel to *ec*, it will only meet *ec* at infinity—that is, an infinitely great force would be required to make it slide; but this is impossible, hence the direction of pull must make an angle to the plane $\theta < (90 - \phi)$ in order that sliding may take place.

We must now consider the case in which a body is dragged down a plane, or simply allowed to slide down. If the angle α be less than ϕ, the body must be dragged down, and if α be

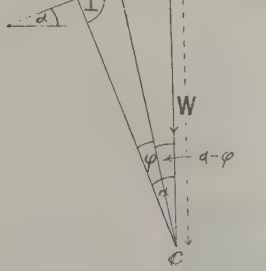


FIG. 287.

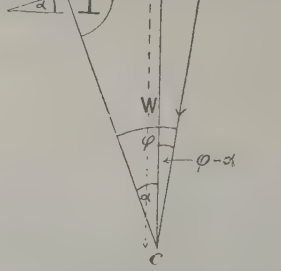


FIG. 288.

NOTE.—The friction now assists the lowering, hence *ec* is set off to the right of *cd*.

greater, a force must be applied to prevent it from being accelerated.

Least pull when body is lowered, $\phi < \alpha$ (Fig. 287).

$$P_{min.} = \overline{be} = W \sin (\alpha - \phi) \text{ and } \theta = \phi \quad . \quad . \quad . \quad (v.)$$

When $\phi > \alpha$, be_1 is the least force required to make the body slide down the plane.

$$P_{min.} = \overline{be_1} = W \sin (\phi - \alpha) \quad . \quad . \quad . \quad . \quad (vi.)$$

when $\phi = \alpha$, $P_{min.}$ is of course zero.

The remaining cases are arrived at in a similar manner; we will therefore simply state them.

	$\phi < \alpha$.	$\phi > \alpha$.
Least pull	$W \sin (\alpha - \phi)$	$W \sin (\phi - \alpha)$
Parallel pull ...	$W (\sin \alpha - \mu \cos \alpha)$	$W (\mu \cos \alpha - \sin \alpha)$
Horizontal pull ...	$W \tan (\alpha - \phi)$	$W \tan (\phi - \alpha)$
General case ...	$\frac{W \sin (\alpha - \phi)}{\cos (\theta + \phi)}$	$\frac{W \sin (\phi - \alpha)}{\cos (\theta - \phi)}$

NOTE.—If the line of pull comes below the plane, the angle θ takes the $-$ sign.

In the case of the parallel pull, it is worth noting that when $\phi < \alpha$, we have—

Total work done = work done in lowering the body − work done in dragging the body through the horizontal distance against friction

and when $\phi > \alpha$ we have the same relation, but the work done is negative, that is, the body has to be retarded.

It should be noticed that the effect of friction on an inclined plane is to increase the steepness when the block is being hauled up the plane, and to decrease it when hauling it down the plane by an amount equal to the friction angle.

Efficiency of Inclined Planes.—If an inclined plane be used as a machine for raising or lowering weights, we have—

$$\text{Efficiency} = \frac{\text{useful work done } (i.e. \text{ without friction})}{\text{actual work done (with friction)}}$$

Inclined Plane when raising a Load.—The maximum efficiency occurs when the pull is least, *i.e.* when $\theta = \phi$. The useful work done without friction is when $\phi = 0$; then—

The work done without friction $= \dfrac{LW \sin \alpha}{\cos \theta}$ from (iv.)

" " with " $= LW \sin (\phi + \alpha)$ from (i.)

where L = the distance through which the body is dragged;
α = the inclination of the plane to the horizon;
θ = the inclination of the force to the plane;
ϕ = the friction angle.

$$\text{Then maximum efficiency} = \frac{\dfrac{LW \sin \alpha}{\cos \theta}}{LW \sin (\phi + \alpha)} = \frac{\sin \alpha}{\cos \theta \sin (\phi + \alpha)} \quad \text{(vii.)}$$

When the pull is horizontal, $\theta = \alpha$, and—

$$\text{Efficiency} = \frac{\sin \alpha}{\cos \alpha \,.\, \tan (\phi + \alpha)} = \frac{\tan \alpha}{\tan (\phi + \alpha)} \quad \text{(viii.)}$$

when the pull is parallel, $\theta = 0$, and $\cos \theta = 1$;

$$\text{Efficiency} = \frac{\sin \alpha \,.\, \cos \phi}{\sin (\alpha + \phi)} \quad \text{(ix.)}$$

General case, when the line of pull makes an angle θ with the direction of sliding—

$$\text{Efficiency} = \frac{\sin \alpha \,.\, \cos (\theta - \phi)}{\cos \theta \,.\, \sin (\phi + \alpha)} \quad \text{(x.)}$$

Friction of Wedge.—This is simply a special case of the inclined plane in which the pull is horizontal, or when it acts normal to W. We then have from equation (ii.) $P = W \tan (\phi + \alpha)$ for a single inclined plane; but here we have two inclined planes, each at an angle α, hence W moves twice as far for any given movement of P as in the single inclined plane; hence—

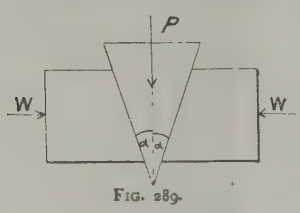


FIG. 289.

$$P = 2W \tan (\phi + \alpha) \text{ for a wedge}$$

The wedge will not hold itself in position, but will spring back, if the angle α be greater than the friction angle ϕ.

From the table on p. 293 we have the pull required to withdraw the wedge—

$$-P = 2W \tan (\alpha - \phi)$$

The efficiency of the wedge is the same as that of the inclined plane, viz.—

$$\text{Efficiency} = \frac{\tan \alpha}{\tan (\alpha + \phi)} \quad \text{when overcoming a resistance} \quad \text{(xi.)}$$

$$\left.\begin{matrix}\text{reversed} \\ \text{efficiency}\end{matrix}\right\} = \frac{\tan (\alpha - \phi)}{\tan \alpha} \quad \text{when withdrawing from a resistance}$$

(see p. 335)

Efficiency of Screws and Worms—Square Thread.—A screw thread is in effect a narrow inclined plane wound round a cylinder; hence the efficiency is the same as that of an inclined plane. We shall, however, work it out by another method.

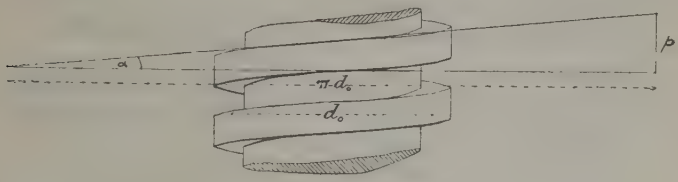


FIG. 290.

Let p = the pitch of the screw;
d_0 = the mean diameter of the threads;
W = the weight lifted.

$$\left.\begin{matrix}\text{The useful work done per revolution} \\ \text{on the nut without friction}\end{matrix}\right\} = W p = W \pi d_0 \tan \alpha$$

$$\left.\begin{matrix}\text{The force applied at the mean radius of} \\ \text{the thread required to raise nut}\end{matrix}\right\} = \left\{\begin{matrix}P = W \tan (\alpha + \phi) \\ \text{(see equation ii.)}\end{matrix}\right.$$

$$\left.\begin{matrix}\text{The work done in turning the nut} \\ \text{through one complete revolution}\end{matrix}\right\} = P \pi d_0$$

$$= W \pi d_0 \tan (\alpha + \phi)$$

$$\text{Efficiency, when raising the weight,} = \frac{W \pi d_0 \tan \alpha}{W \pi d_0 \tan (\alpha + \phi)}$$

$$= \frac{\tan \alpha}{\tan (\alpha + \phi)} = \frac{\sin \alpha \cos (\alpha + \phi)}{\cos \alpha \sin (\alpha + \phi)}$$

$$= \frac{\sin (2\alpha + \phi) - \sin \phi}{\sin (2\alpha + \phi) + \sin \phi} = 1 - \frac{2 \sin \phi}{\sin (2\alpha + \phi) + \sin \phi}$$

This has its maximum value when the fractional part is least, or when $\sin (2\alpha + \phi) = 1$.

Since the sine of an angle cannot be greater than 1, then $2\alpha + \phi = 90^\circ$ and $\alpha = 45 - \frac{\phi}{2}$. Inserting this value,

$$\text{maximum efficiency} = \left(\frac{1 - \tan\frac{\phi}{2}}{1 + \tan\frac{\phi}{2}}\right)^2 = 1 - 2\mu \text{ (nearly).}$$

In addition to the friction on the threads, the friction on the thrust collar of the screw must be taken into account. The thrust collar may be assumed to be of the same diameter as the thread; then the

$$\left.\begin{matrix}\text{efficiency of screw thread} \\ \text{and thrust collar}\end{matrix}\right\} = \frac{\tan\alpha}{\tan(\alpha + 2\phi)} \text{ (approx.)}$$

In the case of a nut the radius at which the friction acts will be about 1½ times that of the threads; we may then say—

$$\left.\begin{matrix}\text{efficiency of screw thread and nut} \\ \text{bedding on a flat surface}\end{matrix}\right\} = \frac{\tan\alpha}{\tan(\alpha + 2{\cdot}5\phi)}$$

If the angle of the thread be very steep, the screw will be reversible, that is, the nut will drive the screw. By similar reasoning to that given above, we have—

$$\text{reversed efficiency} = \frac{\tan(\alpha - \phi)}{\tan\alpha} \text{ (see p. 335)}$$

Such an instance is found in the Archimedian drill brace, and another in the twisted hydraulic crane-post used largely on board ship. By raising and lowering the twisted crane-post, the crane, which is in reality a part of a huge nut, is slewed round as desired.

Triangular Thread.—In the triangular thread the normal pressure on the nut is greater than in the square-threaded screw, in the ratio of $\frac{W_0}{W} = \frac{1}{\cos\frac{\theta}{2}}$, and $W_0 = \frac{W}{\cos\frac{\theta}{2}}$, where θ is the angle of the thread. In the Whitworth thread the angle θ is 55°, hence $W_0 = 1{\cdot}13\ W$. In the Sellers thread $\theta = 60^\circ$ and $W_0 = 1{\cdot}15\ W$; then, taking a mean value of $W_0 = 1{\cdot}14\ W$, we have—

$$\text{efficiency} = \frac{\tan\alpha}{\tan(\alpha + 1{\cdot}14\phi)}$$

In the case of an ordinary bolt and nut, the radius at which the friction acts between the nut and the washer is about $1\frac{1}{2}$ times that of the thread, and, taking the same coefficient of friction for both, we have—

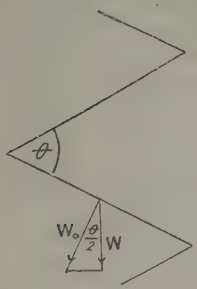


Fig. 291.

$$\left.\begin{matrix}\text{efficiency}\\ \text{of a bolt}\\ \text{and nut}\end{matrix}\right\} = \frac{\tan \alpha}{\tan (\alpha + 2{\cdot}64\phi)} \text{ (approx.)}$$

The following table may be useful in showing roughly the efficiency of screws. In several cases they have been checked by experiments, and found to be fair average values; the efficiency varies greatly with the amount of lubrication :—

Table of Approximate Efficiencies of Screw Threads.

Angle of thread α.	Efficiency per cent. when no friction between nut and washer or a thrust collar.		Efficiency per cent. allowing for friction between nut and washer or a thrust collar.	
	Sq. thread.	V-thread.	Sq. thread.	V-thread.
2°	19	17	11	8
3°	26	23	14	12
4°	32	28	17	16
5°	36	33	21	20
10°	55	52	36	29
20°	67	63	48	42
$45-\frac{\phi}{2}$	79	75	52	44

In the above table ϕ has been taken as 8·5°, or $\mu = 0{\cdot}15$.

For the efficiency of a worm and wheel see page 344.

Rolling Friction.—When a wheel rolls on a yielding material that readily takes a permanent deformation, the resistance is due to the fact that the wheel sinks in and makes a rut. The greater the weight W carried by the wheel, the deeper will be the rut, and consequently the greater will be the resistance to rolling.

When the wheel is pulled along, it is equivalent to constantly mounting an obstacle at A; then we have—

$$P \cdot \overline{BA} = W \cdot \overline{AC}$$

$$\text{or } P = \frac{W \cdot \overline{AC}}{\overline{BA}}$$

Let $AC = K$;

$$\text{Then } P = \frac{W \cdot K}{R - h}$$

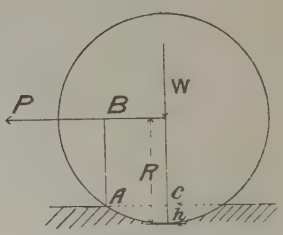


FIG. 292.

But h is usually small compared with R; hence we may write—

$$P = \frac{WK}{R} \text{ (nearly)}$$

P and W, also K and R, must be measured in the same units, or the value of K corrected accordingly. The above treatment is very rough, but the relation holds fairly well in practice. There is much need for further research in this branch of friction.

VALUES OF K.

	K (inches).
Iron or steel wheels on iron or steel rails	0·007–0·015
,, ,, wood	0·06–0·10
,, ,, macadam	0·05–0·20
,, ,, soft ground	3–5
Pneumatic tyres on good road or asphalte	0·02–0·022
,, ,, heavy mud	0·04–0·06
Solid indiarubber tyres on good road or asphalte ...	0·04
,, ,, heavy mud	0·09–0·11

Some years ago Professor Osborne Reynolds investigated the action of rollers passing over elastic materials, and showed clearly that when a wheel rolls on, say, an indiarubber road, it sinks in and compresses the rubber immediately under it, but forces out the rubber in front and behind it, as shown in the sketch. That forced up in the front slides on the surface of the wheel in just the reverse direction to the motion of the wheel, and so hinders its progress. Likewise, as the wheel leaves the heap behind it, the rubber returns to its original

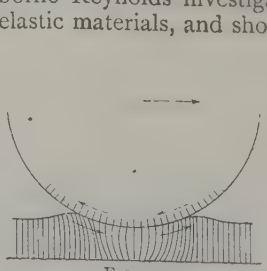


FIG. 293.

place, and again slips on the wheel in the reverse direction to its motion. Thus the resistance to rolling is in reality due to the sliding of the two surfaces. On account of the stretching of the path over which the wheel rolls, the actual length of path rolled over is greater than the horizontal distance travelled by the wheel, hence it does not travel its geometrical distance; the amount it falls short of it or the "slip" depends upon the hardness of the surfaces in contact. Even with the balls in ball bearings the "slip" is quite appreciable.

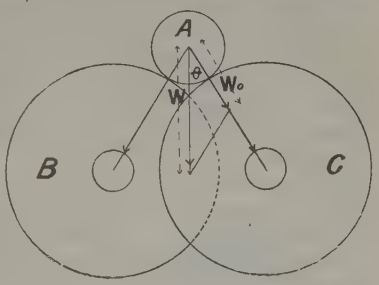


FIG. 294.

Antifriction Wheels.—In order to reduce the friction on an axle it is sometimes mounted on antifriction wheels, as shown. A is the axle in question, B and C are the antifriction wheels. If W be the load on the axle, the load on each antifriction wheel bearing will be—

$$W_0 = \frac{W}{2 \cos \theta}, \text{ and the load on both } \frac{W}{\cos \theta}$$

Let R_a = the radius of the main axle;
R = " " antifriction wheel;
r = " " axle of the antifriction wheel.

The rolling resistance on the surface of the wheels $= \dfrac{WK}{R_a \cos \theta}$

The frictional resistance referred to the surface of the antifriction wheels, or the surface of the main axle $= \dfrac{\mu W r}{R \cdot \cos \theta}$

$$\text{The total resistance} = \frac{W}{\cos \theta}\left(\frac{K}{R_a} + \frac{\mu r}{R}\right)$$

If the main axle were running in plain bearings, the resistance would be μW; hence—

$$\frac{\text{friction with plain bearings}}{\text{friction with antifriction wheels}} = \frac{\mu R \cos \theta}{K\frac{R}{R_a} + \mu r}$$

In some instances a single antifriction wheel is used, the axle A being kept vertically over the axle of the wheel by means of guides. The main trouble with all such devices is that the axle travels in an endwise direction unless prevented by some form of thrust bearing. One British Railway Company has had large numbers of waggons fitted with a single antifriction disc on each axle bearing; the rolling resistance is materially less than when fitted with ordinary bearings, but the discs are liable to get "cross cornered," and to give trouble in other ways.

Roller Bearings.—There are many forms of roller bearings in common use, but unfortunately few of them give really satisfactory results. The friction of even the worst of them is considerably lower than that of bearings provided with ordinary lubrication. If the rollers are not perfectly parallel in themselves (in a cylindrical bearing), and are not kept absolutely parallel with the shaft, they tend to roll in a helical path, but since the cage and casing prevent them from doing so, they press the cage against the flange of the casing and set up what

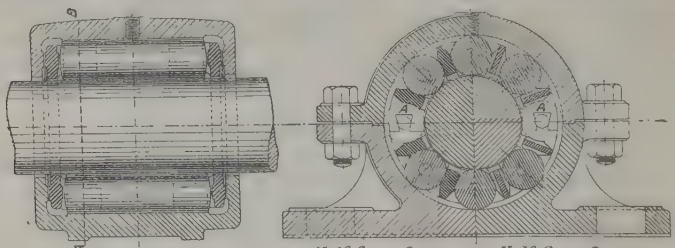


FIG. 295.

is known as "end-thrust," which thereby gives rise to a large amount of friction between the end of the cage and the flange of the bearing, and in other ways disturbs the smoothness of running. Few, if any, roller bearings are entirely free from this defect; it is moreover liable to vary greatly from time to time both as regards its amount and direction. In general, it is less at high speeds than at low, and it increases with the load on the bearing; it does not appear to be greatly affected by lubrication. Bearings in which the end-thrust is high nearly always show a high coefficient of friction, and *vice versâ*. High friction is always accompanied with a large amount of wear and vibration. In order to reduce wear and to ensure smooth

running, the rollers, sleeve, and liner should be of the hardest steel, very accurately ground and finished. In many of the cheaper forms of roller bearing no sleeve is used, hence the rollers are in direct contact with the shaft. The outer casing is usually split to allow of a bearing on a long line of shafting being replaced when necessary without removing the pulleys and couplings, or without taking the shafting down. This is an undoubted advantage which is not possessed by ball bearings. The reason why ball bearings will not run with split or jointed races will be obvious after reading the paragraphs devoted to ball bearings.

The commonest form of roller bearing is that shown in Fig. 295. There is no sleeve on the shaft and no liner in the casing; the steel rollers are kept in position by means of a gun-metal cage, which is split to allow of the rollers being readily removed.

Another cheap form of roller bearing which is extensively used is the Hyatt, in which the rollers take the form of helical springs; they are more flexible than solid rollers, and consequently accommodate themselves to imperfections of alignment and workmanship.

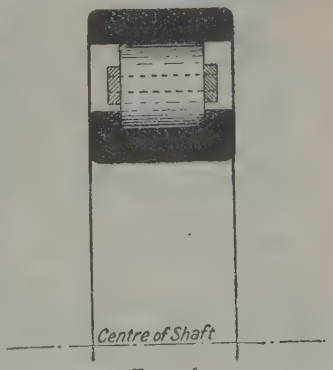


FIG. 296.

In the Hoffmann short roller bearing, Fig. 296, the length of the rollers is equal to their diameter; the roller paths and rollers are of the hardest steel ground to a great degree of accuracy. The end-thrust is almost negligible and the coefficient of friction is low; the bearings will run under a considerably higher load than a ball bearing of similar dimensions. The following table gives fair average results for a friction test of an ordinary roller bearing :—

Total load in lbs.	40 revolutions per minute.		400 revolutions per minute.	
	μ	End thrust in lbs.	μ	End thrust in lbs.
2000	0·0131	82	0·0053	51
4000	0·0094	147	0·0035	89
6000	0·0082	212	0·0029	128
8000	0·0070	270	0·0026	166
10,000	0·0072	340	0·0024	205

A test of a Hoffmann short roller bearing gave the following results :—

Total load in lbs.	μ	End thrust in lbs.	Temperature air at 62° F.
2000	0·0012	None	—
4000	0·0010	,,	—
6000	0·0008	7	76° F.
8000	0·0010	84	89° F.
10,000	0·0011	210	96° F.
12,000	0·0010	270	94° F.
14,000	0·0010	20	95° F.
14,500	0·0010	0	95° F.

Diameter of sleeve ...	4·75 inches	Hardened steel highly finished.
Diameter of rollers ...	1·00 inch	
Length of rollers ...	1·00 inch	
Number of rollers ...	14	
Revolutions per minute ...	400	

For details of other types of roller bearings and results of tests, the reader should refer to a paper by the author on "Roller and Ball Bearings," *Proceedings of the Institution of Civil Engineers*, vol. clxxxix.

Ball Bearings.—The "end-thrust" troubles that are experienced with roller bearings can be entirely avoided by the substitution of balls for rollers. The form of the ball path, however, requires careful consideration.

Various types of ball races are shown in diagrammatic form in Fig. 297. A is known as a four-point radial bearing, the outer cones screw into the casing, and thereby permit of adjustment as the bearing wears. B is a three-point bearing capable of similar adjustment. C is a three-point bearing; the inner coned rings are screwed on to the shaft, and can be tightened up as desired. None of these forms of adjustment are satisfactory for heavy loads. A four-point thrust bearing is shown at F; the races are ground to an angle of 45° with the axle, but since the circumferential speed of the race at *a* is greater than at *b*, the circle *aa* on the ball tends to rotate at a higher speed than the circle *bb*, but since this cannot occur, grinding and scratching of the ball take place. In order to avoid this defect, races were made as shown at G; the circle *aa* was greater than *bb* in the ratio $\frac{R_a}{R_b}$. By this means it was expected that a true rolling motion would occur, but the bearing was

not a success. The three-point bearing H was designed on similar lines, but a considerable amount of grinding of the balls took place.

To return to the radial bearings. A two-point bearing is shown at D; the balls rolled between two plain cylindrical surfaces. A cage was usually provided for holding the balls in their relative position. In E the balls ran in grooved races. In these bearings a true rolling motion is secured, the balls do

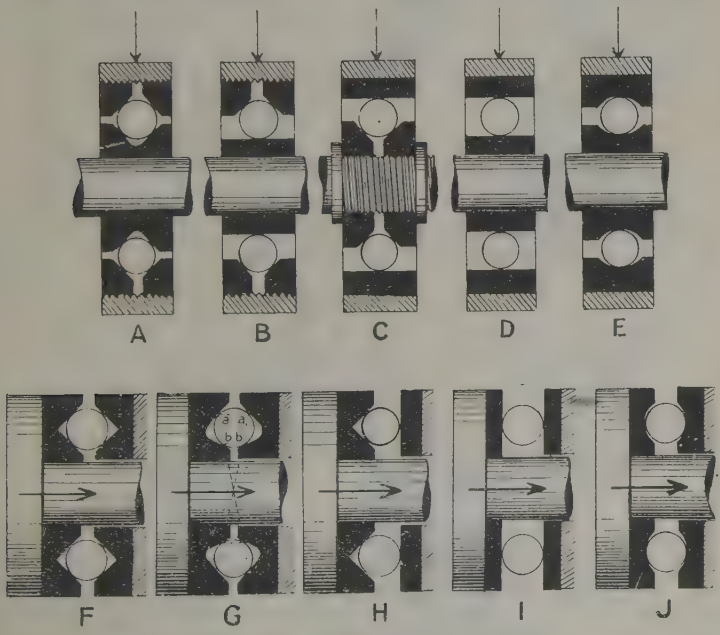


FIG. 297.

not grind or scratch, and the friction is considerably less than in A, B, C. Thrust bearings designed on similar lines are shown at I and J. A more detailed view of such bearings is shown in Fig. 298. The lower race is made with a spherical seat to allow it to swivel in case the shaft gets out of line with its housing, a very wise precaution which greatly increases the life of the bearing.

When designing bearings for very heavy loads, the difficulty is often experienced of placing in one row a sufficient number

of balls of the required diameter. In that case two or more rows or rings may be arranged concentrically, but it is almost impossible to get the workmanship sufficiently accurate, and to reduce the elastic strain on the housings to such an extent as to evenly distribute the load on both rings. The one set should therefore be backed with a sheet of linoleum or other soft material which will yield to a sufficient extent to equalize approximately the load on each ring. Such a bearing is shown in Fig. 299. The sheet-iron casing dips into an oil channel for the purpose of excluding dust. The lower half of the housing is made with a spherical seat.

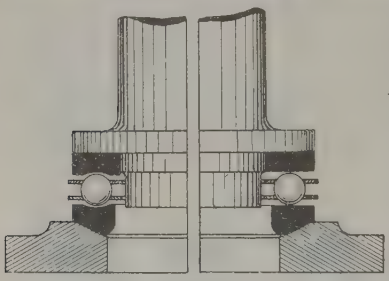


FIG. 298.

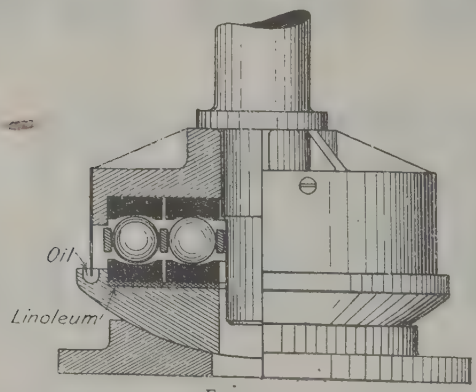


FIG. 299.

Modern cylindrical or radial bearings are almost always made of the two-point type; for special purposes plain cylindrical races may be used, but balls running in grooved races will safely carry much higher loads than when they run on plain cylinders. With plain races there is no difficulty in inserting the full number of balls in the bearing, but when grooved races are used, only about one-half the number can be inserted unless some special device be resorted to. But

since the load-carrying capacity of a bearing depends upon the number of balls it contains, it is evidently important to get the bearing as nearly filled as possible. After packing in and spacing as many balls as possible, the remaining balls are inserted through a transverse slot in one side of the race; the depth of this slot is slightly less than that of the groove in which the balls run, hence it does not in any way affect the smoothness of running. See Fig. 300.

When two or more rows of balls are used in a cylindrical bearing, each row must be provided with a separate ring. The inner ring or sleeve must be rigidly attached to the shaft, and the outer ring should be backed with linoleum in order

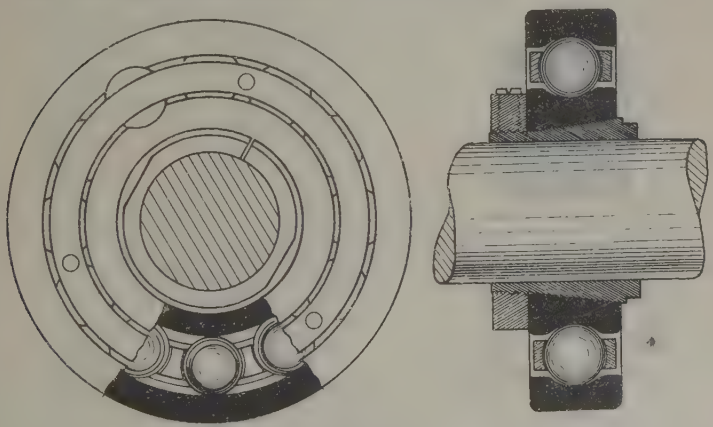


Fig. 300.

to evenly distribute the load on each row of balls. The housing itself should be provided with a spherical seating to allow for any want of alignment. A design for such a bearing is shown in the Author's paper referred to above. A special form of bearing has been designed by the Hoffmann Co. with the same object.

It is of great importance to attach the sleeve, or inner ring, of the bearing rigidly to the shaft. It is sometimes accomplished by shrinking; in that case the shrinkage must not be more than $\frac{\text{diameter of shaft}}{2000}$. If this proportion is exceeded the ring is liable to crack on cooling, or to expand to such an extent as to jamb the balls. Where shrinking is not resorted

to the ring is sometimes made taper in the bore, and is tightened on to the shaft by means of a nut, or by a clamping sleeve, as shown in Fig. 300.

In the Skefco ball bearing shown in Fig. 301 the outer ball race is ground to a spherical surface, and the sleeve is provided with grooved races to receive two rows of balls. By this arrangement the full number of balls can be packed in by tilting the inner race, but the great feature of the bearing is the spherical ball race which enables the bearing to be used on a shaft which does not run true, or on a machine in which the frame springs considerably relative to the shaft.

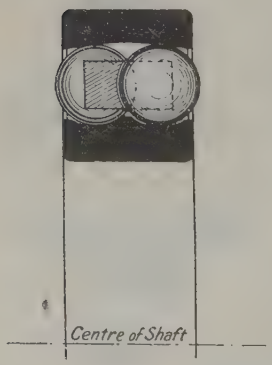


FIG. 301.

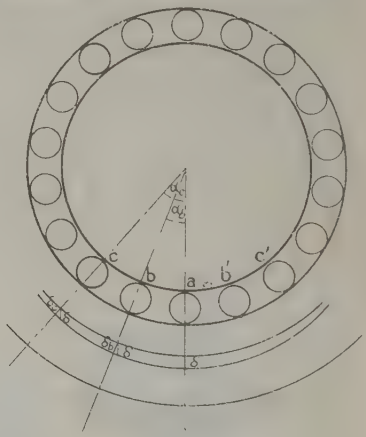


FIG. 302.

Approach of the Balls Race when under Load.—When an elastic ball is placed between two elastic surfaces which are pressed together, the ball yields under the pressure and the surfaces become hollow; the theory of the subject was first enunciated by Hertz, and afterwards by Heerwagen. The results obtained by the two theories are not identical, but there is no material difference between them. Experimental researches on the strain which steel balls undergo when loaded show that the theories are trustworthy within narrow limits. The following expression is due to Hertz.

Let δ = the amount, in inches, the plates approach one another when loaded.

P = the load on the ball in pounds.

d = the diameter of the ball in inches.

Then

$$\delta = \frac{1}{32{,}000}\sqrt[3]{\frac{P^2}{d}}$$

Distribution of the Load on the Balls of a Radial Bearing.—The load on the respective balls in a radial bearing may be arrived at by Stribeck's method. Thus—

When the bearing is loaded the inner race approaches the outer race by an amount δ_a. The load on the ball a immediately in the line of loading is p_a. The load on the adjacent balls b is less, because they are compressed a smaller amount than a, namely, $\delta_b = \delta_a \cos \alpha_b$. Similarly, $\delta_c = \delta_a \cos \alpha_c$. Let m be the number of balls in the bearing, then—

$$\alpha_b = \frac{360^\circ}{m} \text{ and } \alpha_c = \frac{2 \times 360^\circ}{m}$$

Then $\delta_a = \frac{1}{32{,}000}\sqrt[3]{\frac{P_a^2}{d}}$

hence $\frac{P_a^2}{\delta_a^3} = \frac{P_b^2}{\delta_b^3} = \frac{P_c^2}{\delta_c^3}$ = etc., when d does not vary.

$$P_b = \frac{P_a \delta_b^{\frac{3}{2}}}{\delta_a^{\frac{3}{2}}} = \frac{P_a \delta_a^{\frac{3}{2}} \cos^{\frac{3}{2}}\left(\frac{360}{m}\right)}{\delta_a^{\frac{3}{2}}} = P_a \cos^{\frac{3}{2}}\left(\frac{360}{m}\right)$$

and $P_c = P_a \cos^{\frac{3}{2}} 2\left(\frac{360}{m}\right)$.

The total load on the bearing W is—

$$W = P_a + 2P_b + 2P_c + \text{etc.}$$
$$= P_a\left[1 + 2\left\{\cos^{\frac{3}{2}}\left(\frac{360}{m}\right) + \cos^{\frac{3}{2}} 2\left(\frac{360}{m}\right) + \text{etc.}\right\}\right]$$

Examples—

m	10	15	20
$\frac{360}{m}$	36°	24°	18°
$\frac{W}{P_a}$	2·28	3·44	4·58
$m \div \frac{W}{P_a}$	4·38	4·37	4·36

Thus $P_a = \frac{4{\cdot}37W}{m}$

This expression is only true when there is no initial "shake" or "bind" in the bearing and no distortion of the ball races. To allow for such deficiencies Stribeck proposes—

$$P_a = \frac{5W}{m} \quad \text{and} \quad W = \frac{mP_a}{5}$$

Necessity for Accuracy of Workmanship.—The expression for the approach of the plates shows how very small is the amount for ordinary working conditions. In a one-inch ball the approach is about $\frac{1}{600}$ of an inch; hence any combined errors, such as hills on the races or balls to the extent of $\frac{1}{1000}$ of an inch, will increase the load on the ball by 50 per cent. Extreme accuracy in finishing the races and balls is therefore absolutely essential for success.

Friction of Ball Bearings.—The results of experiments tend to show that the friction of a ball bearing—

(1) Varies directly as the load;

(2) Is independent of the speed;

(3) Is independent of the temperature;

(4) The friction of rest is but very slightly greater than the friction of motion;

(5) Is not reduced by lubrication in a clean well-designed bearing.

The following results, which may be regarded as typical, lend support to the statements (1), (2), (3):—

Load, in lbs. ...	1000	2000	3000	4000	5000	6000	7000	8000	9000	10,000
Friction moment, inch-lbs. ...	5·0	6·0	8·4	12·8	17·0	20·4	25·2	30·4	36·0	40·0

Speed, in revs. per min. (approx.)...	5	50	100	500	800	1000
Friction moment, inch-lbs. ...	20·0	19·8	19·7	19·8	20·1	20·0

Temperature, Fahrenheit ...	58°	65°	77°	84°	98°
Friction moment, inch-lbs.	37·6	38·1	39·0	39·0	38·7

The curves given in Fig. 303 were obtained by an autographic recorder in the author's laboratory. They show clearly

that the friction of rest in the case of a ball bearing is practically the same as the friction of motion, and that it is very much less than that of an ordinary bearing.

Although the friction of a ball bearing is not reduced by lubrication, yet a small amount of lubrication is necessary in order to prevent rust and corrosion of the balls and races. Thick grease resembling vaseline, which has been freed from all traces of corrosive agents, is used by many makers; others find that the best lard oil is preferable, but in any case great care must be taken to get a lubricant which will not set up corrosion of the balls and races.

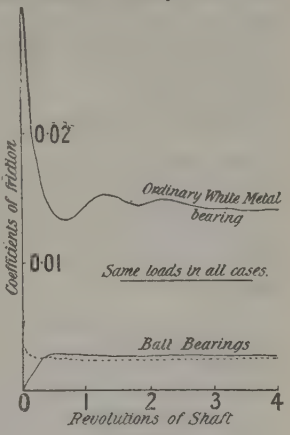


FIG. 303.

Cost of Ball Bearings.—The cost of a ball bearing or a first-class roller bearing is considerably greater than that of an ordinary bearing, but owing to the fact that they are more compact, and that the mechanical efficiency of a machine fitted with ball bearings is much higher than when fitted with ordinary bearings, a considerable saving in metal may be effected by their use; with the result that the first cost of some machines, such as electric motors, is actually less when fitted with ball bearings than with ordinary bearings. The quantity of lubricant required by a ball bearing is practically *nil*, and they moreover require practically no attention. Provided ball bearings are suitably proportioned for the load and speed, and are intelligently fitted and used, they possess great advantages over other types of bearings.

Safe Loads and Speeds for Ball Bearings.—The following expressions are based on the results of a large number of experiments by the author:—

W = The maximum load which may be placed on a bearing in pounds.

m = The number of balls in the bearing.

d = The diameter of the balls in inches.

N = The number of revolutions per minute made by the shaft.

D = The diameter of the ball race, measured to the middle of the balls, in inches.

$$W = \frac{K m d^3}{ND + Cd}$$

where K and C have the following values :—

Type of bearing	K	C
Cylindrical—no grooves	1,000,000	2000
Cylindrical—grooved races	2,000,000 to 2,500,000	2000
Thrust—no grooves	500,000	200
Thrust—grooved races	1,000,000 1,250,000	200

Information on ball bearings can also be found in the following publications :—*Engineering*, April 12, 1901 ; December 26, 1902 ; February 20, 1903. *Proceedings of the Institution of Civil Engineers*, vols. lxxxix. and clxxxix. "Machinery" Handbooks—"Ball Bearings."

Friction of Lubricated Surfaces.—The laws which appear to express the behaviour of well-lubricated surfaces are almost the reverse of those of dry surfaces. For the sake of comparison, we tabulate them below side by side—

Dry Surfaces.	*Lubricated Surfaces.*
1. The frictional resistance is nearly proportional to the normal pressure between the two surfaces.	1. The frictional resistance is almost independent of the pressure with bath lubrication so long as the oil film is not ruptured, and approaches the behaviour of dry surfaces as the lubrication becomes meagre.
2. The frictional resistance is nearly independent of the speed for low pressures. For high pressures it tends to decrease as the speed increases.	2. The frictional resistance of a flooded bearing, when the temperature is artificially controlled, increases (except at very low speeds) nearly as the speed, but when the temperature is not controlled the friction does not appear to follow any definite law. It is high at low speeds of rubbing, decreases as the speed increases, reaches a minimum at a speed dependent upon the temperature and the intensity of pressure ; at higher speeds it appears to increase as the square root of the speed ; and finally, at speeds of over 3000 feet per minute, some believe that it remains constant.

3. The frictional resistance is not greatly affected by the temperature.	3. The frictional resistance depends largely upon the temperature of the bearing, partly due to the variation in the viscosity of the oil, and partly to the fact that the diameter of the bearing increases with a rise of temperature more rapidly than the diameter of the shaft, and thereby relieves the bearing of side pressure.
4. The frictional resistance depends largely upon the nature of the material of which the rubbing surfaces are composed.	4. The frictional resistance with a flooded bearing depends but slightly upon the nature of the material of which the surfaces are composed, but as the lubrication becomes meagre, the friction follows much the same laws as in the case of dry surfaces.
5. The friction of rest is slightly greater than the friction of motion.	5. The friction of rest is enormously greater than the friction of motion, especially if thin lubricants be used, probably due to their being squeezed out when standing.
6. When the pressures between the surfaces become excessive, seizing occurs.	6. When the pressures between the surfaces become excessive, which is at a much higher pressure than with dry surfaces, the lubricant is squeezed out and seizing occurs. The pressure at which this occurs depends upon the viscosity of the lubricant.
7. The frictional resistance is greatest at first, and rapidly decreases with the time after the two surfaces are brought together, probably due to the polishing of the surfaces.	7. The frictional resistance is least at first, and rapidly increases with the time after the two surfaces are brought together, probably due to the partial squeezing out of the lubricant.
8. The frictional resistance is always greater immediately after reversal of direction of sliding.	8. Same as in the case of dry surfaces.

The following instances will serve to show the nature of the experimental evidence upon which the above laws are based.

1. The frictional resistance is independent of the pressure with bath lubrication.

TOWER'S EXPERIMENT

Lubricant.	Pressure in pounds per square inch.						
	100	153	205	310	415	520	625
	Frictional resistance in pounds.						
Olive oil	0·89	0·87	0·82	0·84	—	0·87	—
Lard oil	0 90	0·80	0·86	0·90	0·87	0·87	—
Sperm oil	0·64	0·57	0·55	0·59	—	—	—
Mineral grease ...	—	1·27	1·35	1·24	1·12	1·14	1·25

The results shown in Fig. 304 were obtained from the author's friction testing machine. In the case of the "oil

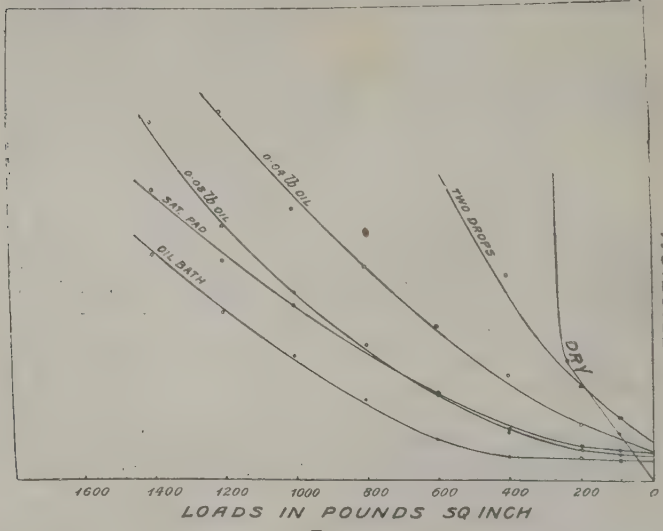


FIG. 304.

bath" the film was ruptured at a pressure of about 400 lbs. per square inch, after which the friction varied in the same manner as a poorly lubricated bearing. It is of interest to note that the friction of a dry bearing is actually less than that of a flooded bearing when the intensity of pressure is low.

2. The manner in which the friction of a flooded bearing varies with the velocity of rubbing is shown in Fig. 305. Curves A and B were obtained from a solid bush bearing such as a lathe neck by Heiman (*Zeitschrift des Vereines Deutscher Ingenieure*, Bd. 49, p. 1161). Curves C and D were

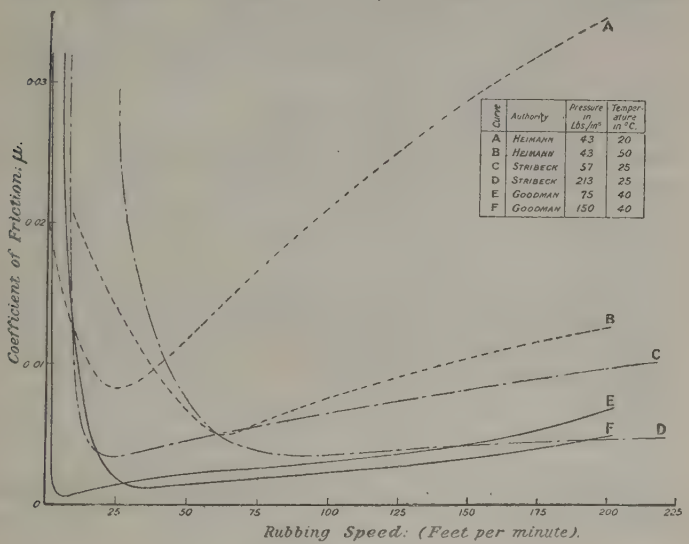


FIG. 305.

obtained by Stribeck (*Z. des V. D. Ing.*, September 6, 1902) with double ring lubrication. Curves E and F were obtained by the Author with bath lubrication (*Proceedings I. C. E.*, vol. clxxxix.).

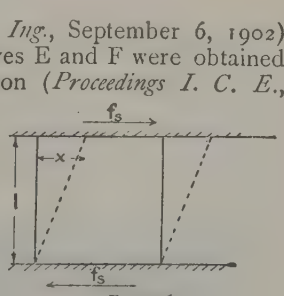


FIG. 306.

The erratic fashion in which the friction varies is due to many complex actions, which have not as yet been reduced to rigid mathematical treatment, although Osborne Reynolds, Sommerfeld, Petroff, and others have done much excellent work in this direction. An examination of the problem on the assumption that the friction is due to the shearing of a viscous film of oil of uniform thickness is of interest, although it does not give results entirely in accord with experiments.

In the theory of the shear of an elastic body we have the relation (see page 376)—

$$\frac{x}{l} = \frac{f_s}{G} = \frac{F_s}{AG}$$

where f_s is the intensity of shear stress.
F_s is the total resistance to shear.
A is the cross-sectional area of the element subjected to shear.
G is the modulus of rigidity.

But in the case of viscous fluids in which the resistance to flow varies directly as the speed S, we have—

$$\frac{S}{l} = \frac{f_s}{K} = \frac{F_s}{AK} \quad \text{and} \quad F_s = \frac{AKS}{l}$$

where K is the coefficient of viscosity.

When a journal runs in a solid cylindrical bush of diameter d and length L with a film of oil of uniform thickness interposed, the friction of the journal is—

$$F_s = \frac{\pi d LKS}{l}$$

If W be the load on the journal, and μ be the coefficient of friction, then

$$\mu = \frac{F_s}{W} = \frac{\pi d LKS}{Wl}$$

If p be the *nominal* intensity of pressure—

$$p = \frac{W}{dL}$$

$$\text{hence } \mu = \frac{\pi KS}{pl} = \frac{cS}{p}$$

From this relation we should expect that the coefficient of friction in an oil-borne brass would vary directly as the speed, as the coefficient of viscosity, also inversely as the intensity of pressure and as the thickness of the film. But owing to disturbing factors this relation is not found to hold in actual bearings. Osborne Reynolds and Sommerfeld have pointed out that the thickness of the oil film on the "on" side of a brass is greater than on the "off" side. The author has

experimentally proved that this is the case by direct measurement, and indirectly by showing that the wear on the "off" side is greater than on the "on" side. The above-mentioned writers have also shown that the difference in the thickness on the two sides depends on the speed of rotation, the eccentricity being greatest at low speeds. Reynolds has shown that the friction increases with the eccentricity; hence at low velocities the effect of the eccentricity is predominant, but as the speed increases it diminishes with a corresponding reduction in the friction until the minimum value is reached (see Fig. 305). After the minimum is passed the effect of the eccentricity becomes less important, and if the temperature of the oil film remained constant the friction would vary very nearly as the speed, but owing to the fact that more heat is generated in shearing the oil film at high speeds than at low the temperature of the film increases, and thereby reduces the viscosity of the oil, also the friction with the result that it increases less rapidly than a direct ratio of the speed. Experiments show that when the temperature is not controlled the friction varies more nearly as the square root of the speed. This square root relation appears to hold between the minimum point and about 500 feet per minute; above that speed it increases less rapidly than the square root, and when it exceeds about 3000 feet per minute the friction appears to remain constant at all speeds. For a flooded bearing in which the temperature is not controlled the friction appears to follow the law—

$$\mu = \frac{c\sqrt{S}}{p}$$

between the above-mentioned limits.

The fact must not be overlooked that the deviation from the straight line law of friction and speed after the minimum is passed is largely due to the fact that the temperature of the film does not remain constant. In some tests made by the author in 1885 (*Proceedings Inst. C. Engineers*, vol. lxxxix., page 449) the friction was found to vary directly as the speed when the temperature was controlled by circulating cooling water through the shaft. The speeds varied from 4 to 200 feet per minute with both bath and pad lubrication, and with brasses embracing arcs from 180° to 30°. Tests of a similar character made on white metals on a large testing machine also showed the straight line law to hold between about 15 and 1000 feet per minute with both bath and pad lubrication.

Where the temperature of the bearing is controlled, the friction-speed relation appears to closely agree with that deduced above from viscosity considerations, viz.—

$$\mu = \frac{cS}{p}$$

3. The curves given in Fig. 307 show the relation between

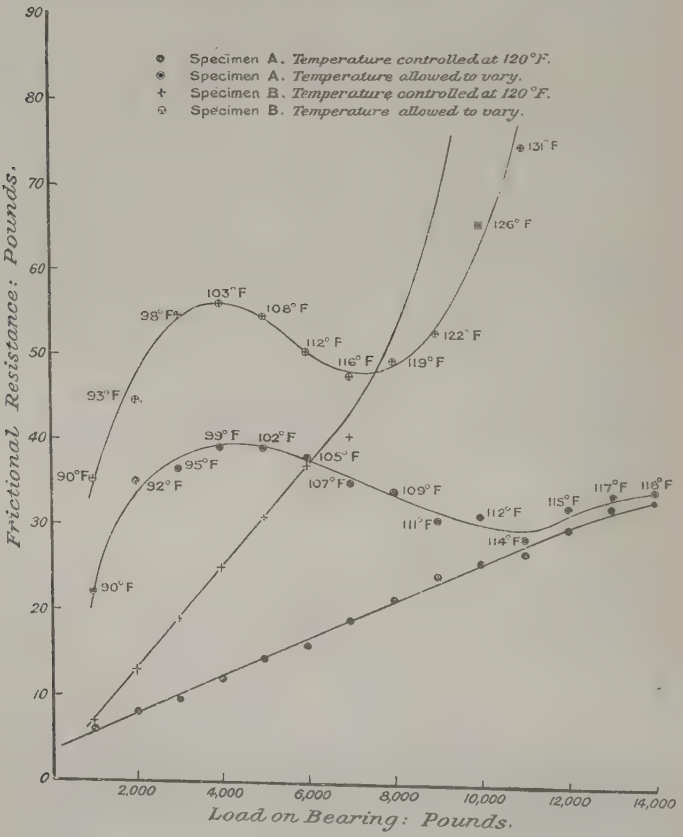


FIG. 307.

the friction and the temperature. When other conditions are kept constant the relation between the coefficient of friction

μ and the temperature t may be represented approximately by the empirical expression—

$$\mu_t = \frac{\text{constant}}{t - 40} + 0{\cdot}001$$

where μ_t is the coefficient of friction at the temperature t F. Thus, if the coefficient at 60° F. is 0·0176 the constant is 0·332, and the coefficient at 120° F. would be 0·0051.

Tower showed that the relation

$$\mu_t = \frac{\text{constant}}{t}$$

closely held for many of his tests.

When making comparative friction tests of bearing metals it is of great importance to control the temperature of the bearing at a predetermined point for all the metals under test.

In all friction testing machines provision should be made for circulating water through the shaft or the bearing for controlling the temperature.

The curves in Fig. 307 shows the effect of controlling the

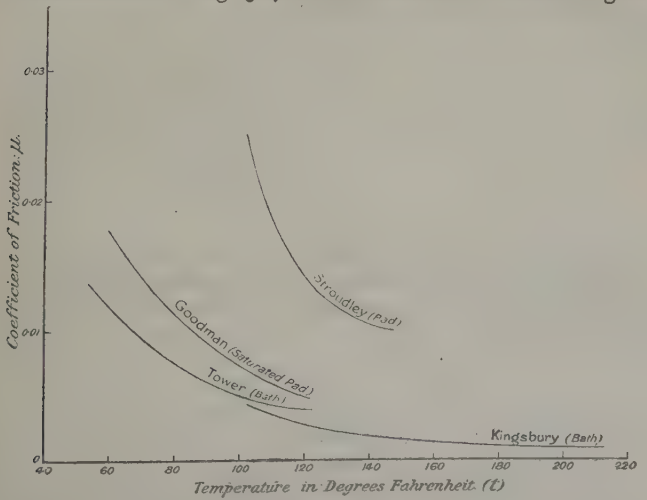


FIG. 307*a*.

temperature when testing white antifriction metals, also of allowing it to vary as the test proceeds.

4. Mr. Tower and others have shown that in the case of a

flooded bearing there is no metallic contact between the shaft and bearing; it is therefore quite evident that under such circumstances the material of which the bearing is composed makes no difference to the friction. When the author first began to experiment on the relative friction of antifriction metals, he used profuse lubrication, and was quite unable to detect the slightest difference in the friction; but on using the smallest amount of oil consistent with security against seizing, he was able to detect a very great deal of difference in the friction. In the table below, the two metals A and B only differed in composition by changing one ingredient, amounting to 0·23 per cent. of the whole.

Load in lbs. sq. inch		150	250	350	450	750	950
Coefficient of friction	A	0·0143	0·0112	0·0091	0·0082	0·0075	0·0083
	B	0·0083	0·0062	0·0054	0·0050	0·0045	0·0047

5. The following tests by Thurston will show how much greater is the friction of rest than of motion :—

Load in lbs. sq. inch		50	100	250	500	750	1000
Coefficient of friction	150 ft. min.	0·013	0·008	0·005	0·004	0·0043	0·009
	At instant of starting	0·07	0·135	0·14	0·15	0·185	0·18

Oil used, sperm.

The ratio between the starting and the running coefficients depends largely upon the viscosity of the oil, as shown by the following tests by the author. See *Proceedings Inst. C. E.*, vol. lxxxix. p. 433.

Lubricant.		Coefficient of friction.		Ratio.
		Running.	At starting.	
Machinery oil		0·0084	0·192	22·9
Thick valve oil	A	0·0329	0·171	5·2
	B	0·0252	0·147	5·8
Grease		0·0350	0·090	2·6

6. Experiments by Tower and others show that a steel shaft in a gun-metal bearing seizes at about 600 lbs. square inch under steady running, whereas when dry the same materials

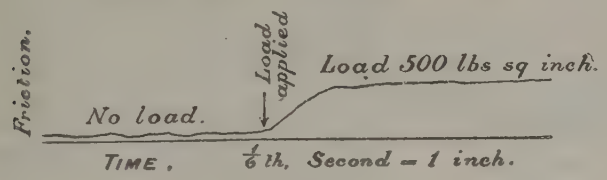


FIG. 308.

seize at about 80 lbs. square inch. The author finds that the seizing pressure increases as the viscosity of the oil increases.

7. Fig. 308 is one of many drawn autographically on the author's machine. The lever which applies the load on the bearing was lifted, and the machine allowed to run with only the weight of the bearing itself upon it; the lever was then suddenly dropped, the friction being recorded automatically.

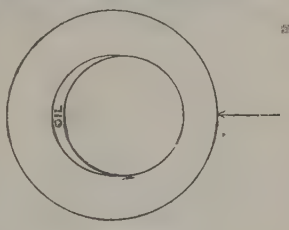


FIG. 309.

An indirect proof of this statement is to be found in the case of connecting-rod ends, and on pins on which the load is constantly reversed; at each stroke the oil is squeezed away from the pressure side of the pin to the other side. Then, when the pressure is reversed, there is a large supply of oil between the bearing and the pin, which gradually flows to the other side. Hence at first the bearing is floating on oil, and the friction is consequently very small; as the oil flows away, the friction increases. This is the reason why a much higher bearing pressure may be allowed in the case of a connecting-rod end than in a constantly revolving bearing.

8. In friction-testing machines it is always found that the temperature and the friction of a bearing is higher after reversal of direction, but in the course of a few hours it gets back to the normal again. Some metals, however, appear to have a grain, as the friction is always much greater when running one way than when running the other way.

Nominal Area of Bearing.—The pressure on a cylindrical bearing varies from point to point; when the lubrication

is very meagre or with a dry bearing it is a maximum at the crown, and is least at the two sides. When the bearing is flooded with oil the distribution of pressure can be calculated from hydrodynamical principles, an account of which will be found in Dr. Nicolson's paper, "Friction and Lubrication," read before the Manchester Association of Engineers, November, 1907.

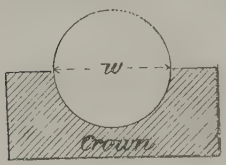


FIG. 310.

For the purpose of comparing roughly the intensity of pressure on two bearings, the pressure is assumed to be evenly distributed over the projected area of the bearing. Thus, if w be the width of the bearing across the chord, and l the length of the bearing, the nominal area is wl, and the nominal pressure per square inch is $\frac{W}{wl}$, where W is the total load on the bearing.

Beauchamp Tower's Experiments.—These experiments were carried out for a research committee of the

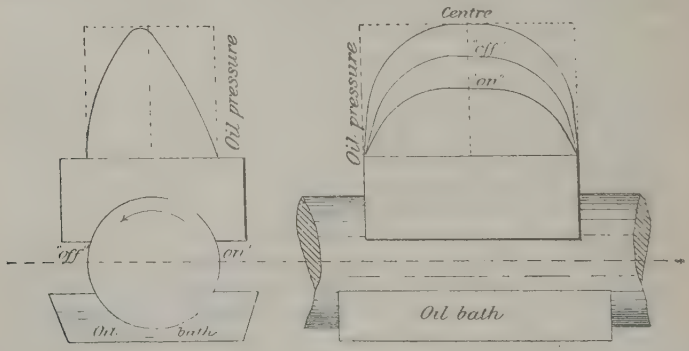


FIG. 311.

Institution of Mechanical Engineers, and deservedly hold the highest place amongst friction experiments as regards accuracy. The reader is referred to the Reports for full details in the Institution *Proceedings*, 1885.

Most of the experiments were carried out with oil-bath lubrication, on account of the difficulty of getting regular lubrication by any other system. It was found that the bearing was completely oil-borne, and that the oil pressure

varied as shown in Fig. 311, the pressure being greatest on the "off" side. In this connection Mr. Tower shows that it is useless—worse than useless—to drill an oil-hole on the resultant line of pressure of a bearing, for not only is it impossible for oil to be fed to the bearing by such means, but oil is also collected from other sources and forced *out* of the hole (Fig. 312), thus robbing the bearings of oil at exactly the spot where it is most required. If oil-holes are used, they must communicate with a part of the bearing where there is little or no pressure (Fig. 313). The position of the point of minimum pressure depends somewhat on the speed of rotation.

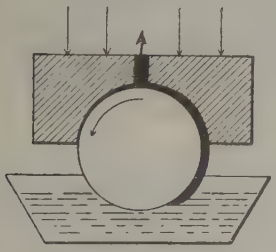


FIG. 312.

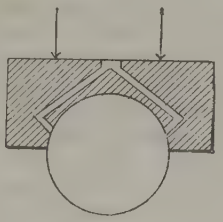


FIG. 313.

A general summary of the results obtained by Mr. Tower are given in the following table. The oil used was rape; the speed of rubbing 150 feet per minute; and the temperature about 90° F.:—

Form of bearing.							
Load at which seizing occurred, in lbs. sq. inch.	100	150	370	550	600	200	90
Coefficient of friction	0·01	0·01	0·006	0·006	0·001	0·01	0·035

Other of Mr. Tower's experiments are referred to in preceding and succeeding paragraphs.

Professor Osborne Reynolds' Investigations.—A theoretical treatment of the friction of a flooded bearing has been investigated by Professor Osborne Reynolds, a full

account of which will be found in the *Philosophical Transactions*, Part I, 1886; see also his published papers, Vol. II. p. 228. In this investigation, he has shown a complete agreement between theory and experiment as regards the total frictional resistance of a flooded bearing, the distribution of oil pressure, and the thickness of the oil film, besides many other points of the greatest interest. Professor Petroff, of St. Petersburg and Sommerfeld have also done very similar work. A convenient summary of the work done by the above-mentioned writers will be found in Archbutt and Deeley's "Lubrication and Lubricants."

The theory of the distribution of oil pressure in a flooded collar bearing has been recently investigated by Michell, who has successfully applied it to the more difficult problem of producing an oil-borne thrust bearing. See a paper by Newbigin in the *Proceedings of the Institution of Civil Engineers*, Session 1913–1914; also *Zeitschrift für Mathematik und Physik*, Vol. 52, 1905.

Goodman's Experiments.—The author, shortly after the results of Mr. Tower's experiments were published, repeated his experiments on a much larger machine belonging to the L. B. & S. C. Railway Company; he further found that the oil pressure could only be registered when the bearing was flooded; if a sponge saturated with oil were applied to the bearing, the pressure was immediately shown on the gauge, but as the oil ran away and the supply fell off, so the pressure fell.

In another case a bearing was provided with an oil-hole on the resultant line of pressure, to which a screw-down valve was attached. When the oil-hole was open the friction on the bearing was very nearly 25 per cent. greater than when it was closed and the oil thereby prevented from escaping.

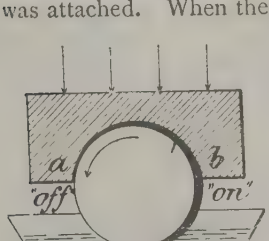


FIG. 314.

Another bearing was fitted with a micrometer screw for the purpose of measuring the thickness of the oil film; in one instance, in which the conditions were similar to those assumed by Professor Reynolds, the thickness by measurement was found to be 0·0004 inch, and by his calculation 0·0006 inch. By the same appliance the author found that the thickness was greater on the "on" side than on the "off" side of the bearing. The wear always takes place where the film is thinnest, *i.e.* on the "off" side of the bearing—

exactly the reverse of what would be expected if the shaft were regarded as a roller, and the bearing as being rolled forwards. When white metal bearings are tested to destruction, the metal always begins to fuse on the "off" side first.

The side on which the wear takes place depends, however, upon the arc of the bearing in contact with the shaft. When the arc subtends an angle greater than about 90° (with white metal bearings this angle is nearer 60°) the wear is on the off side; if less than 90°, on the "on" side. This wear was measured thus: The four screws, *a*, *a*, *a*, *a*, were fitted to an overhanging lip

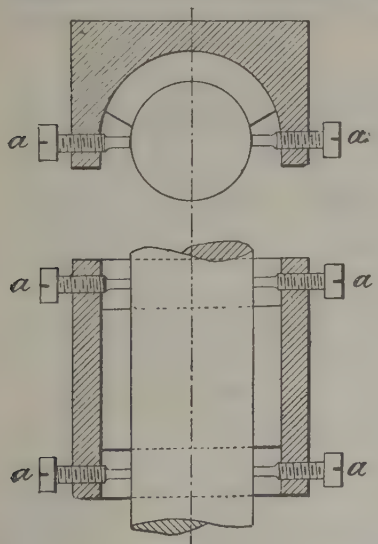


Fig. 315.

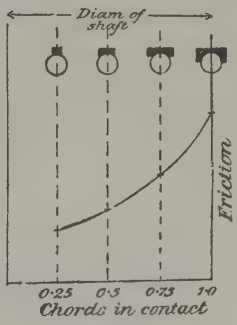


Fig. 316.

on the bearing as shown. They were composed of soft brass. Before commencing a run, they were all tightened up to just touch the shaft; on removing the bearing after some weeks' running, it was seen at once which screws had been bearing and which were free.

Another set of experiments were made in 1885, to ascertain the effect of cutting away the sides of a bearing. The bearings experimented upon were semicircular to begin with, and the sides were afterwards cut away step by step till the width of the bearing was only $\frac{1}{4}$ *d*. The effect of removing the sides is shown in Fig. 316.

The relation may be expressed by the following empirical formula :—

Let R = frictional resistance ;

$$C = \frac{\text{width of chord in contact}}{\text{diameter of journal}}$$

K and N are constants for any given bearing. Then—

$$R = K + NC^2$$

Methods of Lubricating.—In some instances a small force-pump is used to force the oil into the bearing ; it then becomes equivalent to bath lubrication. Many high-speed

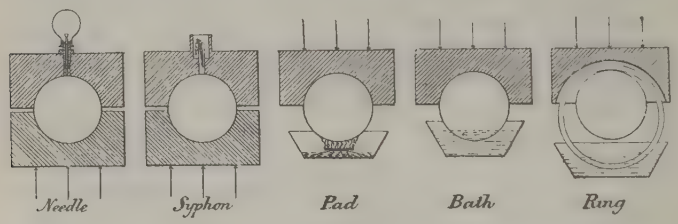


FIG. 317.

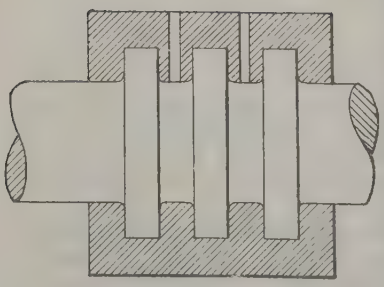


FIG. 318.—Collar bearing.

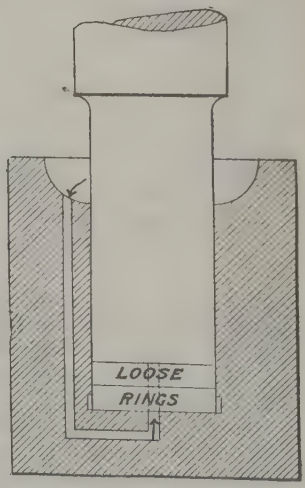


FIG. 319.—Pivot or footstep.

engines and turbines are now lubricated in this way. The oil is forced into every bearing, and the surplus runs back into

a receiver, where it is filtered and cooled. When forced lubrication is adopted with solid bushes, or in bearings in which the load constantly changes in direction, as in connecting rods, the clearance must be considerably greater than when meagre lubrication is supplied. The clearances usually adopted are about one thousandth of the diameter of the shaft for ordinary lubrication, and rather more for flooded bearings.

Relative Friction of Different Systems of Lubrication.

Mode of lubrication.	Tower.	Goodman.
Bath	1·00	1·00
Saturated pad ...	—	1·32
Ordinary pad ...	6·48	2·21
Syphon	7·06	4·20

Seizing of Bearings.—It is well known that when a bearing is excessively loaded, the lubricant is squeezed out, and the friction takes place between metal and metal; the two surfaces then appear to weld themselves together, and, if the bearing be forced round, small pieces are torn out of both surfaces. The load at which this occurs depends much upon the initial smoothness of the surfaces and upon the nature of the material, but chiefly upon the viscosity of the oil. If only the

Fig. 320.

viscosity can be kept up by artificially keeping the bearing cool, by water-circulation or otherwise, the surfaces will not seize until the pressure becomes enormous. The author has had a bearing running for weeks under a load of *two tons per square inch* at a surface velocity of 230 feet per minute with pad lubrication, temperature being artificially kept at 110° Fahr. by circulating water through the axle.[1]

Seizing not unfrequently occurs through unintentional high pressures on the edges of bearings. A very small amount of

[1] In another instance nearly *four* tons per square inch for several hours.

spring will cause a shaft to bear on practically the edge of the bearing (Fig. 320), and thereby to set up a very intense pressure. This can be readily avoided by using spherical-seated bearings. For several examples of such bearings the reader is referred to books on "Machine Design."

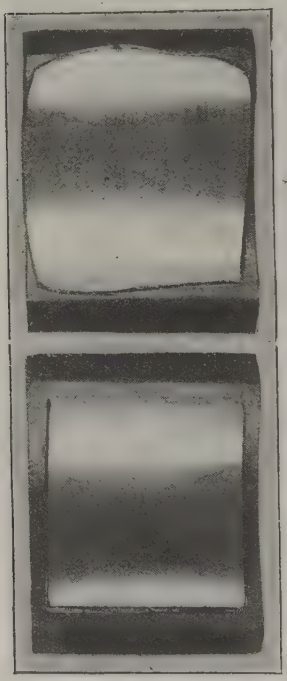


FIG. 321.

Seizing is very rare indeed with soft white metal bearings; this is probably due to the metal flowing and adjusting itself when any uneven pressure comes upon it. This flowing action is seen clearly in Fig. 321, which is from photographs. The lower portion shows the bearing before it was tested in a friction-testing machine, and the upper portion after it was tested. The metal began to flow at a temperature of 370° Fahr., under a pressure of 2000 lbs. per square inch: surface speed, 2094 feet per minute. The conditions under which the oil film ruptures in a flooded bearing will be discussed shortly.

Bearing Metals. — If a bearing can be kept completely oil-borne, as in Tower's oil-bath experiments, the quality of the bearing metal is of very little importance, because the shaft is not in contact with the bearing; but unfortunately, such ideal conditions can rarely be ensured, hence the nature of the bearing metal is one of great importance, and the designer must very carefully consider the conditions under which the bearing will work before deciding upon what metal he will use in any given case. Before going further, it will be well to point out that in the case of bearings running at a moderate speed under moderate loads, practically any material will answer perfectly; but these are not the cases that cause anxiety and give trouble to all concerned: the really troublesome bearings are those that have to run under extremely heavy loads or at very high speeds, and perhaps both.

The first question to be considered is whether the bearing

will be subjected to blows or not; if so, a hard tough metal must be used, but if not, a soft white metal will give far better frictional and wearing results than a harder metal. It would be extremely foolish to put such a metal into the connecting-rod ends of a gas or oil engine, unless it was thoroughly encased to prevent spreading, but for a steadily running journal nothing could be better.

The following is believed to be a fair statement of the relative advantages and disadvantages of soft white metal for bearings :—

SOFT WHITE METALS FOR BEARINGS.

Advantages.	*Disadvantages.*
The friction is much lower than with hard bronzes, cast-iron, etc., hence it is less liable to heat.	Will not stand the hammering action that some shafts are subjected to.
The wear is very small indeed after the bearing has once got well bedded (see disadvantages).	The wear is very rapid at first if the shaft is at all rough; the action resembles that of a new file on lead. At first the file cuts rapidly, but it soon clogs, and then ceases to act as a file.
It rarely scores the shaft, even if the bearing heats.	It is liable to melt out if the bearing runs hot.
It absorbs any grit that may get into the bearing, instead of allowing it to churn round and round, and so cause damage.	If made of unsuitable material it is liable to corrode.

As far as the author's tests go, amounting to over one hundred different metals on a 6-inch axle up to loads of 10 tons, and speeds up to 1500 revolutions per minute, he finds that ordinary commercial lead gives excellent results under moderate pressure: the friction is lower than that of any other metal he has tested, and, provided the pressure does not greatly exceed 300 lbs. per square inch, the wear is not excessive.

A series of tests made by the author for the purpose of ascertaining the effect of adding to antifriction alloys small quantities of metals whose atomic volume differed from that of the bulk, yielded very interesting results. The bulk metal under test consisted of lead, 80; antimony, 15; tin, 5; and the added metal, 0·25. With the exception of one or two metals, which for other reasons gave anomalous results, it was found that the addition of a metal whose atomic volume was greater than that of the bulk caused a diminution in the friction, whereas the addition of a metal whose atomic volume was less than that of the bulk caused an increase in the friction, and

metals of the same atomic volume had apparently no effect on the friction.

All white metals are improved if thoroughly cleaned by stirring in sal ammoniac and plumbago when in a molten state.

Area of Bearing Surfaces.—From our remarks on seizing it will be evident that the safe working pressure for revolving bearings largely depends upon their temperature and the lubricant that is used. If the temperature rise abnormally, the viscosity of the oil is so reduced that it gets squeezed out. The temperature that a bearing attains to depends (1) on the heat generated; (2) on the means for conducting away the heat.

Let S = surface speed in feet per minute;
W = load on the bearing in pounds;
t_u = number of thermal units conducted away per square inch of bearing per minute in order to keep the temperature down to the desired limit.

The thermal units generated per minute $= \frac{\mu WS}{773}$

The nominal area of bearing surface in square inches, viz. dL $\Big\} = \frac{\mu WS}{773 t_u}$

As a first approximation the following values of μ and t_u may be assumed:—

VALUES OF μ AND t_u.

Method of lubrication.	Value of μ.
Bath	0·004
Pad	0·012
Syphon	0·020

Conditions of running.	Values of t_u.			
	Crank and other pins.	Continuous running bearing.	Crank and other pins.	Continuous running bearing.
	Maximum temperature of bearing.			
	140° F.	140° F.	100° F.	100° F.
Exposed to currents of cold air or other means of cooling, as in locomotive or car axles	4-7	1-1·5	2-3·5	0·5 0·75
In tolerably cool places, as in marine and stationary engines	0·75-1	0·3-0·5	0·4-0·5	0·15-0·25
In hot places and where heat is not readily conducted away	0·4-0·5	0·1-0·3	0·2 0·25	—

After arriving at the area by the method given above, it should be checked to see that the pressure is not excessive.

Bearing.	Maximum permissible pressure in lbs. per sq. inch.
Crank-pins.—Locomotive	1500
Marine and stationary	600
Shearing machines	3000
Gudgeon pins.—Locomotive	2000
Marine and stationary	800
Railway car axles	350
Ordinary pedestals.—Gun-metal	200
Good white metal	500
Collar and thrust bearings.—Gun-metal	80
Good white metal	200
Lignum vitæ	50
Slide blocks.—Cast-iron or gun-metal	80
Good white metal	250
Chain and rope pulleys for cranes.—Gun-metal bush ...	1000

Work absorbed in Revolving Bearings.

Let W = total load on bearing in pounds ;
D = diameter of bearing in inches ;
N = number of revolutions per minute ;
L = length of journal in inches.

For Cylindrical Bearings.—

$$\left.\begin{array}{l}\text{Work done per minute}\\ \text{in foot-pounds}\end{array}\right\} = \frac{\mu W \pi D N}{12}$$

$$\text{horse-power absorbed} = \frac{W \pi D N \mu}{12 \times 33{,}000} = \frac{\mu W D N}{126{,}000}$$

A convenient rough-and-ready estimate of the work absorbed by a bearing can be made by assuming that the frictional resistance F on the surface of a bearing is 3 lbs. per square inch for ordinary lubrication, 2 lbs. for pad, 1 lb. for bath, the surface being reckoned on the nominal area.

$$\left.\begin{array}{l}\text{Work done in overcoming the friction}\\ \text{per minute in foot-pounds}\end{array}\right\} = \frac{\pi D^2 L F N}{12}$$

Flat Pivot.—If the thrust be evenly distributed over the whole surface, the intensity of pressure is—

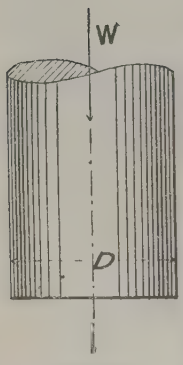
$$p = \frac{W}{\pi R^2}$$

pressure on an elementary ring $= 2\pi rp \,.\, dr$

moment of friction on an elementary ring $= 2\pi r^2 \mu p \,.\, dr$

moment of friction on whole surface $= 2\pi\mu p \int r^2 \,.\, dr$

$$M_f = \frac{2\pi\mu p R^3}{3}$$



Substituting the value of p from above—

$$M_f = \tfrac{2}{3}\mu WR$$

$$\text{work done per minute in foot-pounds} = \frac{\mu WDN}{5{\cdot}73}$$

$$\text{horse-power absorbed} = \frac{\mu WDN}{189{,}000}$$

This result might have been arrived at thus: Assuming the load evenly distributed, the triangle (Fig. 323) shows the distribution of pressure, and consequently the distribution of the friction. The centre of gravity of the triangle is then the position of the resultant friction, which therefore acts at a radius equal to $\frac{2}{3}$ radius of the pivot.

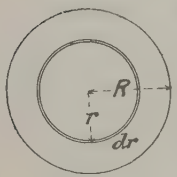


FIG. 322.

If it be assumed that the unequal wear of the pivot causes the pressure to be unevenly distributed in such a manner that the product of the normal pressure p and the velocity of rubbing V be a constant, we get a different value for M_f; the $\frac{2}{3}$ becomes $\frac{1}{2}$. It is very uncertain, however, which is the true value. The same remark also applies to the two following paragraphs.

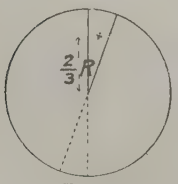


FIG. 323.

Collar Bearing (Fig. 325).—By similar reasoning to that given above, we get—

$$\left.\begin{matrix}\text{Moment of friction} \\ \text{on collar}\end{matrix}\right\} = 2\pi\mu p \int_{r=R_2}^{r=R_1} r^2 \,.\, dr$$

$$M_f = \frac{2\mu W(R_1^3 - R_2^3)}{3(R_1^2 - R_2^2)}$$

Conical Pivot.—The intensity of pressure p all over the surface is the same, whatever may be the angle α.

Let P_0 be the pressure acting on one half of the cone—

$$P_0 = \frac{W}{2 \sin \alpha}$$

The area of half the surface of the cone is—

$$A = \frac{\pi RL}{2} = \frac{\pi R^2}{2 \sin \alpha}$$

$$p = \frac{P_0}{A} = \frac{W \cdot 2 \sin \alpha}{2 \sin \alpha \cdot \pi R^2} = \frac{W}{\pi R^2} = \frac{\text{weight}}{\text{projected area}}$$

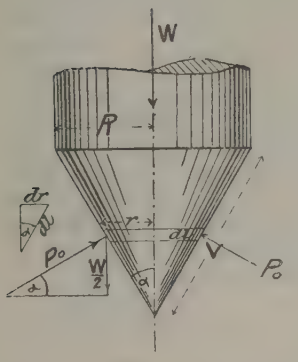


FIG. 324.

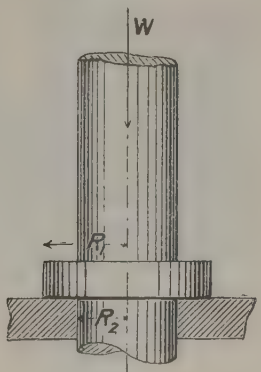


FIG. 325.

Total normal pressure on any elementary ring $= 2\pi rp \,.\, dl$

moment of friction on elementary ring $= 2\pi r^2 \mu p \,.\, dl$

$$\left(\text{but } dl = \frac{dr}{\sin \alpha}\right) = \frac{2\pi r^2 \mu p dr}{\sin \alpha}$$

$$\text{moment of friction on whole surface} = \frac{2\pi\mu p}{\sin \alpha} \int r^2 \,.\, dr$$

$$M_f = \frac{2\pi\mu p R^3}{3 \sin \alpha}$$

Substituting the value of p, we have—

$$M_f = \frac{2\mu WR}{3 \sin \alpha}$$

The angle α becomes 90°, and $\sin \alpha = 1$ when the pivot becomes flat.

By similar reasoning, we get for a truncated conical pivot (Fig. 326)—

$$M_f = \frac{2\mu W(R_1^3 - R_2^3)}{3 \sin \alpha (R_1^2 - R_2^2)}$$

Schiele's Pivot and Onion Bearing (Figs. 327, 328).—Conical and flat pivots often give trouble through heating, probably due to the fact that the wear is uneven, and therefore the contact between the pivot and step is imperfect, thereby giving rise to intense local pressure. The object sought in the Schiele pivot is to secure even wear all over the pivot. As the footstep wears, every point in the pivot will sink a vertical distance h, and the point a sinks to a_1, where $aa_1 = h$. Draw ab normal to the curve at a, and ac normal to the axis. Also draw ba_1 tangential to the dotted curve at b, and ad to the full-lined curve at a; then, if h be taken as very small, ba_1 will be practically parallel to ad, and the two triangles aba_1 and acd will be practically similar, and—

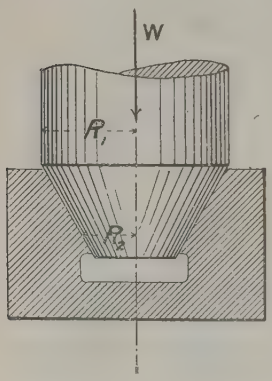


FIG. 326.

$$\frac{ad}{ac} = \frac{aa_1}{ba}, \text{ or } ad = \frac{ac \times aa_1}{ba}$$

$$\text{or } ad = \frac{r \,.\, h}{ba}$$

But ba is the wear of the footstep normal to the pivot, which is usually assumed to be proportional to the friction F between the surfaces, and to the velocity V of rubbing; hence—

$$\overline{ba} \propto \text{FV} \propto \mu p \,.\, 2\pi r\text{N}$$
$$\text{or } \overline{ba} = \text{K}\mu pr$$

where K is a constant for any given speed and rate of wear; hence—

$$ad = \frac{r \,.\, h}{\text{K}\mu pr} = \frac{h}{\text{K}\mu p}$$

But h is constant by hypothesis, and μ is assumed to be constant all over the pivot; p we have already proved to be constant (last paragraph); hence $\overline{ad}$, the length of the tangent to the curve, is constant; thus, if the profile of a pivot be so constructed that the length of the tangent $ad = t$ be constant, the wear will be (nearly) even all over the pivot. Although our

assumptions are not entirely justified, experience shows that such pivots do work very smoothly and well. The calculation of the friction moment is very similar to that of the conical pivot.

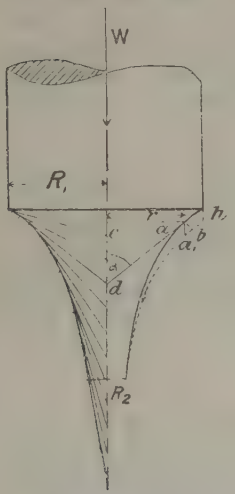


FIG. 327.

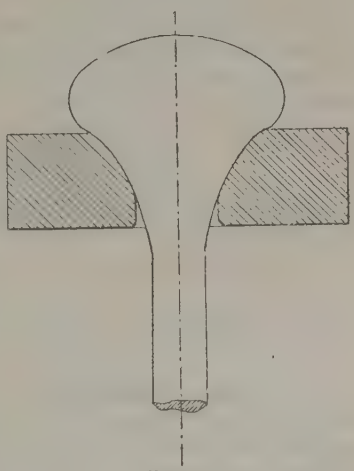


FIG. 328.

The normal pressure at every point is—

$$p = \frac{\text{weight}}{\text{projected area}} = \frac{W}{\pi(R_1^2 - R_2^2)}$$

By similar reasoning to that given for the conical pivot, we have—

$$\text{Moment of friction on an elementary ring of radius } r \Big\} = \frac{2\pi r^2 \mu p dr}{\sin \alpha}$$

$$\left(\text{but } \frac{r}{\sin \alpha} = t\right) \qquad = 2\pi t \mu p r dr$$

$$\text{and moment of friction for whole pivot} = 2\pi t \mu p \int_{r=R_2}^{r=R_1} r \cdot dr$$

$$\therefore M_f = 2\pi t \mu p \frac{R_1^2 - R_2^2}{2}$$

Substituting the value of p, $M_f = W\mu t$

The onion bearing shown in the figure is simply a Schiele pivot with the load suspended from below.

Friction of Cup Leathers.—The resistance of a hydraulic plunger sliding through a cup leather has been investigated by Hick, Tuit, and others. The formula proposed by Hick for the friction of cup leathers does not agree well with experiments; the author has therefore recently tabulated the results of published experiments and others made in his laboratory, and finds that the following formulæ much more nearly agree with experiment:—.

Let F = frictional resistance of a leather in pounds per square inch of water-pressure;
d = diameter of plunger in inches;
p = water-pressure in pounds per square inch.

Then $F = 0{\cdot}08p + \frac{100}{d}$ when in good condition

$F = 0{\cdot}08p + \frac{250}{d}$ „ bad „

Efficiency of Machines.—In all cases of machines, the work supplied is expended in overcoming the useful resistances for which the machine is intended, in addition to the useless or frictional resistances. Hence the work supplied must always be greater than the useful work done by the machine.

Let the work supplied to the machine be equivalent to *lowering* a weight W through a height h;
the useful work done by the machine be equivalent to *raising* a weight W_u through a height h_u;
the work done in overcoming friction be equivalent to *raising* a weight W_f through a height h_f.

Then, if there were no friction—

Supply of energy = useful work done

$$Wh = W_u h_u$$

$$\text{or } \frac{W_u}{W} = \frac{h}{h_u} = R$$

or mechanical advantage = velocity ratio

When there is friction, we have—

Supply of energy = useful work done + work wasted in friction

$$Wh = W_u h_u + W_f h_f$$

and—

$$\text{the mechanical efficiency} = \frac{\text{useful work done}}{\text{total work done}}$$

$$\text{or} = \frac{\text{the work got out}}{\text{the work put in}}$$

Let η = the mechanical efficiency; then—

$$\eta = \frac{W_u h_u}{Wh}, \text{ or } \frac{W_u h_u}{W_u h_u + W_f h_f}$$

η is, of course, always less than unity. The "counter-efficiency" is $\frac{1}{\eta}$, and is always greater than unity.

Reversed Efficiency.—When a machine is reversed, for example, when a load is being lowered by lifting-tackle, the original resistance becomes the driver, and the original driver becomes the resistance; then—

$$\text{Reversed efficiency} = \frac{\text{useful work done in lifting W through } h}{\text{total work done in lowering } W_u \text{ through } h_u}$$

$$\eta_r = \frac{Wh}{W_u h_u} = \frac{W_u h_u - W_f h_f}{W_u h_u}$$

When W acts in the same direction as W_u, *i.e.* when the machine has to be assisted to lower its load, η_r takes the negative sign. In an experiment with a two-sheaved pulley block, the pull on the rope was 170 lbs. when lifting a weight of 500 lbs.; the velocity ratio in this case $R = \frac{h_u}{h} = \frac{1}{4}$.

$$\text{Then } \eta = \frac{W_u h_u}{Wh} = \frac{500 \times 1}{170 \times 4} = 0{\cdot}735$$

The friction work in this case $W_f h_f$ was $170 \times 4 - 500 \times 1$ = 180 foot-lbs. Hence the reversed efficiency $\eta_r = \frac{500 - 180}{500}$ = 0·64, and in order to lower the 500 lbs. weight gently, the backward pull on the rope must be—

$$\frac{500}{4} \times 0{\cdot}64 = 80 \text{ lbs.}$$

If the 80 lbs. had been found by experiments, the reversed efficiency would have been found thus—

$$\eta_r = \frac{80 \times 4}{500 \times 1} = 0{\cdot}64$$

The reversed efficiency must always be less than unity, and may even become negative when the frictional resistance of the machine is greater than the useful resistance. In order to lower the load with such a machine, an additional force acting in the same sense as the load has to be applied; hence such a machine is self-sustaining, *i.e.* it will not run back when left to itself. The least frictional resistance necessary to ensure that it shall be self-sustaining is when $W_f h_f = W_u h_u$; then, substituting this value in the efficiency expression for forward motion, we have—

$$\eta = \frac{W_u h_u}{2 W_u h_u} = \tfrac{1}{2}$$

Thus, in order that a machine which is not fitted with a non-return mechanism may be self-supporting its efficiency cannot be over 50 per cent. This statement is not strictly accurate, because the frictional resistance varies somewhat with the forces transmitted, and consequently is smaller when lowering than when raising the load; the error is, however, rarely taken into account in practical considerations of efficiency.

This self-supporting property of a machine is, for many purposes, highly convenient, especially in hand-lifting tackle, such as screw-jacks, Weston pulley blocks, etc.

Combined Efficiency of a Series of Mechanisms.—If in any machine the power is transmitted through a series of simple mechanisms, the efficiency of each being η_1, η_2, η_3, etc., the efficiency of the whole machine will be—

$$\eta = \eta_1 \times \eta_2 \times \eta_3, \text{ etc.}$$

If the power be transmitted through n mechanisms of the same kind, each having an efficiency η_1, the efficiency of the whole series will be approximately—

$$\eta = \eta_1^n$$

Hence, knowing the efficiency of various simple mechanisms, it becomes a simple matter to calculate with a fair degree of accuracy the efficiency of any complex machine.

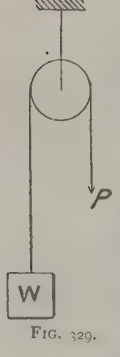


FIG. 329.

Efficiency of Various Machine Elements.

Pulleys.—In the case of a rope or chain passing over a simple pulley, the frictional resistances are due to (1) the resistance of the rope or chain to bending; (2) the friction on the axle. The first varies with the make.

size, and newness of rope; the second with the lubrication. The following table gives a fairly good idea of the total efficiency at or near full load of single pulleys; it includes both resistances 1 and 2:—

Diameter of rope		⅜ in.	¾ in.	1 in.	1½ in.	chain.
Maximum efficiency per cent.	Clean and well oiled	96	93	91	88	95–97
	Dirty	94	91	89	86	93–96
	Clean and well oiled, with stiff new rope	—	91	—	—	—

These figures are fair averages of a large number of experiments. The diameter of the pulley varied from 8 to 12 times the diameter of the rope, and the diameter of the pins from ½ inch to 1¼ inch.

It is useless to attempt to calculate the efficiency with any great degree of accuracy.

Pulley Blocks.—When a number of pulleys are combined for hoisting tackle, the efficiency of the whole may be calculated approximately from the known efficiency of the single pulley. The efficiency of a single pulley does not vary greatly with the load unless it is absurdly low; hence we may assume that the efficiency of each is the same. Then, if the rope passes over n pulleys, each having an efficiency η_1, we have the efficiency of the whole—

$$\eta = \eta_1^n$$

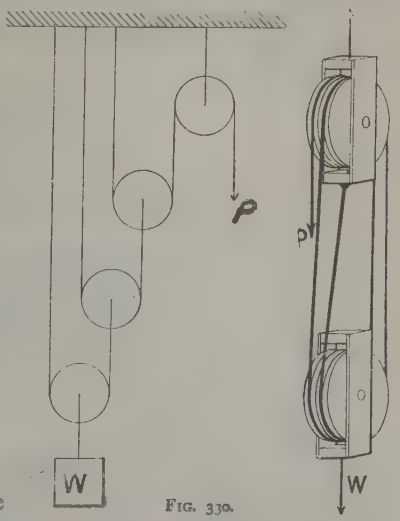


Fig. 330.

The following table will serve to show how the efficiency varies in different pulley blocks.

Loads in pounds.	Single pulley.		Two-sheaved.		Three-sheaved.	
	Old ⅜-in. rope.	New ¾-in. rope.	Old ½-in. rope.	New ½-in. rope.	Old ⅜-in. rope.	New ¾-in. rope.
14	94	90	—	—	—	—
28	94·5	90·5	80	75	30	24
56	·95	91	84	78·5	50	35
112	96	92	86	91·5	60	41
168	—	—	87·5	93	65	44
224	—	—	89	93	69	47
280	—	—	90	94	72	50
336	—	—	—	—	74	53
448	—	—	—	—	78	56

Weston Pulley Block.—This is a modification of the old Chinese windlass; the two upper pulleys are rigidly attached; the radius of the smaller one is r, and of the larger R. Then, neglecting friction for the present, and taking moments about the axle of the pulleys, we have—

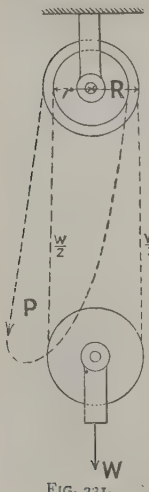


FIG. 331.

$$\frac{W}{2}R = \frac{W}{2}r + PR$$

$$\frac{W}{2}(R - r) = PR$$

and the velocity ratio—

$$v_r = \frac{W}{P} = \frac{2R}{R - r}$$

The pulleys are so chosen that the velocity ratio is from 30 to 40. The efficiency of these blocks is always under 50 per cent., consequently they will not run back when left alone.

From a knowledge of the efficiency of a single-chain pulley, one can make a rough estimate of the relative sizes of pulleys required to prevent such blocks from running back. Taking the efficiency of each pulley as 97 per cent. when the weight is just on the point of running back, the tension in the right-hand chain will be 97 per cent. of that in the left-hand chain due to the friction

on the lower pulley; but due to the friction on the upper pulley only 97 per cent. of the effort on the right-hand chain can be transmitted to the left-hand chain, whence for equilibrium, when P = 0, we have—

$$\frac{W}{2}r = 0{\cdot}97 \times 0{\cdot}97 \times \frac{W}{2}R$$

$$\text{or } r = 0{\cdot}94R$$

$$\text{and the velocity ratio} = \frac{2R}{R - 0{\cdot}94R} = 33$$

which is about the value commonly adopted. The above treatment is only approximate, but it will serve to show the relation between the efficiency and the ratio between the pulleys.

Morris High-efficiency Self-sustaining Pulley Block.—In pulley blocks of the Weston type the efficiency rarely exceeds 45 per cent., but in geared self-sustaining blocks it may reach nearly 90 per cent.

The self-sustaining mechanism is shown in Fig. 332. When hoisting the load the sprocket wheel A together with

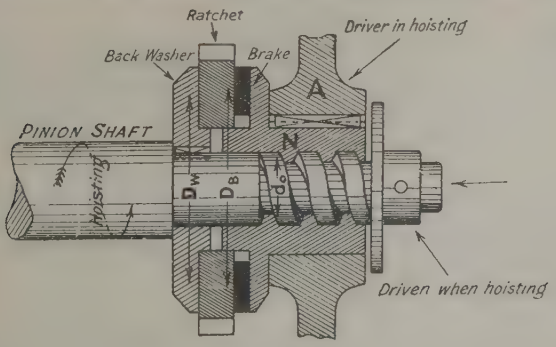


FIG. 332.

By kind permission of Messrs. Herbert Morris, Ltd., Loughborough.

the nut N are rotated by means of an endless hand chain running in a clockwise sense of rotation. The nut traverses the quick running thread until the leather brake ring presses on the face of the ratchet wheel, the friction between these surfaces, also between the back of the ratchet wheel and the back washer, becomes sufficiently great to lock them altogether. The back washer is keyed to the pinion shaft, the pinion gears

into a toothed wheel provided with a pocketed groove for the lifting chain.

Let P_h = the pull on the hand-chain.
D = the diameter of the hand-chain sprocket wheel A.
d_o = the mean diameter of the screw thread (see page 295).
P = the circumferential force acting at the mean diameter of the screw thread when lifting.
W = the axial pressure exerted by the screw when lifting.
e = the mechanical efficiency of the gear from the lifting hook to the brake.
α = the angle of the screw thread.
ϕ = the friction angle for the threads and nut which is always less than α.
μ_b = the coefficient of friction between the brake ring and the ratchet wheel.
μ_w = ditto, back washer and ratchet wheel.
D_b = mean diameter of brake ring.
D_w = mean diameter of back washer bearing surface.

Then
$$P = \frac{P_h D}{e d_o} = W \tan (\alpha + \phi) \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

(see page 295)

$$\frac{P_h D}{e} = W d_o \tan (\alpha + \phi) \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

The axial pressure must be at least sufficient to produce enough friction on the brake ring and on the back washer to prevent the load on the hook from running down when the hand chain is released.

Hence

$W(\mu_b D_b + \mu_w D_w)$ must be greater than $\frac{P_h D}{e}$ (iii.)

In order to provide a margin of safety against the load running back, the friction on the back washer may be neglected; then from (ii.) and (iii.)

$$W \mu_b D_b = W d_o \tan (\alpha + \phi)$$

and
$$\frac{D_b}{d_o} = \frac{\tan (\alpha + \phi)}{\mu_b} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

When these conditions are fulfilled the brake automatically locks on releasing the hand chain. The overall mechanical efficiency of the pulley block can be calculated from the mechanical efficiency of the toothed gearing and the friction of the chain in the pocketed grooves.

When lowering the load the hand-chain wheel revolves in the opposite direction, thus tending to relieve the pressure of the brake. At the same time the sleeve on which the hand wheel is mounted bears against the washer and nut at the end of the pinion shaft, so that a drive in the lowering direction can be obtained through the gears.

General Efficiency Law.—A simple law can be found to represent tolerably accurately the friction of any machine when working under any load it may be capable of dealing with. It can be stated thus: "The total effort F that must be exerted on a machine is a constant quantity K, *plus* a simple function of the resistance W to be overcome by the machine."

The quantity K is the effort required to overcome the friction of the machine itself apart from any useful work. The law may be expressed thus—

$$F = K + Wx$$

The value of K depends upon the type of machine under consideration, and the value of x upon the velocity ratio v_r of

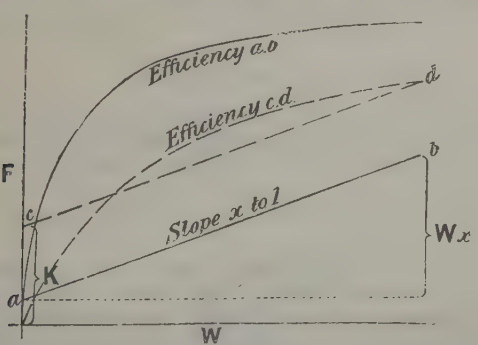


Fig. 333.

the machine. From Fig. 333 it will be seen how largely the efficiency is dependent upon the value of K. The broken and

the full-line efficiency curves are for the same machine, with a large and a small initial resistance.

$$\text{The mechanical efficiency } \eta = \frac{W}{Fv_r} = \frac{W}{(K + Wx)v_r}$$

$$= \frac{1}{\left(\frac{K}{W} + x\right)v_r}$$

Thus we see that the efficiency increases as the load W increases. Under very heavy loads $\frac{K}{W}$ may become negligible; hence the efficiency may approach, but can never exceed—

$$\eta_{max} = \frac{1}{xv_r}$$

The following values give results agreeing well with experiments :—

	x	K	η
Rope pulley blocks	$\frac{1 + 0{\cdot}05v_r}{v_r}$	$2v_rd$ lbs.	$\frac{W}{W(1 + 0{\cdot}05v_r) + Kv_r}$
Chain blocks of the Weston type	$\frac{1 + 0{\cdot}1v_r}{v_r}$	3 lbs.	$\frac{W}{W(1 + 0{\cdot}1v_r) + Kv_r}$
Self-sustaining geared blocks	$\frac{1 + 0{\cdot}009v_r}{v_r}$	1·5 lbs.	$\frac{W}{W(1 + 0{\cdot}009v_r) + Kv_r}$

d = diam. of rope in inches.

Levers.—The efficiency of a simple lever (when used at any other than very low loads) with two pin joints varies from 94 to 97 per cent., the lower value for a short and the higher for a long lever.

When mounted on well-formed knife-edges, the efficiency is practically 100 per cent.

Toothed Gearing.—The efficiency of toothed gearing depends on the smoothness and form of the teeth, and whether lubricated or not. Knowing the pressure on the teeth and the distance through which rubbing takes place (see p. 165), also the μ, the efficiency is readily arrived at; but the latter varies so much, even in the same pair of wheels, that it is very difficult to repeat experiments within 2 or 3 per cent.; hence calculated values depending on an arbitrary choice of μ cannot have any

pretence to accuracy. The following empirical formula fairly well represents average values of experiments:—

For one pair of machine-cut toothed wheels, including the friction on the axles—

$$\eta = 0{\cdot}96 - \frac{1}{2{\cdot}5N}$$

for rough unfinished teeth—

$$\eta = 0{\cdot}90 - \frac{1}{2{\cdot}5N}$$

Where N is the number of teeth in the smallest wheel.

When there are several wheels in one train, let n = the number of pairs of wheels in gear;

$$\text{Efficiency of train } \eta_1 = \eta^n$$

The efficiency increases slightly with the velocity of the pitch lines (see *Engineering*, vol. xli. pp. 285, 363, 581; also Kennedy's "Mechanics of Machinery," p. 579).

Velocity of pitch line in feet per minute ...	10	50	100	150	200
Efficiency	0·940	0·972	0·980	0·984	0·986

Screw and Worm Gearing.—We have already shown

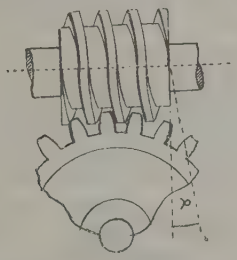


FIG. 334.

how to arrive at the efficiency of screws and worms when the coefficient of friction is known. The following table is taken from the source mentioned above:—

Velocity of pitch line in feet per minute	10	50	100	150	200
	Efficiency per cent.				
Angle of thread α, 45°	87	94	95	96	97
,, ,, 30°	82	90	93	94	95
,, ,, 20°	75	86	90	92	92
,, ,, 15°	70	82	87	89	90
,, ,, 10°	62	76	82	85	86
,, ,, 7°	53	69	76	80	81
,, ,, 5°	45	62	70	74	76

The figure shows an ordinary single worm and wheel. As the angle α increases, the worm is made with more than one thread; the worm and wheel is then known as screw gearing. For details, the reader should refer to books on machine design.

Friction of Slides.—A slide is generally proportioned so that its area bears some relation to the load; hence when the load and coefficient of friction are unknown, the resistance to sliding may be assumed to be proportional to the area; when not unduly tightened, the resistance may be taken as about 3 lbs. per square inch.

Friction of Shafting.—A 2-inch diameter shaft running at 100 revolutions per minute requires about 1 horse-power per 100 feet when all the belts are on the pulleys. The horse-power increases directly as the speed and approximately as the cube of the diameter.

This may be expressed thus—

Let D = diameter of the shafting in inches;
N = number of revolutions per minute;
L = length of the shafting in feet;
F = the friction horse-power of the shafting.

$$\text{Then } F = \frac{NLD^3}{80{,}000}$$

The horse-power that can be transmitted by a shaft is—

$$\text{H.P.} = \frac{ND^3}{64} \text{ to } \frac{ND^3}{27} \text{ (see p. 580)}$$

according to the working stress.

Hence the efficiency of line shafting on which there are numerous pulleys is—

$$\eta = \frac{\text{horse-power transmitted} - \text{friction horse-power}}{\text{horse-power transmitted}}$$

$$\eta = \frac{\frac{ND^3}{64} - \frac{LND^3}{80{,}000}}{\frac{ND^3}{64}}$$

$$= 1 - \frac{L}{1250} \text{ for a working stress of 5000 lbs. sq. inch}$$

and $1 - \frac{L}{2000}$ " " 8000 " "

$1 - \frac{L}{2960}$ " " 12,000 " "

Thus it will be seen that ordinary line shafting may be extremely wasteful in power transmission. The author knows of several instances in which more than one-half the power of the engine is wasted in driving the shafting in engineers' shops; but it must not be assumed from this that shafting is necessarily a wasteful method of transmitting power. Most of the losses in line shafting are due to bending the belts to and fro over the pulleys (see p. 349), and to the extra pressure on the bearings due to the pull on the belts and the weight of the pulleys.

In an ordinary machine shop one may assume that there is, on an average, a pulley and a 3-inch belt at every 5 feet. The load on the bearings due to this belt, together with the weight of the shaft and pulley, will be in the neighbourhood of 500 lbs. The load on the countershaft bearings may be taken at about the same amount or, say, a load on the bearings of 1000 lbs. in all. Let the diameter of the shafting be 3 inches; the 5 feet length will weigh about 120 lbs., hence the load on the bearings due to the pulleys, belts, etc., will be about eight times as great as that of the shaft itself—and considering the poor lubrication that shafting usually gets, one may take the relative friction in the two cases as being roughly in this proportion. Over and above this, there is considerable loss due to the work done in bending the belt to and fro.

We shall now proceed to find the efficiency of shafting, which receives its power at one end and transmits it to a distant point at its other end, *i.e.* without any intermediate pulleys.

Consider first the case of a shaft of the same diameter throughout its entire length.

Let L = the length of the shaft in feet;
R = the radius of the shaft in inches;
W = the weight of the shaft 1 square inch in section and 1 foot long;
μ = the coefficient of friction;
η = the efficiency of transmission;
f_s = the torsional skin stress on the shaft per square inch;

Weight of the shaft $= W\pi R^2 L$ lbs.

moment of the friction $= \mu W\pi R^3 L$ inch-lbs.

the maximum twisting moment at the motor end of the shaft $= \frac{f_s \pi R^3}{2}$ inch-lbs. (see p. 576)

the efficiency of the transmission η $= \frac{\text{the effective twisting moment at the far end}}{\text{the twisting moment at the motor end}}$

$$\eta = \frac{\text{maximum twisting moment} - \text{friction moment}}{\text{maximum twisting moment}}$$

$$= 1 - \frac{\text{friction moment}}{\text{maximum twisting moment}}$$

$$= 1 - \frac{\mu W\pi R^3 L}{\frac{f_s \pi R^3}{2}} = 1 - \frac{2\mu WL}{f_s}$$

For a hollow shaft in which the inner radius is $\frac{1}{n}$ of the outer, this becomes—

$$\eta_h = 1 - \frac{2\mu WL}{f_s}\left(\frac{n^2}{n^2+1}\right)$$

Now consider the case in which the shaft is reduced in diameter in order to keep the skin stress constant throughout its length.

Let the maximum twisting moment at the motor end of the shaft = T_1;

Let the useful twisting moment at the far end of the shaft = T_2.

Then the increase of twisting moment dt due to the friction on an elemental length $dl = \mu W \pi R^3 dl = dt$.

For the twisting moment t we may substitute—

$$t = \frac{f_s \pi R^3}{2} \text{ (see p. 576)}$$

$$\text{or } \pi R^3 = \frac{2t}{f_s}$$

by substitution, we get—

$$dt = \frac{2\mu W t dl}{f_s}$$

$$\text{and } \frac{dt}{t} = \frac{2\mu W dl}{f_s}$$

Integrating—

$$\frac{2\mu WL}{f_s} = \int_{T_2}^{T_1} \frac{1}{t} dt = \log_e \frac{T_1}{T_2}$$

$$\text{and } T_1 = T_2 e^{\frac{2\mu WL}{f_s}}$$

where $e = 2{\cdot}72$, the base of the system of natural logarithms.

$$\text{The efficiency } \eta' = \frac{T_2}{T_1} = \frac{1}{e^{\frac{2\mu WL}{f_s}}}$$

$$\eta' = e^{-\frac{2\mu WL}{f_s}}$$

and for a hollow shaft, such as a series of drawn tubes, which are reduced in size at convenient intervals—

$$\eta_h' = e^{-\frac{2\mu WLn^2}{f_s(n^2+1)}}$$

The following table shows the distance L to which power may be transmitted with an efficiency of 80 per cent. For ordinary bearings we have assumed a high coefficient of friction, viz. 0·04, to allow for poor lubrication and want of accurate alignment of the bearings. For ball bearings we also

take a high value, viz. 0·002. Let the skin stress f, on the shaft be 8000 lbs. sq. inch, and let $n = 1{\cdot}25$.

Form of shafting ...	Parallel.		Taper.	
Kind of bearings ...	Ordinary.	Ball.	Ordinary.	Ball.
	Feet.	Feet.	Feet.	Feet.
Solid shaft with belts [1]	400	—	—	—
,, ,, without belts	6000	120,000	76,600	1,530,000
Hollow ,, ,,	9840	197,000	125,550	2,505,000

These figures at first sight appear to be extraordinarily high, and every engineer will be tempted to say at once that they are absurd. The author would be the last to contend that power can practically be transmitted through such distances with such an efficiency, mainly on account of the impossibility of getting perfectly straight lines of shafting for such distances, and the prohibitive costs; but at least the figures show that very economical transmission may, under convenient circumstances, be accomplished by shafting—and when straight lengths of shafting could be put in they would unquestionably

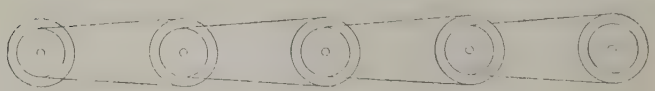


FIG. 335.

be far more economical in transmitting power than could be accomplished by converting the mechanical energy into electrical by means of a dynamo, losing a certain amount of the energy in the mains, and finally reconverting the electrical energy into mechanical by means of a motor; but, of course, in most cases the latter method is the most convenient and the cheapest, on account of the ease of carrying the mains as against that of shafting. The possibility of transmitting power very economically by shafting was first pointed out by Professor

[1] Apart from the loss in bending the belts to and fro as they pass over the pulleys.

Osborne Reynolds, F.R.S., in a series of Cantor Lectures on the transmission of power.

Belt and Rope Transmission.—The efficiency of belt and rope transmission for each pair of pulleys is from 95 to 96 per cent., including the friction on the bearings; hence, if there are n sets of ropes or belts each having an efficiency η, the efficiency of the whole will be, approximately—

$$\eta_1 = \eta^n$$

Experiments by the author on a large number of belts show that the work wasted by belts due to resistance to bending over pulleys, creeping, etc., varies from 16 to 21 foot-lbs. per square foot of belt passed over the pulleys.

Mechanical Efficiency of Steam-engines. — The work absorbed in overcoming the friction of a steam-engine is roughly constant at all powers; it increases slightly as the power increases. A full investigation of the question has been made by Professor Thurston, who finds that the friction is distributed as follows:—

Main bearings	35–47 per cent.
Piston and rod	21–33 ,,
Crank-pin	5–7 ,,
Cross-head and gudgeon-pin	4–5 ,,
Valve and rod ... 2·5 balanced, 22 unbalanced	,,
Eccentric strap	4–5 ,,
Link and eccentric	9 ,,

The following instances may be of interest in illustrating the approximate constancy of the friction at all powers:—

EXPERIMENTAL ENGINE, UNIVERSITY COLLEGE, LONDON.

Syphon Lubrication.

I.H.P.	2·75	9·25	10·23	11·14	12·34	13·95	14·29
B.H.P.	0·0	5·63	7·50	7·66	9·09	11·09	11·25
Friction H.P. ...	2·75	3·62	2·73	3·48	3·25	2·86	3·04

EXPERIMENTAL ENGINE, THE UNIVERSITY, LEEDS.

Syphon and Pad Lubrication.

I.H.P. ...	2·48	5·16	6·83	8·30	11·50	13·84	17·02	22·30
B.H.P. ...	0 0	2·35	3·94	5·61	8·70	10·82	13·89	19·09
Friction H.P.	2·48	2·81	2·89	2·69	2·80	3·02	3·13	3·21

BELLISS ENGINE, BATH (FORCED) LUBRICATION.

(See *Proc. I.M.E.*, 1897.)

I.H.P.	49·8	102·7	147·1	193·6	217·5
B.H.P.	44·5	97·0	140·6	186·0	209·5
Friction H.P. ...	5·3	5·7	6·5	7·6	8·0

Friction Pressure.—The friction of an engine can be conveniently expressed by stating the pressure in the working cylinder required to drive the engine when running light. Under the best conditions it may be as low as 1 lb. per square inch (see *Engineer*, May 30, 1913, p. 574). In ordinary steam engines in good condition the friction pressure amounts to $2\frac{1}{2}$ to $3\frac{1}{2}$ lbs. square inch, but in certain bad cases it may amount to 5 lbs. square inch. It has about the same value in gas and oil engines per stroke, or say from 10 to 14 lbs. square inch, reckoned on the impulse strokes when exploding at every cycle, or twice that amount when missing every alternate explosion.

Thus, if the mean effective pressure in a steam-engine cylinder were 50 lbs. square inch, and the friction pressure 3 lbs. square inch, the mechanical efficiency of the engine would be $\frac{50-3}{50} = 94$ per cent. if double-acting, and $\frac{50-6}{50} = 88$ per cent. if single-acting.

The mean effective pressure in a gas-engine cylinder seldom exceeds 75 lbs. square inch. Thus the mechanical efficiency is from 81 to 87 per cent.

The friction horse-power, as given in the above tables, can also be obtained in this manner.

MECHANICAL EFFICIENCY PER CENT. OF VARIOUS MACHINES.

(From experiments in all cases with more than quarter full load.)

Weston pulley block ($\frac{1}{2}$ ton)	30–40
,, ,, ,, (larger sizes)	40–47
Epicycloidal pulley block	40–45
Morris	75–85
One-ton steam hoists or windlasses	50–70
Hydraulic windlass	60–80
,, jack	80–90
Cranes (steam)	60–70
Travelling overhead cranes	30–50

Locomotives	$\frac{\text{draw bar H.P.}}{\text{I.H.P.}}$	65–75
Two-ton testing-machine, worm and wheel, screw and nut, slide, two collars		2–3
Screw displacer–hydraulic pump and testing-machine, two cup leathers, toothed-gearing four contacts, three shafts (bearing area, 48 sq. inches), area of flat slides, 18 sq. inches, two screws and nuts		2–3
Lancashire Cotton Mills (see *Proceedings of the Manchester Association of Engineers*, 1892)	About 1000 I.H.P. engines, spur-gearing, and engine friction	74
	Rope drives	70
	Belt ,, ...	71
	Direct (400-H.P. engines) ...	76

Belts.

Coil Friction.—Let the pulley in Fig. 336 be fixed, and a belt or rope pass round a portion of it as shown. The weight W produces a tension T_1; in order to raise the weight W, the tension T_2 must be greater than T_1 by the amount of friction between the belt and the pulley.

Let F = frictional resistance of the belt;
p = normal pressure between belt and pulley at any point.

Then, if μ = coefficient of friction—

$$F = T_2 - T_1 = \Sigma \mu p$$

Let the angle α embraced by the belt be divided into a great number, say n, parts, so that $\frac{\alpha}{n}$ is very small; then the tension on both sides of this very small angle is nearly the same. Let the mean tension be T; then, expressing α in circular measure, we have—

$$p = T \frac{\alpha}{n}$$

The friction at any point is (neglecting the stiffness of the belt)—

$$\mu p = \mu T \frac{\alpha}{n} = T_2' - T_1'$$

But we may write $\frac{\alpha}{n}$ as $\delta\alpha$; also $T_2' - T_1'$ as δT. Then—

$$\mu T \, . \, \delta\alpha = \delta T$$

which in the limit becomes—

$$\mu T \,.\, d\alpha = dT$$

$$\frac{dT}{T} = \mu \,.\, d\alpha$$

We now require the sum of all these small tensions expressed in terms of the angle embraced by the belt:—

$$\int_{T_1}^{T_2} \frac{dT}{T} = \mu \int_0^{\alpha} d\alpha$$

$$\log_e T_2 - \log_e T_1 = \mu\alpha$$

$$\log_e \frac{T_2}{T_1} = \mu\alpha$$

$$\frac{T_2}{T_1} = e^{\pm\mu\alpha}$$

$$\text{or } \log\left(\frac{T_2}{T_1}\right) = \pm 0{\cdot}4343\mu\alpha$$

where $e = 2{\cdot}72$, the base of the system of natural logarithms, and $\log e = 0{\cdot}4343$.

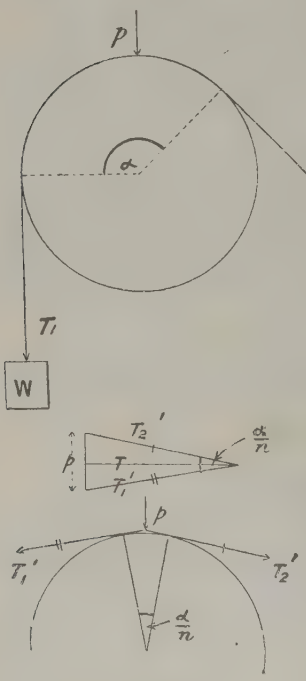


FIG. 336.

When W is being raised, the + sign is used in the index, and when lowered, the − sign. The value of μ for leather, cotton, or hemp rope on cast iron is from 0·2 to 0·4, and for wire rope 0·5.

If a wide belt or plaited rope be used as an absorption dynamometer, and be thoroughly smeared with tallow or other thick grease, the resistance will be greatly increased, due to the shearing of the film of grease between the wheel and the rope. By this means the author has frequently obtained an apparent value of μ of over 1—a result, of course, quite impossible with perfectly clean surfaces.

Power transmitted by Belts.—Generally speaking, the power that can be transmitted by a belt is limited by the friction between the belt and the pulley. When excessively loaded, a belt usually slips rather than breaks, hence the

friction is a very important factor in deciding upon the power that can be transmitted. When the belt is just on the point of slipping, we have—

$$\text{Horse-power transmitted} = \frac{FV}{33{,}000} = \frac{(T_2 - T_1)V}{33{,}000}$$

$$= \frac{T_2\left(1 - \frac{1}{e^{\mu a}}\right)V}{33{,}000}$$

where the friction F is expressed in pounds, and V = velocity in feet per minute. Substituting the value of e, and putting $\mu = 0{\cdot}4$ and $a = 3{\cdot}14$ (180°), we have the tension on the tight side 3·5 times that on the slack side.

$$\text{H.P.} = \frac{0{\cdot}72\,T_2V}{33{,}000}$$

For single-ply belting T_2 may be taken as about 80 lbs. per inch of width, allowing for the laced joints, etc.

Let w = width of belt.

$$\text{Then } T_2 = 80w$$

$$\text{and H.P.} = \frac{0{\cdot}72 \times 80wV}{33{,}000} = \frac{wV}{600}$$

for single-ply belting;

$$\text{and H.P} = \frac{wV}{300}$$

for double-ply belting.

The number of square feet of belt passing over the pulleys per minute is $\frac{wV}{12}$.

Hence the number of square feet of belt required per minute per horse-power is—

$$\frac{\frac{wV}{12}}{\frac{wV}{600}} = 50 \text{ square feet per minute for single-ply, and } 25 \text{ square feet per minute for double-ply}$$

This will be found to be an extremely convenient expression for committal to memory.

Centrifugal Action on Belts.—In Chapter VI. we showed that the two halves of a flywheel rim tended to fly apart due to the centrifugal force acting on them; in precisely the same manner a tension is set up in that portion of a belt wrapped round a pulley. On p. 202 we showed that the stress due to centrifugal force was—

$$f = \frac{W_r V_w^2}{g}$$

where W_r is the weight of 1 foot of belting 1 square inch in section. $W_r = 0{\cdot}43$ lb., and V_w = the velocity in feet per *second*; V = velocity in feet per minute; hence—

$$f = \frac{0{\cdot}43 V_w^2}{32{\cdot}2} = \frac{V_w^2}{75} = \frac{V^2}{270{,}000}$$

and the effective tension for the transmission of power is—

$$T_2 = \frac{V^2}{270{,}000}$$

The usual thickness of single-ply belting is about 0·22 inch, and taking the maximum tension as 80 lbs. per inch of width, this gives $\frac{80}{0{\cdot}22} = 364$ lbs. per square inch of belt, and the power transmitted per square inch of belt section is—

$$P = T_2 V - \frac{V^3}{270{,}000}$$

$$\frac{dP}{dV} = T_2 - \frac{3V^2}{270{,}000}$$

For maximum power

$$T_2 = \frac{3V^2}{270{,}000} = 364$$

and V = 5700 feet per minute.

The tension in the belt when transmitting the maximum power is therefore—

$$T_2 - \frac{5700^2}{270{,}000} = 364 - 121 = 243 \text{ lbs. per square inch.}$$

and the maximum horse-power transmitted per square inch of belt section—

$$H.P_{max.} = \frac{0{\cdot}72 \times 243 \times 5700}{33{,}000} = 30 \text{ nearly.}$$

For ropes we have taken the weight per foot run as 0·35 lb.

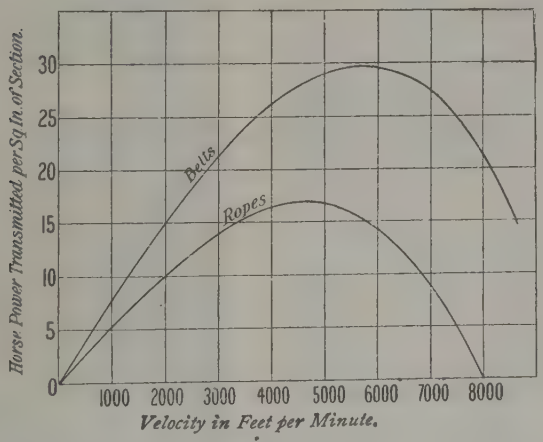


FIG. 337.

per square inch of section, and the maximum permissible stress as 200 lbs. per square inch. On this basis we get the maximum horse-power transmitted when V = 4700 feet per minute, and the maximum horse-power per square inch of rope = 17·1.

The curves in Fig. 337 show how the horse-power transmitted varies with the speed.

The accompanying figure (Fig. 338), showing the stretch of a belt due to centrifugal tension, is from a photograph of an indiarubber belt running at a very high speed; for comparison

the belt is also shown stationary. The author is indebted to his colleague Dr. Stroud for the photograph, taken in the Physics laboratory at the Leeds University.

Creeping of Belts.—The material on the tight side of a belt is necessarily stretched more than that on the slack side, hence a driving pulley always receives a greater length of belt than it gives out; in order to compensate for this, the belt creeps as it passes over the pulley.

Let l = unstretched length of belt passing over the pulleys in feet per minute;
l_2 = stretched length on the T_2 side;
l_1 = " " " T_1 "
N_1 = revolutions per minute of driven pulley;
N_2 = " " " driving "
d_1 = diameter of driven pulley } measured to the middle
d_2 = " driving " } of the belt;
x = stretch of belt in feet;
E = Young's modulus;
f_1 and f_2 = stresses corresponding to T_1 and T_2 in lbs. square inch.

$$\text{Then } x = \frac{f_2 l}{E}$$

$$l_2 = l + x = l\left(1 + \frac{f_2}{E}\right) = \pi d_2 N_2$$

$$l_1 = l\left(1 + \frac{f_1}{E}\right) = \pi d_1 N_1$$

$$\frac{N_1}{N_2} = \frac{l\left(1 + \frac{f_1}{E}\right) d_2}{l\left(1 + \frac{f_2}{E}\right) d_1}$$

$$\frac{N_1}{N_2} = \frac{(E + f_1) d_2}{(E + f_2) d_1}$$

If there were no creeping, we should have—

$$\frac{N_1}{N_2} = \frac{d_2}{d_1}$$

E = from 8,000 to 10,000 lbs. per square inch. Taking

$T_2 = 80$ lbs. per inch of width, and the thickness as 0·22 inch, we have when $\alpha = 3{\cdot}14$—

$$f_2 = \frac{80}{0{\cdot}22} = 364 \text{ lbs. per square inch}$$

$$\text{and } T_1 = \frac{80}{2{\cdot}72^{0{\cdot}4 \times 3{\cdot}14}} = \frac{80}{3{\cdot}5} = 23 \text{ lbs. per inch width}$$

$$f_1 = \frac{23}{0{\cdot}22} = 104 \text{ lbs. per square inch}$$

$$\text{Hence } \frac{E + f_1}{E + f_2} = \frac{10{,}000 + 104}{10{,}000 + 364} = 0{\cdot}975$$

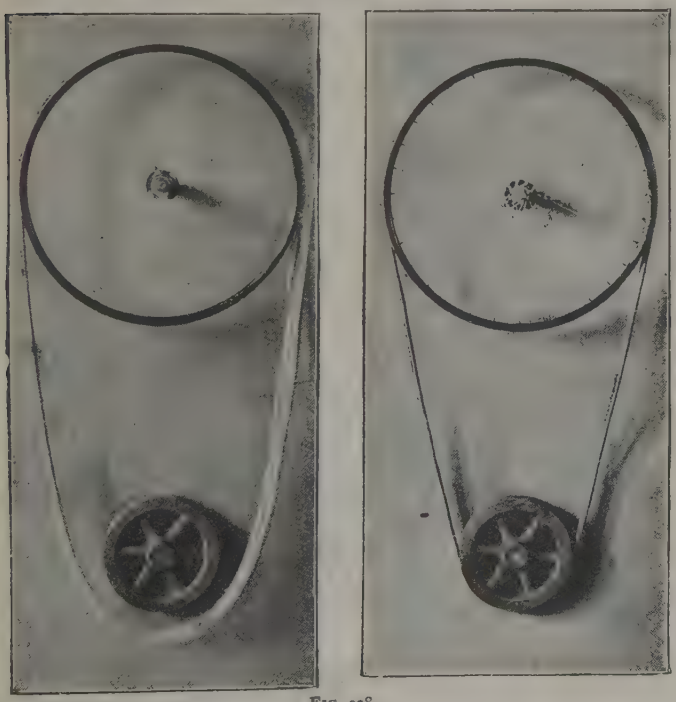


FIG. 338.

or the belt under these conditions creeps or slips 2·5 per cent.

When a belt transmits power, however small, there must be some slip or creep.

When calculating the speed of pulleys the diameter of the pulley should always be measured to the centre of the belt; thus the effective diameter of each pulley is $D + t$, where t is the thickness of the belt. In many instances this refinement is of little importance, but when small pulleys are used and great accuracy is required, it is of importance. For example, the driving pulley on an engine is 6 feet diameter, the driven pulley on the countershaft is 13 inches, the driving pulley on which is 3 feet 7 inches diameter, and the driven pulley on a dynamo is 8 inches diameter; the thickness of the belt is 0·22 inches; the creep of each belt is 2·5 per cent.; the engine runs at 140 revolutions per minute: find the speed of the dynamo. By the common method of finding the speed of the dynamo, we should get—

$$\frac{140 \times 72 \times 43}{13 \times 8} = 4168 \text{ revolutions per minute}$$

But the true speed would be much more nearly—

$$\frac{140 \times 72{\cdot}22 \times 43{\cdot}22 \times 0{\cdot}975 \times 0{\cdot}975}{13{\cdot}22 \times 8{\cdot}22} = 3822 \text{ revs. per minute}$$

Thus the common method is in error in this case to the extent of 9 per cent.

Chain Driving.—In cases in which it is important to prevent slip, chain drives should be used. They moreover possess many advantages over ordinary belt driving if they are properly designed. For the scientific designing of chains and sprocket wheels, the reader is referred to a pamphlet on the subject by Mr. Hans Renold, of Manchester.

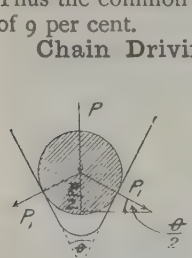


FIG. 339.

Rope Driving.—When a rope does not bottom in a grooved pulley, it wedges itself in, and the normal pressure is thereby increased to—

$$P_1 = \frac{P}{\sin \frac{\theta}{2}}$$

The angle θ is usually about 45°; hence $P_1 = 2{\cdot}6P$.

The most convenient way of dealing with this increased pressure is to use a false coefficient 2·6 times its true value. Taking $\mu = 0{\cdot}3$ for a rope on cast iron, the false μ for a grooved pulley becomes $2{\cdot}6 \times 0{\cdot}3 = 0{\cdot}78$.

The value of $e^{\mu a}$ now becomes 10·1 when the rope embraces half the pulley. The factor of safety on driving-ropes is very large, often amounting to about 80, to allow for defective splicing, and to prevent undue stretching. The working strength in pounds may be taken from $10c^2$ to $16c^2$, where c is the circumference in inches.

Then, by similar reasoning to that given for belts, we get for the horse-power that may be transmitted *per rope* for the former value—

$$\text{H.P.} = \frac{c^2V}{3740}, \text{ or } \frac{d^2V}{374}$$

where d = diameter of rope in inches.

The reader should refer to a paper on rope driving by Mr. Coombe, *Inst. Mech. Engrs. Proceedings*, 1889.

Coefficients of Friction.

The following coefficients obtained on large bearings will give a fair idea of their friction :—

Ball bearings with plain cylindrical ball races,	2-point bearing,	0·0012 to 0·0018
Thrust bearings: Flat ball races	,, ,,	0·0008 to 0·0012
Thrust bearings: One flat, one V race,	3 ,, ,,	mean 0·0018
Thrust bearings: Two V races,	4 ,, ,,	,, 0·0055
Gun-metal bearings tested by Mr. Beauchamp Tower for the Institution of Mechanical Engineers: Plain cylindrical journals with bath lubrication		0·001
Gun-metal bearings tested by Mr. Beauchamp Tower for the Institution of Mechanical Engineers: Plain cylindrical journals with ordinary lubrication		0·01
Gun-metal bearings tested by Mr. Beauchamp Tower for the Institution of Mechanical Engineers: Thrust or collar bearing well lubricated		0·03
Good white metal (author) with very meagre lubrication		0·003
Poor white metal under same conditions		0·008

Reference-books on Friction.

"Lubrication and Lubricants," Archbutt and Deeley.

"Friction and Lubrication," Dr. J. T. Nicolson, Manchester Association of Engineers, 1907–1908.

"Cantor Lectures on Friction," by Dr. Hele-Shaw, F.R.S. Published by the Society of Arts.

CHAPTER X.

STRESS, STRAIN, AND ELASTICITY.

Stress.—If, on any number of sections being made in a body, it is found that there is no tendency for any one part of it to move relatively to any other part, that body is said to be in a *state of ease;* but when one part tends to move relatively to the other parts, we know that the body is acted upon by a system of equal and opposite forces, and the body is said to be in a *state of stress*. Thus, if, on making a series of saw-cuts in a plate of metal, the cuts were found to open or close before the saw had got right through, we should know that the plate was in a state of stress, because the one part tends to move relatively to the other. The stress might be due either to external forces acting on the plate, or to internal initial stresses in the material, such as is often found in badly designed castings.

Intensity of Stress.—The intensity of direct stress on any given section of a body is the total force acting normal to the section divided by the area of the section over which it is distributed; or, in other words, it is the amount of force per unit area.

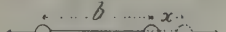
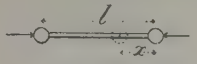
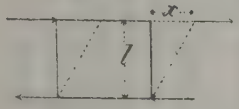
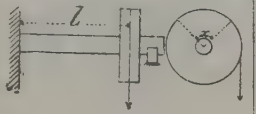
$$\left.\begin{array}{l}\text{Intensity of stress in}\\ \text{pounds per sq. inch}\end{array}\right\} = \frac{\text{the given force in pounds}}{\text{area of the section over which the force acts in sq. inches}}$$

For brevity the word "stress" is generally used for the term "intensity of stress."

The conditions which have to be fulfilled in order that the intensity of stress may be the same at all parts of the section are dealt with in Chapter XV.

Strain.—The strain of a body is the change of form or dimensions that it undergoes when placed in a state of stress. No bodies are absolutely rigid; they all yield, or are strained more or less, when subjected to stress, however small in amount.

The various kinds of stresses and strains that we shall consider are given below in tabular form.

	Nature of load.	The external forces act.	Kind of *stress*.	Kind of *strain*.	When the ratios below are equal to unity, the strain is termed "unit strain."
			Produced in the direction of the applied forces.		
	Pull	*From* one another in the same straight line	Tensile stress or tension	Extension or stretch (x)	$\frac{\text{Extension}}{\text{Original length}} = \frac{x}{l}$
	Push or thrust	*Towards* one another in the same straight line	Compressive stress or pressure	Compression or contraction (x)	$\frac{\text{Compression}}{\text{Original length}} = \frac{x}{l}$
	Shear	*Parallel* to one another	Shearing stress	Shear or slide (x) or distortion	$\frac{\text{Shear}}{\text{Length}} = \frac{x}{l}$
	Bending	*Parallel* to one another	Bending stress	Deflection (x)	
	Twisting	In planes *parallel* with one another	Torsion or shearing stress	Twist, usually measured in degrees	$\frac{\text{Surface twist}}{\text{Length}} = \frac{x}{l}$

Elasticity.—A body is said to be elastic when the strain entirely disappears on the removal of the stress that produced it. Very few materials can be said to be perfectly elastic except for very low stresses, but a great many are approximately so over a wide range of stress.

Permanent Set.—That part of the strain that does not entirely disappear on the removal of the stress is termed "permanent set."

Elastic Limit.—The stress at which a marked permanent set occurs is termed the elastic limit of the material. We use the word *marked* because, if very delicate measuring instruments be used, very slight sets can be detected with much lower stresses than those usually associated with the elastic limit. In elastic materials the strain is usually proportional to the stress; but this is not the case in all materials that fulfil the conditions of elasticity laid down above. Hence there is an objection to the definition that the elastic limit is that point at which the strain ceases to be proportional to the stress.

Plasticity.—If none of the strain disappears on the removal of the stress, the body is said to be plastic. Such bodies as soft clay and wax are almost perfectly plastic.

Ductility.—If only a small part of the strain be elastic, but the greater part be permanent after the removal of the stress, the material is said to be ductile. Soft wrought iron, mild steel, copper, and other materials, pass through such a stage before becoming plastic.

Brittleness.—When a material breaks with a very low stress and deforms but a very small amount before fracture, it is termed a brittle material.

Behaviour of Materials subjected to Tension.

Ductile Materials.—If a bar of ductile metal, such as wrought iron or mild steel, be subjected to a low tensile stress, it will stretch a certain amount, depending on the material; and if the stress be doubled, the stretch will also be doubled, or the stretch will be proportional to the stress (within very narrow limits). Up to this point, if the bar be relieved of stress, it will return to its original length, *i.e.* the bar is elastic; but if the stress be gradually increased, a point will be reached when the stretch will increase much more rapidly than the stress; and if the bar be relieved of stress, it will not return to its original length—in other words, it has taken a "permanent set." The stress at which this occurs is, as will be seen from our definition above, the *elastic limit* of the material.

Let the stress be still further increased. Very shortly a

point will be reached when the strain will (in good wrought iron and mild steel) suddenly increase to 10 or 20 times its previous amount. This point is termed the *yield point* of the material, and is always quite near the elastic limit. For all commercial purposes, the elastic limit is taken as being the same as the yield point. Just before the elastic limit was reached, while the bar was still elastic, the stretch would only be about $\frac{1}{1000}$ of the length of the bar; but when the yield point is reached, the stretch would amount to $\frac{1}{100}$, or $\frac{1}{50}$ of the length of the bar.

The elastic extensions of specimens cannot be taken by direct measurements unless the specimens are very long indeed; they are usually measured by some form of extensometer. That shown in Fig. 340 was designed by the author

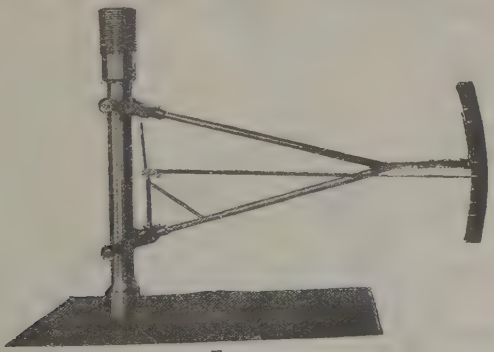


FIG. 340.

some years ago, and gives entirely satisfactory results; it reads to $\frac{1}{10000}$ of an inch; it is simple in construction, and does not get out of order with ordinary use. It consists of suitable clips for attachment to the specimen, from which a graduated scale is supported; the relative movement of the clips is read on the scale by means of a pointer on the end of a 100 to 1 lever.

In Fig. 341 several elastic curves are given. In the case of wrought iron and steel, the elastic lines are practically straight, but they rapidly bend off at the elastic limit. In the case of cast iron the elastic line is never straight; the strains always increase more rapidly than the stresses, hence Young's modulus is not constant. Such a material as copper takes a "permanent set" at very low loads, it is almost impossible to say exactly where the elastic limit occurs.

As the stress is increased beyond the yield point, the strain continues to increase much more rapidly than before, and the material becomes more and more ductile; and if the stress be now removed, almost the whole of the strain will be found to be permanent. But still a careful measurement will show that a very small amount of the strain is still elastic.

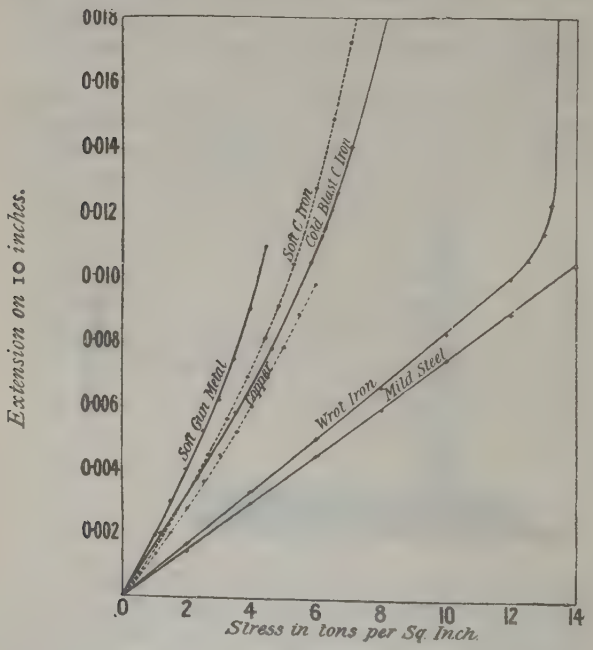


FIG. 341.

Just before the maximum stress is reached, the material appears to be nearly perfectly plastic. It keeps on stretching without any increase in the load. Up to this point the strain on the bar has been evenly distributed (approximately) along its whole length; but very shortly after the plastic state has been reached the bar extends locally, and "stricture" commences, *i.e.* a local reduction in the diameter occurs, which is followed almost immediately by the fracture of the bar. The extension before stricture occurs is termed the "proportional" extension, and that after fracture the "final" extension, which

is known simply as the "extension" in commercial testing. We shall return to this point later on.

The stress-strain diagram given in Fig. 342 will illustrate clearly the points mentioned above.

Brittle Materials.—Brittle materials at first behave in a similar manner to ductile materials, but have no marked elastic limit or yield point. They break off short, and have no ductile or plastic stage.

Extension of Ductile Materials.—We pointed out above that the final extension of a ductile bar consisted of two parts—(1) An extension evenly distributed along the whole length of the bar, the total amount of which is consequently

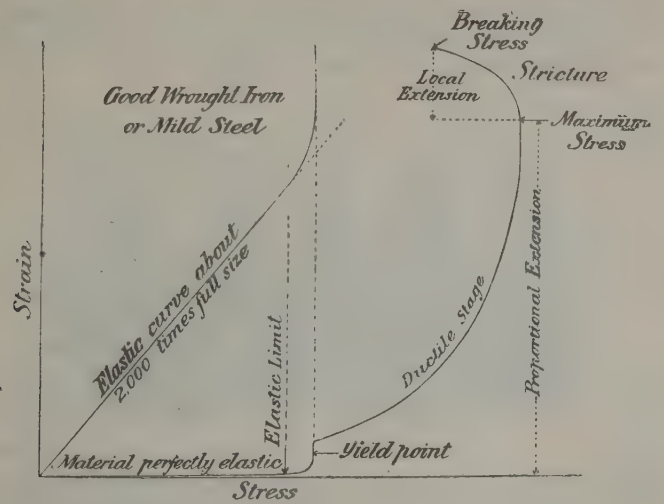


FIG. 342.

proportional to the length of the bar; (2) A local extension at fracture, which is very much greater per unit length than the distributed or proportional extension, and is independent (nearly so) of the length of the bar. Hence, on a short bar the local extension is a very much greater proportion of the whole than on a long bar. Consequently, if two bars of the same material but of different lengths be taken, the percentage of extension on the short bar will be much greater than on the long bar.

The following results were obtained from a bar of Lowmoor iron :—

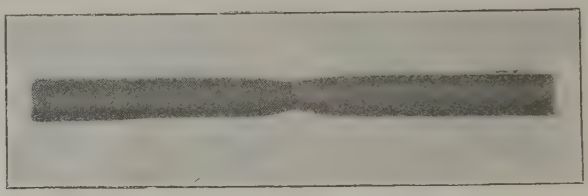


FIG. 343.

The local extension in this bar was 54 per cent. on 2 inches.

The final extensions reckoned on various lengths, each including the fracture, were as follows :—

Length	10″	8″	6″	4″	2″
Percentage of extension ...	22	24·5	34	41	54

(See papers by Mr. Wicksteed in *Industries*, Sept. 26, 1890, and by Professor Unwin, I.C.E., vol. clv.) Hence it will be seen that the length on which the percentage of extension is measured must always be stated. The simplest way of obtaining comparative results for specimens of various lengths is to always mark them out in inches throughout their whole length, and state the percentage of extension on the 2 inches at fracture as well as on the total length of the bar. A better method would be to make all test specimens of similar form, *i.e.* the diameter a fixed proportion of the length; but any one acquainted with commercial testing knows how impracticable such a suggestion is.

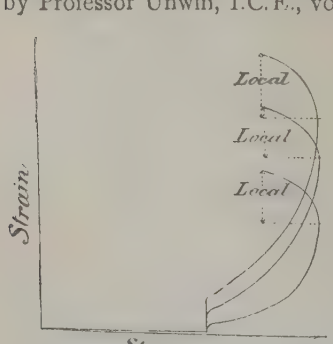


FIG. 344.

Load-strain diagrams taken from bars of similar material, but of different lengths, are somewhat as shown in Fig. 344.

If L = original length of a test bar between the datum points;

L_1 = stretched length of a test bar between the datum points;

Then $L_1 - L = x$, the extension

The percentage of extension is—

$$\frac{L_1 - L}{L} \times 100 = \frac{100x}{L}$$

In Fig. 345 we show some typical fractures of materials tested in tension.

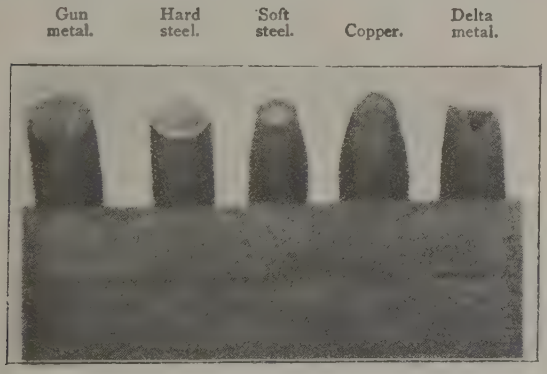


FIG. 345.

If specimens are marked out in inches prior to testing, and after fracture they are measured up to give the extension on various lengths, always including the fracture, they will, on plotting, be found to give approximately a law of the form—

$$\text{Total extension} = K + nL$$

where K is a constant depending upon the material and the diameter of the bar, and n is some function of the length. Several plottings for different materials are given in Fig. 346.

Reduction in Area of Ductile Materials.—The volume of a test bar remains constant within exceedingly small limits, however much it may be strained; hence, as it extends

the sectional area of the bar is necessarily reduced. The reduction in area is considered by some authorities to be the best measure of the ductility of the material.

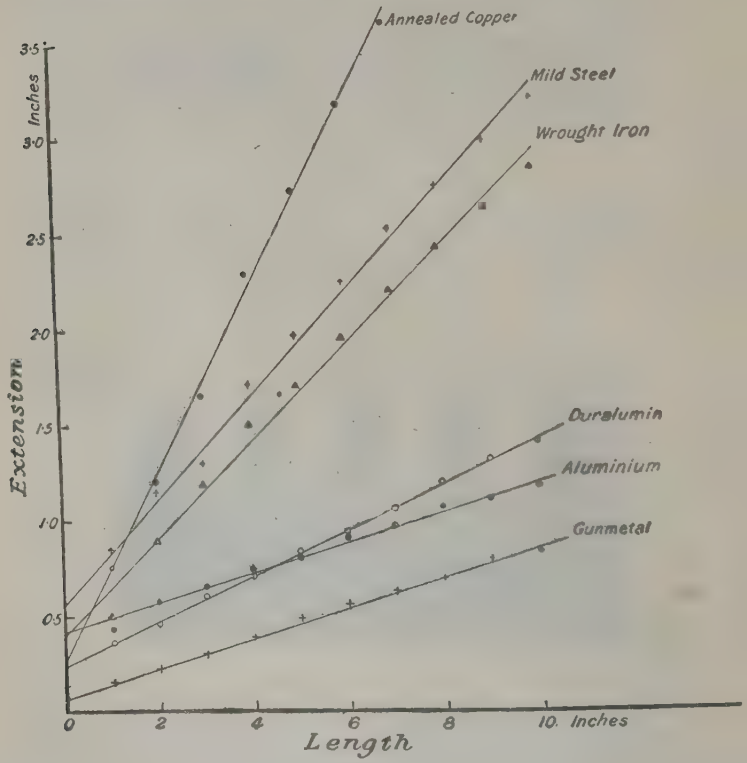


FIG. 346.

Let A = the original sectional area of the bar;
A_1 = the final area at the fracture.

Then the percentage of reduction in area is $\frac{A - A_1}{A} \times 100$

If a bar remained parallel right up to the breaking point, as some materials approximately do, the reduction in area can be calculated from the extension, thus:

The volume of the bar remains constant; hence—

$$LA = L_1A_1, \text{ or } A_1 = \frac{LA}{L_1}$$

and the reduction in area is—

$$\frac{A - A_1}{A}$$

Then, substituting the value of A_1, we have—

$$\frac{L_1A - LA}{L_1A} = \frac{L_1 - L}{L_1} = \frac{x}{L_1}$$

Thus the reduction in area in the case of a test bar which remains parallel is equal to the extension on the bar *calculated on the stretched length.* This method should never be used for calculating the reduction in area, but it is often a useful check. The published account of some tests of steel bars gave the following results:—

Length of bar, 2 inches; extension, 6·0 per cent.; reduction in area, 4·9 per cent.;

Then x in this case was $\frac{6 \times 2}{100} = 0{\cdot}12$ inch

and $L_1 = 2{\cdot}12$ inches

Reduction in area $= \frac{0{\cdot}12 \times 100}{2{\cdot}12} = 5{\cdot}66$ per cent.

Thus there is probably an error in measurement in getting the 4·9 per cent., for the reduction in area could not have been less than 5·66 per cent. unless there had been a hard place in the metal, which is improbable in the present instance.

Real and Nominal Stress in Tension.—It is usual to calculate the tensile stress on a test bar by dividing the maximum load by the area of the original section. This method, though convenient and always adopted for commercial purposes, is not strictly accurate, on account of the reduction of the area as the bar extends.

Using the same notation as before for the lengths and areas—

Let W = the load on the bar at any instant;

S = the nominal stress on the bar, viz. $\frac{W}{A}$;

S_1 = the real stress on the bar, viz. $\frac{W}{A_1}$.

Then, as the volume of the bar remains constant—

$$LA = L_1A_1, \text{ and } \frac{L_1}{L} = \frac{A}{A_1}$$

$$\text{and } \frac{S_1}{S} = \frac{\frac{W}{A_1}}{\frac{W}{A}} = \frac{A}{A_1} = \frac{L_1}{L}$$

$$\text{or the real stress } S_1 = \frac{SL_1}{L}$$

The diagram of real stress may be conveniently constructed as in Fig. 347 from the ordinary stress-strain diagram.

The construction for one point only is given. The length

Fig. 347.

L of the specimen is set off along the strain axis, and the stress ordinate *de* is projected on to the stress axis, viz. *ao*. The line *ba* is then drawn to meet *ed* produced in *c*, which gives us one point on the curve of real stress. For by similar triangles we have—

$$\frac{S_1}{S} = \frac{L_1}{L}, \text{ or } S_1 = \frac{SL_1}{L}$$

which we have shown above to be the real stress.

The last part of the diagram, however, cannot be obtained

thus, as the above relation only holds as long as the bar remains parallel; but points on the real stress diagram between

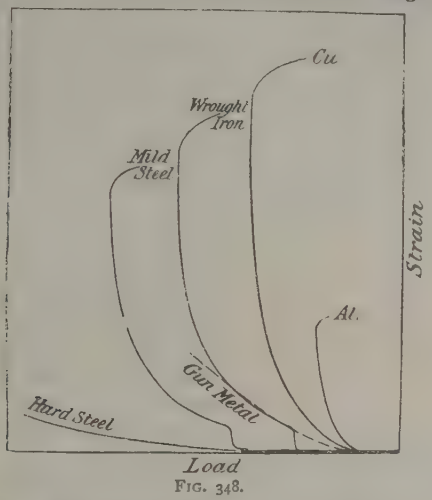


FIG. 348.

g and f can be obtained by stopping the test at intervals, noting the load and the corresponding diameter of the bar in the stricture: the load divided by the corresponding stricture area gives the real stress at the instant.

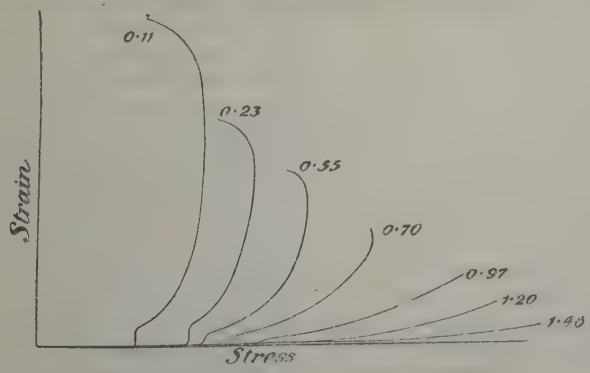


FIG 349.—Steel containing several percentages of carbon.

Typical Stress. Strain Curves for Various Materials in Tension.—The curves shown in Figs. 348,

349, were drawn by the author's autographic recorder (see *Engineering*, December 19, 1902), from bars of the same length and diameter.

Some of the curves in Fig. 350 are curiously serrated, *i.e.* the metal does not stretch regularly (these serrations are not due to errors in the recording apparatus, such as are obtained by recorders which record the faults of the operator as well as the characteristics of the material). The author finds that all alloys containing iron give a serrated diagram when cold and a smooth diagram when hot, whereas steel does the reverse. This peculiar effect, which is disputed by some, has been independently noticed by Mons. Le Chatelier.

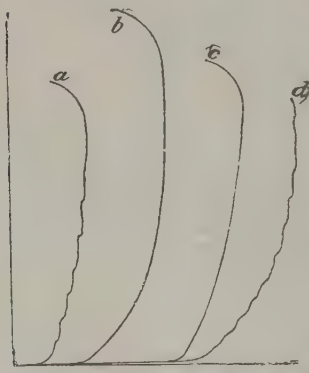


FIG. 350.—(*a*) Rolled aluminium; (*b*) rolled copper: (*c*) rolled "bull" metal, temp. 400° Fahr.; (*d*) ditto 60° Fahr. N.B.—Bull metal and delta metal behave in practically the same way in the testing-machine.

Artificial Raising of the Elastic Limit.—The form of a stress-strain curve depends much upon the physical state of the metal, and whether the elastic limit has been artificially raised or not. It has been known for many years that if a piece of metal be loaded beyond the elastic limit, and the load be then released, the next time the material is loaded, the elastic limit will approximately coincide with the previous load. In the diagram in Fig. 351, the metal was loaded up to the point *c*, and then released; on reloading, the elastic limit occurred at the stress *cd*, whereas the original elastic limit was at the stress *ab*. Now, if in manufacture, by cold rolling, drawing, or otherwise, the limit had been thus artificially raised, the stress-strain diagram would have been *dce*.

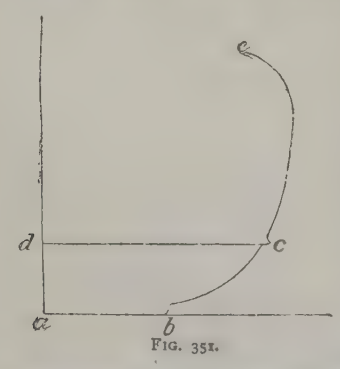


FIG. 351.

Young's Modulus of Elasticity (E).—We have already

stated that experiments show that the strain of an elastic body is proportional to the stress. In some elastic materials the strain is much greater than in others for the same intensity of stress, hence we need some means of concisely expressing the amount of strain that a body undergoes when subjected to a given stress. The usual method of doing this is to state the intensity of stress required to strain the bar by an amount equal to its own length, *assuming the material to remain perfectly elastic.* This stress is known as Young's modulus of (or measure of) elasticity. We shall give another definition of it shortly.

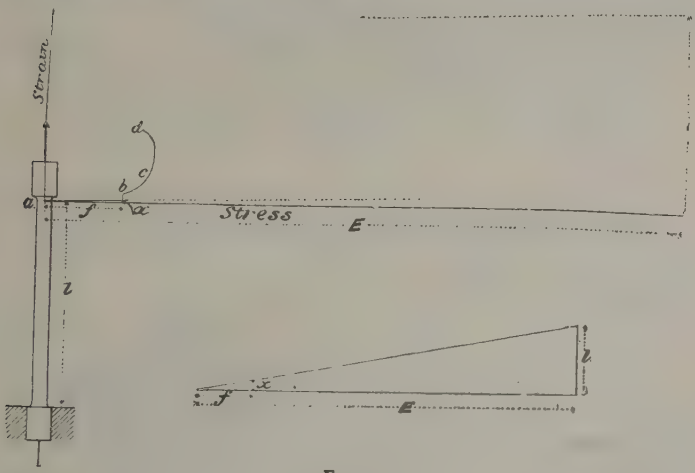


FIG. 352.

In the diagram in Fig. 352 we have shown a test-bar of length l between the datum points. The lower end is supposed to be rigidly fixed, and the upper end to be pulled; let a stress-strain diagram be plotted, showing the strain along the vertical and the stress along the horizontal. As the test proceeds we shall get a diagram *abcd* as shown, similar to the diagrams shown on p. 365. Produce the *elastic* line onward as shown (we have had to break it in order to get it on the page) until the *elastic* strain is equal to l; then, if x be the elastic strain at any point along the elastic line of the diagram corresponding to a stress f, we have by similar triangles—

$$\frac{x}{l} = \frac{f}{E}$$

The stress E is termed "Young's modulus of elasticity," and sometimes briefly "The modulus of elasticity." Thus in tension we might have defined the modulus of elasticity as *The stress required to stretch a bar to twice its original length, assuming the material to remain perfectly elastic.* It need hardly be pointed out that no constructive materials used by engineers do remain perfectly elastic when pulled out to twice their original length; in fact, very few materials will stretch much more than the *one-thousandth* of their length and remain elastic. It is of the highest importance that the elastic stretch should not be confused with the stretch beyond the elastic limit. It will be seen in the diagram above that the part *bcd* has *nothing whatever* to do with the modulus of elasticity.

We may write the above expression thus:

$$E = \frac{f}{\frac{x}{l}}$$

Then, if we reckon the strain per unit length as on p. 361, we have $\frac{x}{l}$ = unit strain, and we may write the above relation thus:—

$$\text{Young's modulus of elasticity} = \frac{\text{unit stress}}{\text{unit strain}}$$

Thus Young's modulus is often defined as the ratio of the unit stress to the unit strain while the material is perfectly elastic, or we may say that it is that stress at which the strain becomes unity, assuming the material to remain perfectly elastic.

The first definition we gave above is, however, by far the clearest and most easily followed.

For compression the diagram must be slightly altered, as in Fig. 353.

In this case the lower part of the specimen is fixed and the upper end pushed down; in other respects the description of the tension figure applies to this diagram, and here, as before, we have—

$$\frac{x}{l} = \frac{f}{E}$$

For most materials the value of E is the same for both tension and compression; the actual values are given in tabular form on p. 427.

Occasionally in structures we find the combination of two or more materials having very different coefficients of elasticity; the problem then arises, what proportion of the total load is

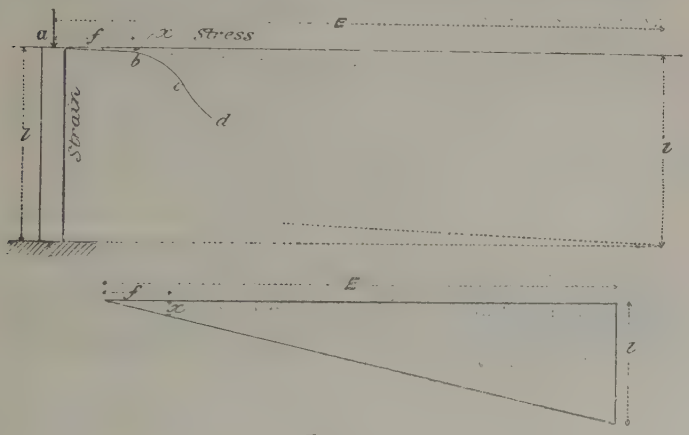


FIG. 353.

borne by each? Take the case of a compound tension member.

Let E_1 = Young's modulus for material 1;
E_2 = " " " 2;
A_1 = the sectional area of 1;
A_2 = " " 2;
f_1 = the tensile stress in 1;
f_2 = " " 2;
W = the total load on the bar.

Then—

$$W = f_1A_1 + f_2A_2 = W_1 + W_2$$

But $f = \frac{Ex}{l}$

whence $\frac{f_1}{f_2} = \frac{E_1x_1l_2}{E_2x_2l_1} = \frac{E_1}{E_2}$.

since the components of the member are attached together at both ends, and therefore the proportional strain is the same in both;

$$\text{and } \frac{A_1 f_1}{A_2 f_2} = \frac{A_1 E_1}{A_2 E_2} = \frac{W_1}{W_2}$$

which gives us the proportion of the load borne by each of the component members;

$$\text{and } W_1 = \frac{W}{1 + \frac{A_2 E_2}{A_1 A_2}}$$

$$W_2 = \frac{W}{1 + \frac{A_1 E_1}{A_2 E_2}}$$

By similar reasoning the load in each component of a bar containing three different materials can be found.

The Modulus of Transverse Elasticity, or the Coefficient of Rigidity (G).—The strain or distortion of an

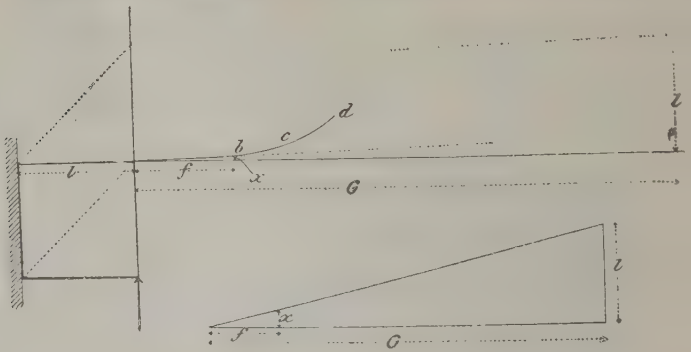


FIG. 354.

element subjected to shear is measured by the slide, x (see p. 361). The shear stress required to make the slide x equal to the length l is termed the modulus of transverse elasticity, or the coefficient of rigidity, G. Assuming, as before, that the material remains perfectly elastic, we can also represent this graphically by a diagram similar to those given for direct elasticity.

In this case the base of the square element in shear is rigidly fixed, and the outer end sheared, as shown.

From similar triangles, we have—

$$\frac{x}{l} = \frac{f}{G} \qquad G = \frac{f}{\frac{x}{l}} = \frac{\text{stress}}{\text{strain}}$$

Relation between the Moduli of Direct and Transverse Elasticity.—Let *abcd* be a square element in a perfectly elastic material which is to be subjected to—

(1) Tensile stress equal to the modulus stress; then the length l of the line *ab* will be stretched to $2l$, viz. ab_1, and the strain reckoned on unit length will be $\frac{2l - l}{l} = 1$.

(2) Shearing stress also equal to the modulus stress; then the length l of the line *ab* will be stretched to $\sqrt{l^2 + l^2} = \sqrt{2}l = 1{\cdot}41l$, and the strain reckoned on unit length will be $\frac{1{\cdot}41l - l}{l} = 0{\cdot}41$.

Thus, when the modulus stress is reached in shear the strain is 0·41 of the strain when the modulus stress is reached

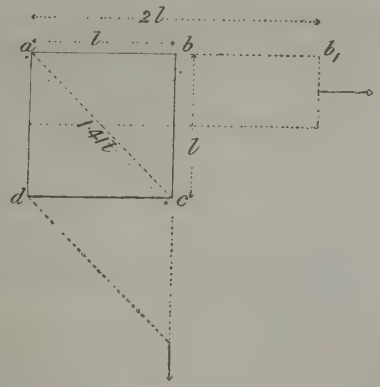


FIG. 355.

in tension; but the stress is proportional to the strain, therefore the modulus of transverse elasticity is 0·41, or $\frac{2}{5}$ nearly, of the modulus of direct elasticity.

The above proof must be regarded rather as a popular demonstration of this relation than a scientific treatment. The orthodox treatment will be given shortly.

Strength of Wire.—Surprise is often expressed that the

strength of wire is so much greater than that of the material from which it was made; the great difference between the two is, however, largely due to the fact that the nominal tensile strength of a piece of material is very much less than the real strength reckoned on the final area. The process of drawing wire is equivalent to producing an elongated stricture in the material; hence we should expect the strength of the wire to approximate to that of the real strength of the material from which it was made (Fig. 356). That it does so will be clear from the following diagrams. In addition to this the skin of the wire is under very severe tensile stress, due to the punishing action of the draw-plate, which causes a compression of the core, with the result that the density of the wire is slightly increased with a corresponding increase in strength. Evidence will shortly be given to show that the skin is in tension and the core in compression.

FIG. 356.

The process of wire-drawing very materially raises the elastic limit, and if several passes be made without annealing the wire, the elastic limit may be raised right up to the breaking point; the permanent stretch of the wire is then extremely small. If a given material will stretch, say, 50 per cent. in the stricture before fracture, and a portion of the material be stretched, say, 48 per cent., by continual passes through the draw-plate without being annealed, that wire will only stretch roughly the remainder, viz. 2 per cent., before fracture. We qualify this remark by saying *roughly*, because there are other disturbing factors; the statement is, however, tolerably accurate. If a piece of wire be annealed, the strength will be reduced to practically that of the original material, and the proportional extension before fracture will also approximate to that of the undrawn material.

If a number of wires of various sizes, all made from the same material, be taken, it will be found that the real stress on the final area is very nearly the same throughout, although the nominal strength of a small, hard, *i.e.* unannealed, wire is considerably greater than that of a large wire.

The above remarks with regard to the properties of wire also apply to the case of cold drawn tubes and extruded metal bars.

The curves given in Fig. 357 clearly show the general effects of wire-drawing on steel; it is possible under certain conditions to get several "passes" without annealing.

The range of elastic extension of wires is far greater than

that of the material in its untreated state; in the latter case the elastic extension is rarely more than $\frac{1}{1000}$ of the length of the bar, but in wires it may reach $\frac{6}{1000}$. The elastic extension curve for a sample of hard steel wire is given in Fig. 358. In some investigations by the author it was found that Young's Modulus for wires was considerably lower than that for the undrawn material. On considering the matter, he concluded that the highly stressed skin of the wire acted as an elastic tube tightly stretched over a core of material, which thereby compressed it transversely, and caused it to elongate longitudinally. If this theory be correct, annealing ought to increase Young's Modulus for the wire. On appealing to experiment it was found that such was the case. A further series

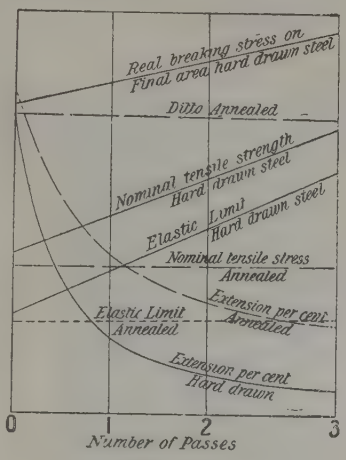


FIG 357.

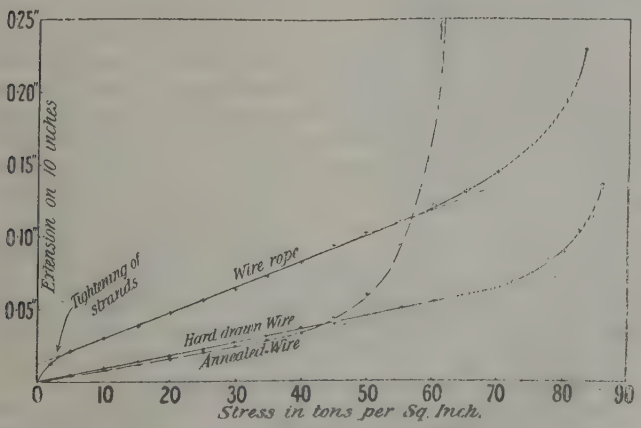


FIG. 358.

of experiments was made on wires of different sizes, all made

from the same billet of steel; the results corroborated the former experiments. In every case the hard-drawn steel wires had a lower modulus than the same wires after annealing. The results were—

Diameter in inches.	Elastic limit.	Maximum stress.	Extension per cent. on 10 ins.	Extension per cent. on 2 ins.	Reduction in area.	E. Pounds sq. inch.	Remarks.
	Pounds per sq. inch.						
0·160	181,700	216,000	1·8	—	34·5	25,430,000	Hard drawn
0·160	59,600	102,400	4·5	—	54·0	28,500,000	Annealed
0·210	78,000	126,700	9·6	19·0	46·8	27,520,000	Rolled rod
0·210	63,800	110,100	10·0	19·5	52·3	27,800,000	Ditto annealed
0·174	125,000	164,800	2·0	6·5	31·5	25,410,000	Rod after one "pass"
0·174	71,600	124,800	4·9	9·3	51·0	27,500,000	Ditto annealed
0·146	160,000	189,700	1·1	5·5	30·4	25,300,000	Rod after two "passes"
0·146	71,300	111,000	3·5	7·5	57·1	27,200,000	Ditto annealed
0·115	188,000	218,900	1·0	5·0	30·2	25,330,000	Rod after three "passes"
					52·8	27,000,000	Ditto annealed
0·115	72,000	113,800	3·1	6·7		21,400,000	Ditto after breaking

Wire Ropes.—The form in which wire is generally used for structural purposes is that of wire ropes. The wires are suitably twisted into strands, and the strands into ropes, either in the same or in the opposite sense as the wires according to the purpose for which the rope is required; for details, special treatises on wire ropes must be consulted.

The hauling capacity of a wire rope entirely depends upon the strength at its weakest spot, which is usually at the attachment of the hook or shackle. The terminal attachment, or the "capping," as it is generally termed, can be accomplished in many ways, but, unfortunately, very few of the methods are at all satisfactory. In the table below the average results of a large number of tests by the author are given.

The method adopted for testing purposes in the Leeds University Machine is shown in Fig. 359. After binding the rope with wire, and tightly "serving" with thick tar band in order to keep the strands in position, the ends are frayed out,

cleaned thoroughly with emery cloth, and finally a hard white metal[1] end is cast on. With ordinary ropes high efficiencies are obtained, but with very hard steel wires, which only stretch a very small amount before breaking, the wires have not the same chance of adjusting themselves to the variable tension in each (due to imperfect manufacture and capping), and consequently tend to break piecemeal at a much lower load than they would if each bore its full share of the load.

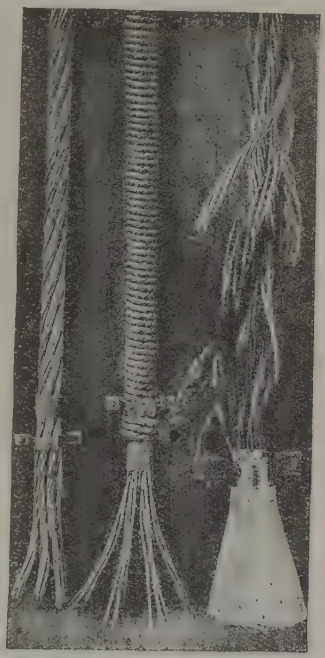


FIG. 359.—Method of capping wire ropes.

On first loading a new wire rope the strain is usually large, due to the tightening up of the strands and wires on one another (see Fig. 359)., but the rope shortly settles down to an elastic condition, then passes an ill-defined elastic limit, and ultimately fractures. From its behaviour during the elastic stage, a value for Young's modulus can be obtained which is always very much lower than that of the wires of which it is composed. This low value is largely due to the tightening of the strands, which continues more or less even up to the breaking load. In old ropes which have taken a permanent "set" the tightening effect is reduced to a minimum, and consequently Young's modulus is greater than for the same rope when new.

The value of E for old ropes varies from 8000 to 10,000 tons per square inch. The strength is often seriously reduced by wear, corrosion, and occasionally by kinks.

[1] A mixture of lead 90 per cent., antimony 10 per cent.

Number of wires.	Diameter of wires.	Section of rope.	E. Tons sq. in.	Breaking load.			Tensile strength of rope.	
				Rope.	Sum of wires.	Ratio.		
	ins.	sq. ins.					tons sq. in.	
20	0·0895	0·126	6000	14·0	14·6	0·96	117	Steel wire ropes
24	0·085	0·136	6870	18·0	18·5	0·97	85	
30	0·091	0·195	5840	23·5	24·6	0·96	127	
36	0·085	0·187	5880	8·25	9·1	0·91	49	
36	0·136	0·522	7200	47·2	47·5	0·99	91	
42	0·081	0·220	5400	20·0	20·4	0·98	90	
56	0·023	0·232	6330	30·9	31·75	0·98	133	
108	0·065	0·358	6020	41·0	42·3	0·97	115	
222	0·044	0·337	5560	35·4	46·3	0·76	105	
222	0·039	0·264	6530	24·9	30·0	0·83	95	
19	0·231	0·796	3770	6·95	8·27	0·84	8·7	Aluminium cable
7	0·231	0·293	3590	2·25	3·03	0·74	7·7	

Work done in fracturing a Bar.—Along one axis a load-strain diagram shows the resistance a bar offers to being pulled apart, and along the other the distance through which this resistance is overcome; hence the product of the two, viz. the area of the diagram, represents the amount of work done in fracturing the bar.

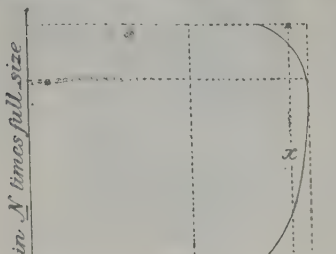


FIG. 360.

Let a = the area of the diagram in square inches;

l = the length of the bar in inches (between datum points);

A = the sectional area of the bar.

If the diagram were drawn 1 inch = 1 ton, and the strain were full size, then a would equal the work done in fracturing the bar; but, correcting for scales, we have—

$$\frac{am}{N} = \text{work done in inch-tons in fracturing the bar}$$

and $$\frac{am}{lAN} = \text{work done in inch-tons per cubic inch in fracturing the bar}$$

Sir Alexander Kennedy has pointed out that the curve during the ductile and plastic stages is a very close approximation to a parabola. Assuming it to be so, the work done can be calculated without the aid of a diagram, thus:

Let L = the elastic limit in tons per square inch;
M = the maximum stress in tons per square inch;
x = the extension in inches.

Then the work done in inch-tons per square inch of section of bar $= Lx + \frac{2}{3}(M - L)x$

$$= \frac{x}{3}(L + 2M)$$

work done in inch-tons per cubic inch $= \frac{x}{3l}(L + 2M)$

But $\frac{x}{l} \times 100 = e$, the percentage of extension

hence the work done in inch-tons per cubic inch $= \frac{e}{300}(L + 2M)$

The work done in inch-tons per cubic inch is certainly by far the best method of measuring the capacity of a given material for standing shocks and blows. Strictly speaking, in order to get comparative results from bars of various lengths, that part of the diagram where stricture occurs should be omitted, but with our present system of recording tests such a procedure would be inconvenient.

The value of the expression for the "work done" in fracturing a bar is evident when one considers the question of bolts which are subjected to jars and vibration. It was pointed out many years ago that ordinary bolts are liable to break off short in the thread when subjected to a severe blow or to long continued vibration, and further, that their life may be greatly increased by reducing the sectional area of the shank down to that of the area at the bottom of the thread. The reason for this is apparent when one calculates the work done in fracturing the bolt in the two cases; it is necessarily very small if the section of the shank be much greater than that at the bottom of the thread, because the bolt breaks before the shank has even passed the elastic limit, consequently all the extension is localized in the short length at the bottom of the thread, but when the area of the shank is reduced the extension is evenly distributed along the bolt. The following tests will serve to emphasize this point:—

Diameter of bolt.	Length.	Work done in fracturing the bolt.	Remarks.
1 in.	13·2 ins.	10·4 inch-tons	Ordinary state
1 ,,	13·2 ,,	39·9 ,, ,,	Turned shank
$\frac{7}{8}$ in.	14 ins.	2·15 inch-tons	Ordinary state
$\frac{7}{8}$,,	14 ,,	16·8 ,, ,,	Turned shank
$\frac{3}{4}$ in.	14 ins.	1·75 inch-tons	Ordinary state
$\frac{3}{4}$,,	14 ,,	7·4 ,, ,,	Turned shank

In this connection it may be useful to remember that the sectional area at the bottom of the thread is—

$$\frac{d(d-1)}{100} \text{ sq. inches (very nearly)}$$

where d is the diameter of the bolt expressed in eighths of an inch. The author is indebted to one of his former students, Mr. W. Stevenson, for this very convenient expression.

Behaviour of Materials subjected to Compression.

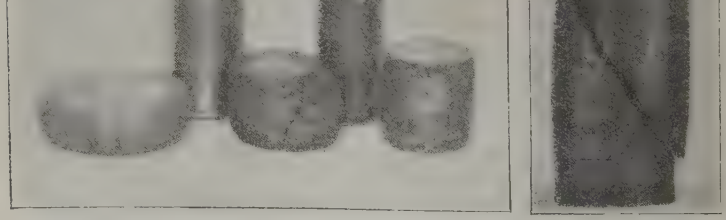


FIG. 361

Ductile Materials.—In the chapter on columns it is shown that the length very materially affects the strength of a piece of material when compressed, and for getting the true compressive strength, very short specimens have to be used in order to

prevent buckling. Such short specimens, however, are inconvenient for measuring accurately the relations between the stress and the strain.

Up to the elastic limit, ductile materials behave in much the same way as they do in tension, viz. the strain is proportional to the stress. At the yield point the strain does not increase so suddenly as in tension, and when the plastic stage is reached, the sectional area gradually increases and the metal spreads. With very soft homogeneous materials, this spreading goes on until the metal is squeezed to a flat disc without fracture. Such materials are soft copper, or aluminium, or lead.

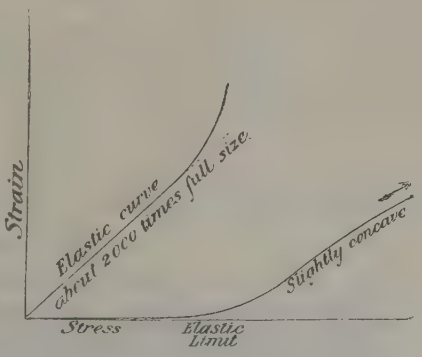


FIG. 362.

In fibrous materials, such as wrought iron and wood, in which the strength across the grain is much lower than with the grain, the material fails by splitting sideways, due to the lateral tension.

The usual form of the stress-strain curve for a ductile material is somewhat as shown in Fig. 362.

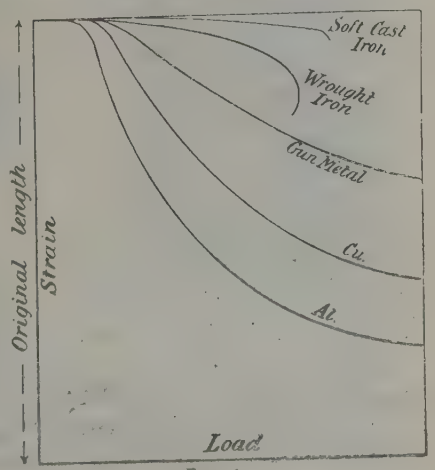


FIG. 363.

If the material reached a perfectly plastic stage, the real stress, *i.e.* $\frac{\text{load at any instant (W)}}{\text{sectional area at that instant } (A_1)}$,

would be constant, however much the material was compressed; then, using the same notation as before—

$$A_1 = \frac{lA}{l_1}$$

$$\text{and from above, } \frac{W}{A_1} = \text{constant}$$

Substituting the value of A_1 from above, we have—

$$\frac{Wl_1}{lA} = \text{constant};$$

But lA, the volume of the bar, is constant;

$$\text{hence } Wl_1 = \text{constant}$$

or the stress-strain curve during the plastic period is a hyperbola. The material never is perfectly plastic, and therefore never perfectly complies with this, but in some materials it very nearly approaches it. For example, copper and aluminium (author's recorder) (Fig. 363). The constancy of the real stress will be apparent when we draw the real stress curves.

FIG. 364.—Asphalte.

Brittle Materials.—Brittle materials in compression, as in tension, have no marked elastic limit or plastic stage. When crushed they either split up into prisms or, if of cubical form, into pyramids, and sometimes by the one-half of the specimen shearing over the other at an angle of about 45°. Such a fracture is shown in Fig. 361 (cast iron).

The shearing fracture is quite what one might expect from purely theoretical reasoning. In Fig. 365 let the sectional area = A;

$$\text{then the stress on the cross-section } S = \frac{W}{A}$$

and the stress on an oblique section *aa*, making an angle α

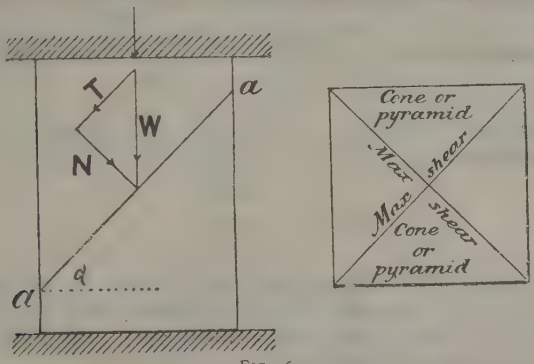


FIG. 365.

FIG. 366.—Portland cement.

with the cross-section, may be found thus: resolve W into components normal $N = W \cos \alpha$, and tangential $T = W \sin \alpha$.

The area of the oblique section $aa = A_0 = \frac{A}{\cos a}$

the normal stress $= \frac{N}{A_0} = \frac{W \cos a}{\frac{A}{\cos a}} = \frac{W \cos^2 a}{A} = S \cdot \cos^2 a$

tangential or shearing stress $= \frac{T}{A_0} = \frac{W \sin a}{\frac{A}{\cos a}} = \frac{W \cos a \sin a}{A}$

$= S \cos a \sin a$

If we take a section at right angles to *aa*, T becomes the normal component, and N the tangential, and it makes an angle of $90 - a$ with the cross-section; then, by similar reasoning to the above, we have—

The area of the oblique section $= A_0' = \frac{A}{\cos (90 - a)} = \frac{A}{\sin a}$

normal stress $= S \sin^2 a$

tangential stress or shearing stress $= S \sin a \cos a$

So that the tangential stress is the same on two oblique sections at right angles, and is greatest when $a = 45°$; it is then $= S \times 0{\cdot}71 \times 0{\cdot}71 = 0{\cdot}5\ S$.

From this reasoning, we should expect compression specimens to fail by shearing along planes at 90° to one another, and a cylindrical specimen to form two cones top and bottom, and a cube to break away at the sides and become six pyramids. That this does occur is shown by the illustrations in Figs. 364–366.

Real and Nominal Stress in Compression (Fig. 367).—In the paragraph on real and nominal stress in tension, we showed how to construct the curve of real stress from the ordinary load-strain diagram. Then, assuming that the volume remains constant and that the compression specimen remains parallel (which is not quite true, as the specimens always become barrel-shaped), the same method of constructing the real stress curve serves for compression. As in the tension curve, it is evident that (see Fig. 347)—

$$\frac{S_1}{S} = \frac{L_1}{L}$$

$$S_1 = \frac{SL_1}{L}$$

Behaviour of Materials subjected to Shear. Nature of Shear Stress.—If an originally square plate or block be

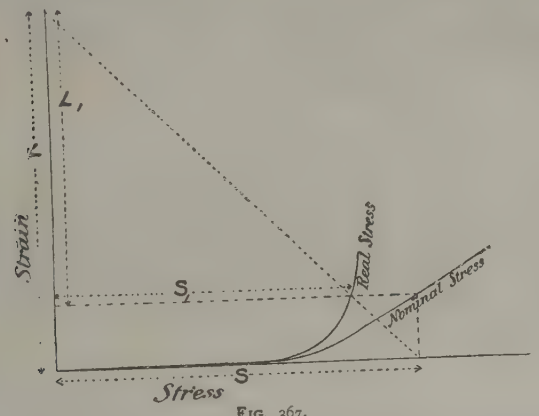


Fig. 367.

acted upon by forces P parallel to two of its opposite sides, the square will be distorted into a rhombus, as shown in Fig. 368,

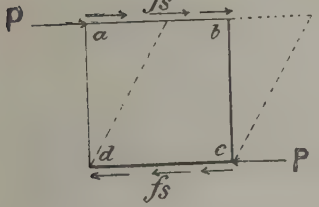


Fig. 368

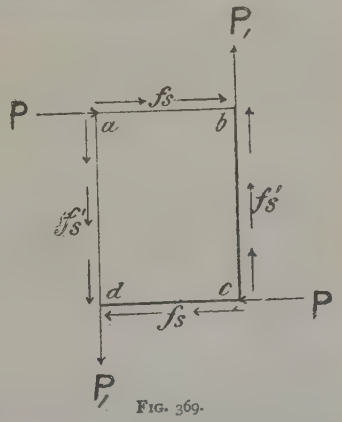


Fig. 369.

and the shearing stress will be $f_s = \frac{P}{ab}$, taking it to be of unit thickness. This block, however, will spin round due to the couple P . *ad* or P . *bc*, unless an equal and opposite couple be applied to the block. In order to make the following remarks

perfectly general, we will take a rectangular plate as shown in Fig. 369.

The plate is acted upon by a clockwise couple, P . *ad*, or f_s . *ab* . *ad*, and a contra-clockwise couple, P_1 . *ab* or f_s' . *ad* . *ab*, but these must be equal if the plate be in equilibrium ;

$$\text{then } f_s \,.\, ab \,.\, ad = f_s' \,.\, ad \,.\, ab$$
$$\text{or } f_s = f_s'$$

i.e. the intensity of stress on the two sides of the plate is the same.

Now, for convenience we will return to our square plate. The forces acting on the two sides P and P_1 may be resolved into forces R and R_1 acting along the diagonals as shown in Fig. 370. The effect of these forces will be to distort the square into a rhombus exactly as before. (N.B.—The rhombus

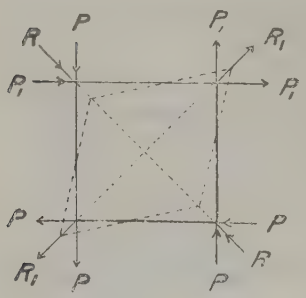


Fig. 370.

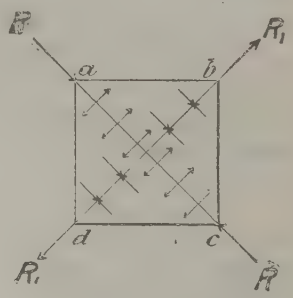


Fig. 371.

in Fig. 368 is drawn in a wrong position for simplicity.) These two forces act at right angles to one another ; hence we see that a shear stress consists of two equal and opposite stresses, a tension and a compression, acting at right angles to one another.

In Fig. 371 it will be seen that there is a tensile stress acting normal to one diagonal, and a compressive stress normal to the other. The one set of resultants, R, tend to pull the two triangles *abc*, *acd* apart, and the other resultants to push the two triangles *abd*, *bdc* together.

Let f_0 = the stress normal to the diagonal.

$$\text{Then } f_0 ac = f_0 \sqrt{2}\, ab, \text{ or } f_0 \sqrt{2}\, bc = R$$
$$\text{But } \sqrt{2}P, \text{ or } \sqrt{2} f_s \,.\, ab, \text{ or } \sqrt{2} f_s \,.\, bc = R$$
$$\text{hence } f_0 = f_s = f_s'$$

Thus the intensity of shear stress is equal on all the four edges and the tension and compression on the two diagonals of a rectangular plate subjected to shear.

Materials in Shear.—When ductile materials are sheared, they pass through an elastic stage similar to that in tension and compression. If an element be slightly distorted, it will return to its original form on the removal of the stress, and during this period the strain is proportional to the stress; but after the elastic limit has been reached, the plate becomes permanently deformed, but has not any point of sudden alteration as in tension. On continuing to increase the stress, a ductile and plastic stage is reached, but as there is no alteration of area under shear, there is no stage corresponding with the stricture stage in tension.

The shearing strength of ductile materials, both at the elastic limit and at the maximum stress, is about $\frac{4}{5}$ of their tensile strength (see p. 400).

Ductile Materials in Shear.—The following results, obtained in a double-shear shearing tackle, will give some idea of the relative strengths of the same bars when tested in tension and in shear; they are averages of a large number of tests :—

Material.	Nominal tensile strength.	Shearing strength.	Shearing strength. / Tensile strength.	Work done per sq. in. of metal sheared through.
Cast iron—hard, close grained	14·6	13·5	0·92	—
,, ordinary . . .	10·9	12·9	1·18	—
,, soft, open grained	7·9	10·7	1·36	—
Best wrought iron	22·0	18·1	0·82	5·7
Mild steel	26·6	20·9	0·79	6 9
Hard steel	48·0	34·0	0·71	3·7
Gun-metal	13·5	15·2	1 13	0·9
Copper	15·0	11·0	0·73	4·5
Aluminium	8·8	5·7	0·65	1·0

From autographic shearing and punching diagrams, it is found that the maximum force required occurs when the shearing tackle is about $\frac{1}{6}$ of the way through the bar, and when the punch is about $\frac{1}{3}$ of the way through the plate.

From a series of punching tests it was found that where—

$$f_s = \frac{\text{load on punch}}{\text{circumference of hole} \times \text{thickness of plate}}$$

	f_s	f_t	Ratio.
Wrought iron . .	19·8	24·8	0·80
Mild steel . . .	22·2	28·4	0·78
Copper	10·4	14·7	0·71

Brittle materials in shear are elastic, although somewhat imperfectly in some cases, right up to the point of fracture; they have no marked elastic limit.

It is generally stated in text-books that the shearing strength of brittle materials is much below $\frac{4}{5}$ of the tensile strength, but this is certainly an error, and has probably come about through the use of imperfect shearing tackle, which has caused double shear specimens to shear first through one section, and then through the other. In a large number of tests made in the author's laboratory, the shearing strength of cast iron has come out rather higher than the tensile stress in the ratio of 1·1 to 1.

Shear combined with Tension or Compression.—We have shown above that when a block or plate, such as *abcd*, is subjected to a shear, there will be a direct stress acting normally to the diagonal *bd*. Likewise if the two sides *ad*, *bc* are subjected to a normal stress, there will be a direct stress acting normally to the section *ef*; but when the block is subjected to both a direct stress and a shear, there will be a direct stress acting normally to a section occupying an intermediate position, such as *gh*.

Consider the stresses acting on the triangular element shown, which is of unit thickness. The intensity of the shear stress on the two edges will be equal (see p. 390) Hence—

The total shear stress on the face $gi = f_s \cdot gi = P_1$
" " " " $hi = f_s \cdot hi = P$
" direct " " $gi = f_t \cdot gi = T$

Let the resultant direct stress on the face *gh*, which we are about to find in terms of the other stresses, be f_n. Then the total direct stress acting normal to the face $gh = f_n \cdot gh = P_2$.

Now consider the two horizontal forces acting on the

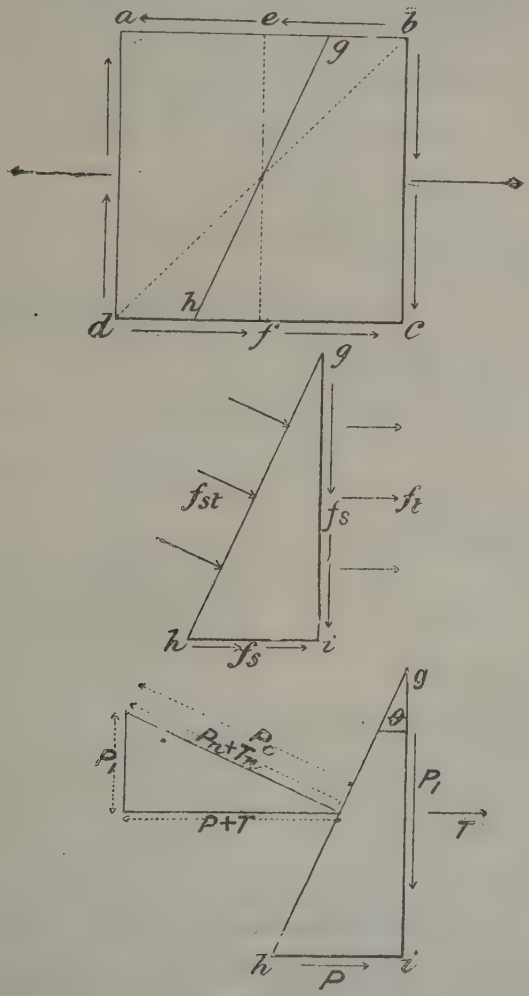


FIG. 372.

element, viz. T and P, and resolve them normally to the face *gh* as shown, we get T_n and P_n.

$$\text{But } P_n + T_n = \frac{P + T}{\cos\theta} = \frac{f_s \cdot \overline{hi} + f_t \cdot \overline{gi}}{\cos\theta}$$

$$\text{also } P_n + T_n = P_0 = f_{st}\overline{gh}$$

hence, substituting the above values, we have—

$$f_t \cdot \overline{gi} + f_s \cdot \overline{hi} = f_{st}\overline{gh}\cos\theta = f_{st} \cdot \overline{gi}$$

$$\text{and } f_t + \frac{f_s \cdot \overline{hi}}{\overline{gi}} = f_{st}$$

Next consider the vertical force acting on the element, viz. P_1.

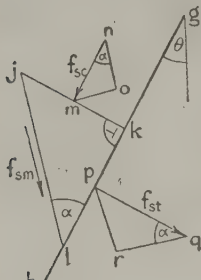


FIG. 373.

$$P_1 = P_0 \sin\theta = f_{st}\overline{gh}\sin\theta = f_{st}\overline{hi}$$

$$\text{or } f_s \cdot \overline{gi} = f_{st}\overline{hi} \text{ and } \frac{f_s \cdot \overline{gi}}{f_{st}} = \overline{hi}$$

$$\frac{\overline{hi}}{\overline{gi}} = \frac{f_s}{f_{st}} = \tan\theta$$

Substituting this value of *hi* in the equation above, we have—

$$f_t + f_s\frac{f_s \cdot \overline{gi}}{f_{st} \cdot \overline{gi}} = f_{st}$$

$$\text{or } f_{st}f_t + f_s^2 = f_{st}^2$$

$$f_{st}^2 - f_{st}f_t = f_s^2$$

Solving, we get

$$f_{st} = \frac{f_t}{2} \pm \sqrt{\frac{f_t^2}{4} + f_s^2}$$

The maximum tensile stress on the face *gh*

$$f_{st} = \frac{f_t}{2} + \sqrt{\frac{f_t^2}{4} + f_s^2}$$

and the compressive stress on the face at right angles to *gh*, viz. *jk*

$$f_{sc} = \frac{f_t}{2} - \sqrt{\frac{f_t^2}{4} + f_s^2}$$

Some materials are more liable to fail by shear than by tension, hence it is necessary to find the maximum shear stress. Draw *jk* at right angles to *gh*, and *jl* making an angle α with it, the value of which will be determined. The compressive stress on *jk* is f_{sc}, and the total stress represented by *mn* is $jk \cdot f_{sc}$. The maximum shear stress f_{sm} occurs on the face *jl*, and the total shear stress on *jl* is $f_{sm} \cdot jl$. The tensile stress on the face *kl* is f_{st} and the total stress is $kl \cdot f_{st}$ which is represented by *pq*. If the tensile stress be + the compressive stress will be −. Since

we want to find the shear stress we must resolve these stresses in the direction of the shear. Resolve *pq*, also *mn*, parallel and normal to *jl*.

$$pr = pq \sin \alpha = kl \cdot f_{st} \sin \alpha$$

$$on = mn \cos \alpha = -jk \cdot f_{sc} \cos \alpha$$

$$f_{sm} jl = pr - on = kl \cdot f_{st} \sin \alpha - jk \cdot f_{sc} \cos \alpha$$

$$f_{sm} = \frac{kl}{jl} f_{st} \sin \alpha - \frac{jk}{jl} f_{sc} \cos \alpha$$

$$= f_{st} \cos \alpha \sin \alpha - f_{sc} \cos \alpha \sin \alpha = (f_{st} - f_{sc}) \cos \alpha \sin \alpha$$

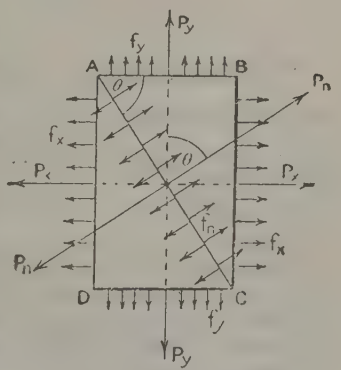
$$= \frac{f_{st} - f_{sc}}{2} \sin 2\alpha.$$

This is a maximum when $2\alpha = 90^\circ$ and $\sin 2\alpha = 1$.

Then $$f_{sm} = \frac{f_{st} - f_{sc}}{2} = \frac{\frac{f_t}{2} + \sqrt{\frac{f_t^2}{4} + f_s^2} - \frac{f_t}{2} + \sqrt{\frac{f_t^2}{4} + f_s^2}}{2}$$

$$f_{sm} = \sqrt{f_s^2 + \frac{f_t^2}{4}}$$

This is the maximum intensity of shear stress which occurs when a piece of elastic material is subjected to a direct stress f_t and a shear stress f_s. The direction of the most stressed section is inclined at 45° to that on which the maximum intensity of tensile stress occurs.



Compound Stresses.—Let the block ABCD, of uniform thickness be subjected to tensile stresses f_x and f_y acting normal to the mutually perpendicular faces AD and DC or BC and AB. The force

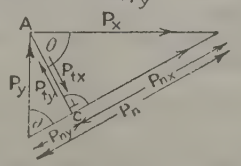


FIG. 374.

$P_x = f_x \cdot AD$ or $f_x \cdot BC$
$P_y = f_y \cdot AB$ or $f_y \cdot DC$
$P_n = f_n \cdot AC$
$P_t = f_t \cdot AC$. (f_t is the tangential or shear stress).

It is required to find (i) the intensity and direction of the

normal stress f_n acting on the face AC, the normal to which makes an angle θ with the direction of P_y. (ii) the intensity of the shear or tangential stress f_t acting on the face AC.

Resolving perpendicular to AC we have—

$$P_n = P_{ny} + P_{nx} = P_y \cos\theta + P_x \sin\theta$$
$$f_n AC = f_y AB \cos\theta + f_x BC \sin\theta$$
$$f_n = f_y \frac{AB}{AC} \cos\theta + f_x \frac{BC}{AC} \sin\theta$$
$$f_n = f_y \cos^2\theta + f_x \sin^2\theta$$

also
$$P_t = P_{ty} - P_{tx}$$
$$P_t = P_y \sin\theta - P_x \cos\theta$$
$$f_t AC = f_y AB \sin\theta - f_x BC \cos\theta$$
$$f_t = f_y \frac{AB}{AC} \sin\theta - f_x \frac{BC}{AC} \cos\theta$$
$$f_t = f_y \cos\theta \sin\theta - f_x \sin\theta \cos\theta$$
$$f_t = (f_y - f_x) \cos\theta \sin\theta = \frac{f_y - f_x}{2} \sin 2\theta$$

This is a maximum when $\theta = 45°$. The tangential or shear stress is then $= \frac{f_y - f_x}{2}$. Thus the maximum intensity of shear stress is equal to one half the difference between the two direct stresses.

The same result was obtained on page 395 by a different process.

Poisson's Ratio.—When a bar is stretched longitudinally, it contracts laterally; likewise when it is compressed longitudinally, it bulges or spreads out laterally. Then, terming stretches or spreads as positive (+) strains, and compressions or contractions as negative (−) strains, we may say that when the longitudinal strain is positive (+), the lateral strain is negative (−).

Let the lateral strain be $\frac{1}{n}$ of the longitudinal strain. The fraction $\frac{1}{n}$ is generally known as Poisson's ratio, although in reality Poisson's ratio is but a special value of the fraction, viz. $\frac{1}{4}$.

Strains resulting from Three Direct Stresses acting at Right Angles.—In the following paragraph it will

be convenient to use suffixes to denote the directions in which the forces act and in which the strains take place. Thus any force P which acts, say, normal to the face x will be termed P_x, and the strain per unit length $\frac{X_x}{l_x}$ will be termed S_x, and the stress on the face f_x; then $S_x = \frac{f_x}{E}$ (see p. 374).

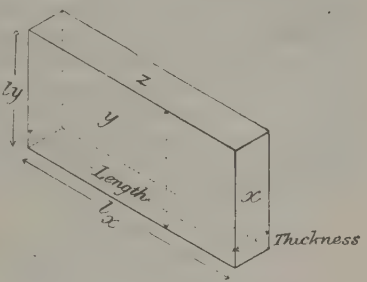


FIG. 375.

Every applied force which produces a stretch or a + strain in its own direction will be termed +, and *vice versâ*.

The strains produced by forces acting in the various directions are shown in tabulated form below.

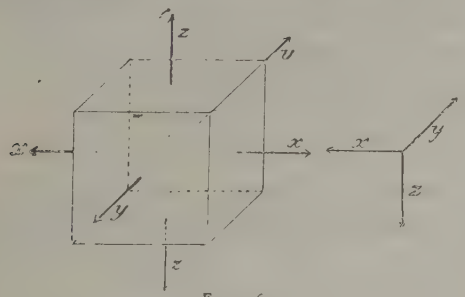


FIG. 376.

Force acting on face of cube.	Strain in direction x. S_x.	Strain in direction y. S_y.	Strain in direction z. S_z.
P_x	$\frac{f_x}{E}$	$-\frac{f_x}{En}$	$-\frac{f_x}{En}$
P_y	$-\frac{f_y}{En}$	$\frac{f_y}{E}$	$-\frac{f_y}{En}$
P_z	$-\frac{f_z}{En}$	$-\frac{f_z}{En}$	$\frac{f_z}{E}$

These equations give us the strains in any direction due to the stresses f_x, f_y, f_z acting alone; if two or more act together, the resulting strain can be found by adding the separate strains, due attention being paid to the signs.

Shear.—We showed above (p. 390) that a shear consists of two equal stresses of opposite sign acting at right angles to one another. The resulting strain can be obtained by adding the strains given in the table above due to the stresses f_1 and f_2, which are of opposite sign and act at right angles to one another.

The strains are—

$$\frac{f_1}{E} + \frac{f_2}{nE} = \frac{f}{E}\left(1 + \frac{1}{n}\right) \text{ in the direction (1)}$$

$$-\frac{f_2}{E} - \frac{f_1}{nE} = -\frac{f}{E}\left(1 + \frac{1}{n}\right) \quad \text{,,} \quad \text{,,} \quad (2)$$

$$-\frac{f_1}{nE} + \frac{f_2}{nE} = 0 \quad \text{,,} \quad \text{,,} \quad (3)$$

Thus the strain in two directions has been increased by $\frac{1}{n}$ due to the superposition of the two stresses, and has been reduced to zero in the third direction.

Let $S = \frac{f}{E}\left(1 + \frac{1}{n}\right)$.

If a square *abcd* had been drawn on the side of the element, it would have become the rhombus *a'b'c'd'* after the strain, the

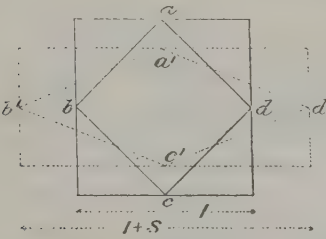


Fig. 377.

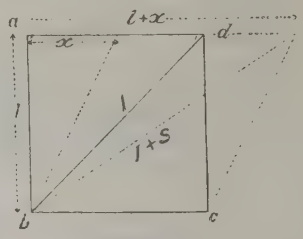


Fig. 378.

long diagonal of the rhombus being to the diagonal of the square as $1 + S$ to 1. The two superposed are shown in Fig. 378. Then we have—

$$l^2 + (l + x)^2 = (1 + S)^2$$

$$\text{or } 2l^2 + 2lx + x^2 = 1 + 2S + S^2$$

But as the diagonal of the square = 1, we have—

$$2l^2 = 1$$

And let $\frac{x}{l} = S_0$; $x = lS_0$; then by substitution we have—

$$2l^2 + 2l^2S_0 + l^2S_0^2 = 1 + 2S + S^2$$

$$\text{and } 1 + S_0 + \frac{S_0^2}{2} = 1 + 2S + S^2$$

Both S and S_0 are exceedingly small fractions, never more than about $\frac{1}{1000}$, and their squares will be still smaller, and therefore negligible. Hence we may write the above—

$$1 + S_0 = 1 + 2S$$

$$\text{or } S_0 = 2S = \frac{2f}{E}\left(1 + \frac{1}{n}\right)$$

$$\text{But } G = \frac{f}{S_0} = \frac{fE}{2f\left(1 + \frac{1}{n}\right)} = \frac{E}{2}\left(\frac{1}{1 + \frac{1}{n}}\right) = \frac{En}{2(n+1)}$$

$$\text{hence } n = \frac{2G}{E - 2G}, \text{ and } E = \frac{2G(n+1)}{n}$$

When $n = 5$, $G = \frac{5}{12}E = 0{\cdot}42E$.
$n = 4$, $G = \frac{2}{5}E = 0{\cdot}40E$.
$n = 3$, $G = \frac{3}{8}E = 0{\cdot}38E$.
$n = 2$, $G = \frac{1}{3}E = 0{\cdot}33E$.

Some values of n will be given shortly.

We have shown above that the maximum strain in an element subject to shear is—

$$S = \frac{f}{E}\left(1 + \frac{1}{n}\right)$$

but the maximum strain in an element subject to a direct stress in tension is—

$$S_t = \frac{f}{E}$$

$$\text{hence } \frac{S}{S_t} = \frac{\frac{f}{E}\left(1 + \frac{1}{n}\right)}{\frac{f}{E}} = \left(1 + \frac{1}{n}\right)$$

$$\text{or } S = S_t\left(1 + \frac{1}{n}\right)$$

	$\frac{S}{S_t}$	$\frac{\text{Safe shear stress}}{\text{safe tensile stress}}$
When $n = 5$	$\frac{6}{5}$	$\frac{5}{6}$
,, $n = 4$	$\frac{5}{4}$	$\frac{4}{5}$
,, $n = 3$	$\frac{4}{3}$	$\frac{3}{4}$
,, $n = 2$	$\frac{3}{2}$	$\frac{2}{3}$

Taking $n = 4$, we see that the same material will take a permanent set, or will pass the elastic limit in shear with $\frac{4}{5}$ of the stress that it will take in tension; or, in other words, the shearing strength of a material is only $\frac{4}{5}$ of the tensile strength. Although this proof only holds while the material is elastic, yet the ratio is approximately correct for the ultimate strength.

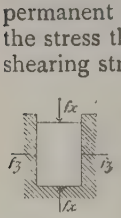


FIG. 379.

Bar under Longitudinal Stress, but with Lateral Strain prevented in one Direction.—Let a bar be subjected to longitudinal stress, f_x, in the direction x, and be free in the direction y, but be held in the direction z.

The strain in the direction z due to the stresses f_z and f_x is $\frac{f_z}{E} - \frac{f_x}{nE} = 0$, because the strain is prevented in this direction.

$$\text{Hence } \frac{f_z}{E} = \frac{f_x}{nE}$$

The strain in the direction x due to these stresses is—

$$\frac{f_x}{E} - \frac{f_z}{nE} = \frac{f_x}{E} - \frac{f_x}{n^2E} = \frac{f_x}{E}\left(1 - \frac{1}{16}\right) = \frac{15}{16}\frac{f_x}{E}$$

Thus the longitudinal strain of a bar held in this manner is only $\frac{15}{16}$ as great as when the bar is free in both lateral directions.

Bar under Longitudinal Stress with Lateral Strain prevented in both Directions.

$$\text{Strain in direction } x = \frac{f_x}{E} - \frac{f_y}{nE} - \frac{f_z}{nE}$$

$$y = \frac{f_y}{E} - \frac{f_x}{nE} - \frac{f_z}{nE} = 0 \quad \left.\begin{array}{l}\text{because the strain is pre-}\\ \text{vented in these}\\ \text{directions.}\end{array}\right.$$

$$z = \frac{f_z}{E} - \frac{f_x}{nE} - \frac{f_y}{nE} = 0$$

Then $\frac{f_y}{E} = \frac{f_x}{nE} + \frac{f_z}{nE}$

and $f_x = nf_y - f_z = f_y(n-1)$ since $f_y = f_z$

The strain in direction x is—

$$\frac{f_x}{E} - \frac{2f_x}{nE(n-1)} = \frac{f_x}{E}\left(1 - \frac{2}{n(n-1)}\right) = \frac{5}{6}\frac{f_x}{E}$$

Or the longitudinal strain of a bar held in this manner is only $\frac{5}{6}$ as great as when the bar is free.

Anticlastic Curvature.—When a beam is bent into a circular arc some of the fibres are stretched and some compressed (see the chapter on beams), the amount depending upon their distance from the N.A.; due to the extension of the fibres, the tension side of the beam section contracts laterally and the compression side extends. Then, corresponding to the strain at EE on the beam profile, we have $\frac{1}{n}$ as much at E'E' on the section; also, the proportional strain—

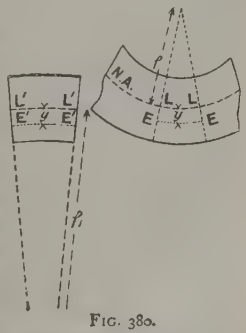


FIG. 380.

$$\frac{EE - LL}{LL} = \frac{y}{\rho}$$

$$\text{and } \frac{E'E' - L'L'}{L'L'} = \frac{y}{\rho_1}$$

$$\text{But } \frac{y}{\rho_1} = \frac{1}{n} \cdot \frac{y}{\rho}$$

$$\text{hence } \rho_1 = n\rho = 4\rho$$

This relation only holds when all the fibres are free laterally, which is very nearly the case in deep narrow sections; but if the section be shallow and wide, as in a flat plate, the layers which would contract sideways are so near to those which would extend sideways, that they are to a large extent prevented from moving laterally; hence the material in a flat wide beam is nearly in the state of a bar prevented from contracting laterally in one direction. Hence the beam is stiffer in the ratio of $\frac{n^2}{n^2-1} = \frac{16}{15}$ than if the section were narrow.

Boiler Shell.—On p. 421 we show that $P_x = 2P_z$; $f_x = 2f_z$; $f_z = \frac{f_x}{2}$.

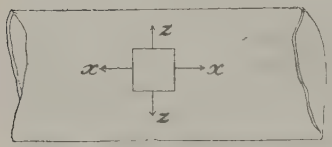


FIG. 381.

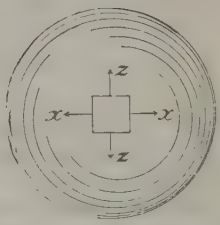


FIG. 382.

Strains.

Let $n = 4$.

$$S_x = \frac{f_x}{E} - \frac{f_z}{En} = \frac{f_x}{E}\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{4}\frac{f_x}{E}$$

$$S_y = -\frac{f_x}{En} - \frac{f_z}{En} = -\frac{f}{E}\left(\frac{1}{2n} + \frac{1}{n}\right) = -\frac{3}{8}\frac{f_x}{E}$$

$$S_z = -\frac{f_x}{En} + \frac{f_z}{E} = \frac{f_z}{E}\left(1 - \frac{1}{2n}\right) = \frac{7}{8}\frac{f_z}{E}$$

Thus the maximum strain is in the direction S_z.

By the thin-cylinder theory we have the maximum strain $= \frac{f_z}{E}$; thus the real strain is only $\frac{7}{8}$ as great, or a cylindrical boiler shell will stand $\frac{8}{7} = 1{\cdot}14$ or 14 per cent. more pressure before the elastic limit is reached than is given by the ordinary ring theory.

Thin Sphere subjected to Internal Fluid Pressure.—In the case of the sphere, we have $P_x = P_z$; $f_x = f_z$.

Strains.—

$$S_x = \frac{f_x}{E} - \frac{f_z}{En} = \frac{f_x}{E}\left(1 - \frac{1}{n}\right) = \frac{3}{4}\frac{f_x}{E}$$

$$S_y = -\frac{f_x}{En} - \frac{f_z}{En} = -\frac{f_x}{E}\left(\frac{1}{n} + \frac{1}{n}\right) = -\frac{1}{2}\frac{f_x}{E}$$

$$S_z = -\frac{f_x}{En} + \frac{f_z}{E} = \frac{f_z}{E}\left(1 - \frac{1}{n}\right) = \frac{3}{4}\frac{f_x}{E}$$

But f_x in this case $= \frac{f_x}{2}$ in the case given above;

$$\therefore S_x \text{ in this case} = \tfrac{3}{4} \times \frac{f_x}{2E} = \frac{3f_x}{8E}$$

$$\frac{\text{Maximum strain in sphere}}{\text{maximum strain in boiler shell}} = \frac{\frac{3}{8}}{\frac{7}{8}} = \frac{3}{7}$$

Hence, in order that the hemispherical ends of boilers should enlarge to the same extent as the cylindrical shells when under pressure, the plates in the ends should be $\frac{3}{7}$ thickness of the plates in cylindrical portion. If the proportion be not adhered to, bending will be set up at the junction of the ends and the cylindrical part.

Cylinder exposed to Longitudinal Stress when under Internal Pressure.—When testing pipes under pressure it is a common practice to close the ends by flat plates held in position by one long bolt passing through the pipe and covers, or several long bolts outside the pipe. The method may be convenient, but it causes the pipes to burst at lower pressures than if flanged covers were used. In the case of pipes with no flanges a long rod fitted with two pistons and cup leathers can be inserted in the pipe, and the water pressure admitted between them through one of the pistons, which produces a pure ring stress, with the result that the pipes burst at a much higher pressure than when tested as described above.

(i) Strain with simple ring stress $= \frac{f_x}{E}$

Pressure required to burst a thin cylinder $p = \frac{2ft}{d}$

(ii) Strain with flanged covers $= \frac{7}{8}\frac{f_x}{E}$

Pressure required to burst a thin cylinder $= p = \frac{8}{7}\frac{2ft}{d}$

(iii) When the covers are held in position by a longitudinal bolt, the load on which is P_1, the longitudinal compression in the walls of the pipe is $\frac{P}{\pi dt} = f_c$.

The circumferential ring strain due to internal pressure and longitudinal compression—

$$\frac{f_x}{E} + \frac{f_o}{nE} = \frac{f_x}{E}\left(1 + \frac{m}{n}\right)$$

where $m = \dfrac{f_o}{f_x}$

and $p = \dfrac{n}{n+m} \times \dfrac{2ft}{d}$

Example.—Let $d = 8$ ins. $t = 0{\cdot}5$ in. $f_x = 10{,}000$ lbs. sq. in.

Simple ring stress $p = 1250$ lbs. sq. in.

With flanged covers $p = 1428$ lbs. sq. in.

With longitudinal bolt and let $m = 2$. This is a high value but is sometimes experienced } $p = 830$ lbs. sq. in.

Thus if a pipe be tested with longitudinal bolts under the extreme condition assumed, the bursting pressure will be only 58 per cent. of that obtained with flanged covers.

Alteration of Volume due to Stress.—If a body were placed in water or other fluid, and were subjected to pressure, its volume would be diminished in proportion to the pressure.

Let V = original volume of body;

δv = change of volume due to change of pressure;

δp „ pressure.

Then $K = \dfrac{\delta p}{-\dfrac{\delta v}{V}} = \dfrac{\text{change of pressure}}{\text{change of volume per cubic unit of the body}}$

$$K = \frac{p}{v}$$

K is termed the coefficient of elasticity of volume.

The change of volume is the algebraic sum of all the strains produced. Then, putting $p = f_x = f_y = f_z$ for a fluid pressure, we have, from the table on p. 397, the resulting strains—

$$\frac{p}{E} - \frac{2p}{En} + \frac{p}{E} - \frac{2p}{En} + \frac{p}{E} - \frac{2p}{En} = \frac{3p}{E} - \frac{6p}{En} \text{ or } v = \frac{3p}{E} - \frac{6p}{En}$$

$$K = \frac{p}{v} = \frac{pEn}{3pn - 6p} = \frac{En}{3n - 6} = \frac{EG}{9G - 3E}$$

But $E = \dfrac{2G(n+1)}{n}$ (p. 399)

hence $K = \dfrac{2G(n+1)}{3n-6} = \frac{1}{3}\left(\dfrac{n}{n-2}\right)E$ and $n = \dfrac{2G + 6K}{3K - 2G}$

The following table gives values of K in tons per square inch also of n:—

Material.	K.	n.
Water	140	—
Cast iron ...	6,000	3·0 to 4·7
Wrought iron	8,800	3·6
Steel	11,000	3·6 to 4·6
Brass	6,400	3·1 to 3·3
Copper ...	10,500	2·9 to 3·0
Flint glass ..	2,400	3·9
Indiarubber ...	—	2·0

The n given above has not been calculated by the above formula, but is the mean of the most reliable published experiments.

Strain Energy Stored in a Plate.—Let the plate be subjected to stresses f_x and f_y.

$$\text{Strain in direction } x = \frac{f_x}{E} - \frac{f_y}{nE}$$

$$\text{,, ,, } y = \frac{f_y}{E} - \frac{f_x}{nE}$$

$$\text{Energy stored per cubic inch} = \frac{\text{strain} \times \text{stress}}{2}$$

$$= \left(\frac{f_x}{E} - \frac{f_y}{nE}\right)\frac{f_x}{2} + \left(\frac{f_y}{E} - \frac{f_x}{nE}\right)\frac{f_y}{2}$$

$$= \frac{1}{2E}\left(f_x^2 + f_y^2 - \frac{2f_xf_y}{n}\right)$$

Strength of Flat Plates.—An attempt at an exact treatment of the strength of flat plates subjected to fluid pressure is long and tedious. See Morley's "Strength of Materials." The following approximate treatment yields results sufficiently accurate for practical purposes.

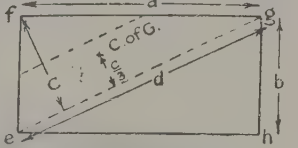


FIG. 383.

Rectangular Plate of thickness t. Consider a diagonal section d.

The moment of the water pressure about the diagonal acting on the triangle efg $= \frac{pab}{2} \times \frac{c}{3}$

The moment of the reactions of the edges of the plate when freely supported, the resultant is assumed to act at the middle of each side $= \frac{pab}{2} \times \frac{c}{2}$

The resulting moment is equal to the moment of resistance of the plate to bending across the diagonal, hence—

$$p\frac{ab}{2} \times \frac{c}{2} - p\frac{ab}{2} \times \frac{c}{3} = f\frac{dt^2}{6} \quad \text{(See Chapter XI)}$$

$$\frac{pabc}{2dt^2} = f$$

But $d = \sqrt{a^2 + b^2}$ and $c = \frac{ab}{\sqrt{a^2 + b^2}}$

Hence $\frac{pa^2b^2}{2(a^2 + b^2)t^2} = f$ when freely supported.

and $\frac{pa^2b^2}{3(a^2 + b^2)t^2} = f$ when the edges are rigidly held.

For a square plate of side a this becomes

$$\frac{pa^2}{4t^2} = f \text{ when freely supported.}$$

and $\frac{pa^2}{6t^2} = f$ when the edges are securely bolted.

From experiments on such plates, Mr. T. H. Bryson, of Troy, U.S.A., arrived at the expression

$$\frac{pa^2}{7t^2} = f$$

When the length of a is very great as compared with b, the support from the ends is negligible. When the edges are rigidly held, the plate simply becomes a built-in beam of breadth a, depth t, span b—

$$p \cdot ab \cdot \frac{b}{12} = f\frac{at^2}{6}$$

$$\frac{pb^2}{2t^2} = f$$

For a circular plate of diameter d and radius r

The moment of the water pressure about a diameter $= \frac{\pi r^2 p}{2} \times \frac{4r}{3\pi}$

The moment of the reactions of the edge of the plate when freely supported $= \frac{\pi r^2 p}{2} \times \frac{2r}{\pi}$

Then $\frac{\pi r^2 f}{2}\left(\frac{2r}{3\pi}\right) = f\frac{2rt^2}{6}$ $\frac{pd^2}{4t^2} = f$

When the edges are rigidly held $\frac{pd^2}{6t^2} = f$

Riveted Joints.

Strength of a Perforated Strip.—If a perforated strip of width w be pulled apart in the testing-machine, it will break through the hole, and if the material be only very slightly ductile, the breaking load will be (approximately) $W = f_t(w - d)t$, where f_t is the tensile strength of the metal, and t the thickness of the plate. If the metal be very ductile the breaking load will be higher than this, due to the fact that the tensile strength is always reckoned on the original area of the test bar, and not on the final area at the point of fracture. The difference between the real and the nominal tensile strength, therefore, depends upon the reduction in area. If we could prevent the area from contracting, we should raise the nominal tensile strength. In a perforated bar the minimum area of the section—through the hole—is surrounded by metal not so highly stressed, hence the reduction in area is less and the nominal tensile strength is greater than that of a plain bar. This apparent increase in strength does not occur until the stress is well past the elastic limit, hence we have no need to take it into account in the design of riveted joints.

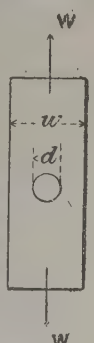


FIG. 384.

Strength of an Elementary Pin Joint.—If a bar, perforated at both ends as shown, were pulled apart through the medium of pins, failure might occur through the tearing of the plates, as shown at *aa* or *bb*, or by the shearing of the pins themselves.

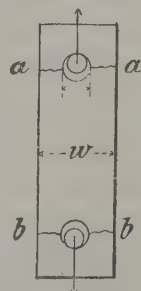


FIG. 385.

Let d = the diameter of the pins;
c = the clearance in the holes.

Then the diameter of the holes $= d + c$

Let f_s = the shearing strength of the material in the pins.

For simplicity at the present stage, we will assume the pins to be in single shear. Then, for equal strength of plate and pins, we have—

$$\{w - (d + c)\}tf_t = \frac{\pi d^2}{4} f$$

If the holes had been punched instead of drilled, a thin ring of metal all round the hole would have been damaged by the rough treatment of the punch. This damaging action can be very clearly seen by examining the plate under a microscope, and its effect demonstrated by testing two similar strips, in one of which the holes are drilled, and in the other punched, the latter breaking at a lower load than the former.

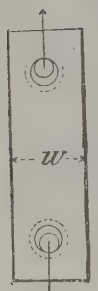


FIG. 386.

Let the thickness of this damaged ring be $\frac{K}{2}$; then the equivalent diameter of the hole will be $d+c+K$, and for equal strength of plate and pins—

$$\{w-(d+c+K)\}tf_t=\frac{\pi d^2}{4}f_s$$

A riveted joint differs from the pin joint in one important respect: the rivets, when closed, completely (or ought to) fill the holes, hence the diameter of the rivet is $d+c$ when closed. In speaking of the diameter of a rivet, we shall, however, always mean the original diameter before closing.

Then, allowing for the increase in the diameter of the rivet, the above expressions become, when the plates are not damaged as in drilling—

$$\{w-(d+c)\}tf_t=\frac{\pi(d+c)^2}{4}f_s$$

When the plates are punched—

$$\{w-(d+c+K)\}tf_t=\frac{\pi(d+c)^2}{4}f_s$$

The value of c, the clearance of a rivet-hole, may be taken at about $\frac{1}{20}$ of the original diameter of the rivet, or $c=0{\cdot}05d$.

The diameter of the rivet is rarely less than $\frac{3}{4}$ inch or greater than $1\frac{1}{4}$ inch for boiler work; hence, when convenient, we may write $c=0{\cdot}05$ inch.

The equivalent thickness of the ring damaged by punching may be taken at $\frac{1}{10}$ of an inch, or $K=0{\cdot}2$ inch. This value has been obtained by very carefully examining all the most recent published accounts of tests of riveted joints.

The relation between the diameter of the rivet and the thickness of the plate is largely a matter of practical convenience. In the best modern practice somewhat smaller rivets are used than was the custom a few years ago; taking $d = 1{\cdot}1\sqrt{t}$ and $(d + c)^2 = 1{\cdot}33t$ appears to agree with present practice.

Instead of using f_s, we may write $\frac{4}{5}f_r$ (see 400), where f_r is the tensile strength of the rivet material.

The values of f_b, f_r, and f_t may be taken as follows, but if in any special case they differ materially, the actual values should be inserted.

Material.	f_b[1]	f_r	f_t	
Iron ...	50	24	21	tons per square inch.
Steel ...	50	28	28	

Ways in which a Riveted Joint may fail.—Riveted joints are designed to be equally strong in tearing the plate

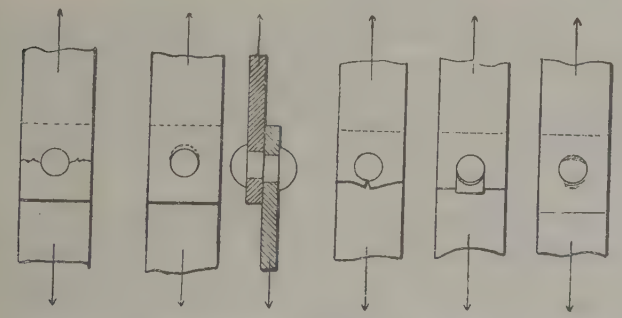

Tearing plate through rivet-holes.
Shearing of rivet.
Bursting through edge of plate.
Shearing through edge of plate.
Crushing of rivet.

FIG. 387.

and in shearing the rivet; the design is then checked, to see that the plates and rivets are safe against crushing.

Failure through bursting or shearing of the edge of the plate is easily avoided by allowing sufficient margin between the edge of the rivet-hole and the edge of the plate; usually this is not less than the diameter of the rivet.

[1] See paragraph on Bearing Pressure.

Lap and Single Cover-plate Riveted Joints.—When such joints are pulled, the plates bend, as shown, till the two

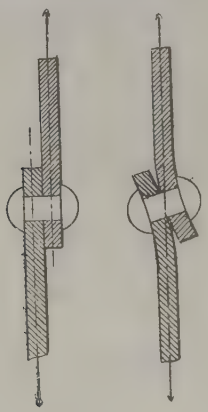


FIG. 388.

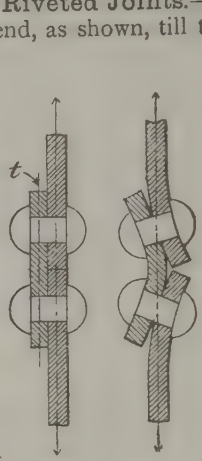


FIG. 389.

lines of pull coincide. This bending action very considerably increases the stress in the material, and consequently weakens the joint. It is not usual to take this bending action into account, although it is as great or greater than the direct stress, and is the cause of the dangerous grooving so often found in lap-riveted boilers.[1]

[1] The bending stress can be approximately arrived at thus: The maximum bending moment on the plates is $\frac{Pt}{2}$ (see p. 505), where P is the total load on a strip of width w. This bending moment decreases as the plates bend. Then, f_t being the stress in the metal between the rivet-holes, the stress in the metal where there are no holes is $f_t\left(\frac{w-d}{w}\right)$; hence—

$$P = wtf_t\left(\frac{w-d}{w}\right)$$

and the bending moment $= \frac{t^2 f_t(w-d)}{2}$

The plate bends along lines where there are no holes; hence

$$Z = \frac{wt^2}{6} \text{ (see Chap. XI.)}$$

and the skin stress due to bending—

$$f = 3f_t\left(\frac{w-d}{w}\right)$$

Single Row of Rivets.

Punched iron—

$$(w - d - c - \mathrm{K})tf_t = \frac{\pi}{4}(d + c)^2\frac{4f_r}{5}$$

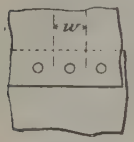


FIG. 390.

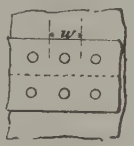


FIG. 391.

Then, substituting 0·05 inch for c on the left-hand side of the equation, and putting in the numerical value of K as given above, also putting $(d + c)^2 = 1{\cdot}33t$, we have on reduction—

$$w - d = 0{\cdot}83\frac{f_r}{f_t} + 0{\cdot}25 \text{ inch}$$

$$w - d = 1{\cdot}20 \text{ inches}$$

Thus the *space between the rivet-holes* $(w - d)$ of all punched iron plates with single lap or cover joints is 1·20 inch, or say $1\frac{1}{4}$ inch for all thicknesses of plate.

By a similar process, we get for—

Punched steel—

$$w - d = 1{\cdot}09 \text{ inches}$$

In the case of drilled plates the constant K disappears, hence $w - d$ is 0·2 inch less than in the case of punched plates; then we have for —

Drilled iron—

$$w - d = 1{\cdot}0 \text{ inch}$$

Drilled steel—

$$w - d = 0{\cdot}89 \text{ inch}$$

Double Row of Rivets.—In this case two rivets have to

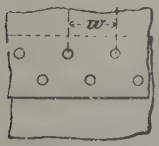


FIG. 392.

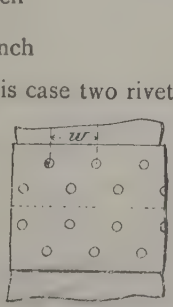


FIG. 393.

be sheared through per unit width of plate w; hence we have—

Punched iron—

$$(w - d - c - \mathrm{K})tf_t = \frac{2\pi}{4}(d + c)^2\frac{4f_r}{5}$$

$$w - d = 1{\cdot}91\frac{f_r}{f_t} + 0{\cdot}25 \text{ inch}$$

$$w - d = 2{\cdot}16 \text{ inches}$$

Punched steel—

$$w - d = 1{\cdot}92 \text{ inches}$$

Drilled iron—

$$w - d = 1{\cdot}96 \text{ inches}$$

Drilled steel—

$$w - d = 1{\cdot}72 \text{ inches}$$

Double Cover-plate Joints.—In this type of joint there is no bending action on the plates. Each rivet is in double shear; therefore, with a single row of rivets the space between the rivet-holes is the same as in the lap joint with two rows of rivets. The joint shown in Fig. 394 has a single row of rivets (*i.e.* in each plate).

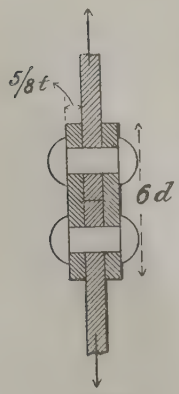


FIG. 394.

Double Row of Rivets.—In this case there are two rivets in double shear, which is equivalent to four rivets in single shear for each unit width of plate w.

Punched iron—

$$(w - d - c - \mathrm{K})tf_t = \frac{4\pi}{4}(d + c)^2\frac{4f_r}{5}$$

$$w - d = 3{\cdot}34\frac{f_r}{f_t} + 0{\cdot}25$$

$$w - d = 4{\cdot}07 \text{ inches}$$

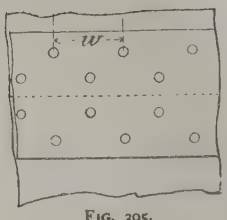


FIG. 395.

Punched steel—

$$w - d = 3{\cdot}59 \text{ inches}$$

Drilled iron—

$$w - d = 3{\cdot}87 \text{ inches}$$

Drilled steel—

$$w - d = 3{\cdot}39 \text{ inches}$$

Diamond Riveting.—In this case there are five rivets in double shear, which is equivalent to ten rivets in single shear,

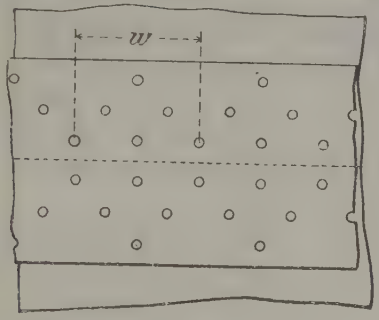


FIG. 396.

in each unit width of plate w; whence we have for drilled steel[1] plates—

$$(w - d - c)tf_t = \frac{10\pi}{4}(d + c)^2\frac{4f_r}{5}$$

$$w - d = 8{\cdot}4 \text{ inches}$$

Combined Lap and Cover-plate Joint.—This joint may fail by—

(1) Tearing through the outer row of rivet-holes.

(2) Tearing through the inner row of rivet-holes and shearing the outer rivet (single shear).

(3) Shearing three rivets in single shear (one on outer row, and two on inner).

Making (1) = (3), we have for drilled steel—

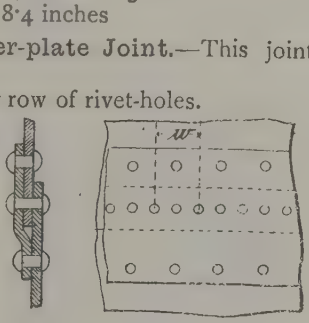


FIG. 397.

$$(w - d - c)tf_t = \frac{3\pi}{4}(d + c)^2\frac{4f_r}{5}$$

$$w - d = 2{\cdot}71 \text{ inches}$$

If we make (3) = (2), we get

$$(w - 2d - 2c)tf_t + \frac{\pi}{4}(d + c)^2\frac{4f_r}{5} = \frac{3\pi}{4}(d + c)^2\frac{4f_r}{5}$$

On reduction, this becomes—

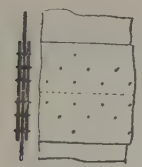
$$w - 2d = 2{\cdot}07 \text{ inches}$$

[1] This joint is rarely used for other than drilled steel plates.

If the value of d be supplied, it will be seen that the joint will not fail by (2), hence such joints may be designed by making (1) = (3).

Pitch of Rivets.—The pitch of the rivets, *i.e.* the distance from centre to centre, is simply w; in certain cases, which are very readily seen, the pitch is $\frac{w}{2}$. The pitch for a number of joints is given in the table below. The diameters of the rivets to the nearest $\frac{1}{16}$ of an inch are given in brackets.

Type of joint.	Thickness of plate, t.	Diameter of rivet. $d = 1{\cdot}1\sqrt{t}$	Iron plates and rivets. Punched. $1{\cdot}20 + d$.	Iron plates and rivets. Drilled. $1 + d$.	Steel plates and rivets. Punched. $1{\cdot}09 + d$.	Steel plates and rivets. Drilled. $0{\cdot}89 + d$.
	in.	in.				
A	$\frac{3}{8}$	0·67 ($\frac{11}{16}$)	1·87	1·67	1·76	1·56
	$\frac{1}{2}$	0·78 ($\frac{13}{16}$)	1·98	1·78	1·87	1·67
	$\frac{5}{8}$	0·87 ($\frac{7}{8}$)	2·07	1·87	1·96	1·76
	$\frac{3}{4}$	0·95 ($\frac{15}{16}$)	2·15	1·95	2·04	1·84
			$2{\cdot}16 + d$.	$1{\cdot}96 + d$.	$1{\cdot}92 + d$.	$1{\cdot}72 + d$.
B (a)	$\frac{3}{8}$	0·67 ($\frac{11}{16}$)	(a) 2·63 2·83	(a) 2·43 2·63	(a) 2·21 2·59	(a) 2·01 2·39
	$\frac{1}{2}$	0 78 ($\frac{13}{16}$)	2·94	2·74	(a) 2·51 2·70	(a) 2·31 2·50
	$\frac{5}{8}$	0·87 ($\frac{7}{8}$)	3·03	2·83	(a) 2·76 2·79	(a) 2·56 2·59
	$\frac{3}{4}$	0·95 ($\frac{15}{16}$)	3·11	2·91	2·87	2·67
	$\frac{7}{8}$	1·03 (1)	3·19	2·99	2·95	2·75
	1	1·10 ($1\frac{1}{8}$)	3·26	3·06	3·02	2·82
			$4{\cdot}07 + d$.	$3{\cdot}87 + d$.	$3{\cdot}59 + d$.	$3{\cdot}39 + d$.
C	$\frac{1}{2}$	0·78 ($\frac{13}{16}$)	4·68 ~~4·85~~	4·65	3·99 ~~4·37~~	3·79 ~~4·17~~
	$\frac{5}{8}$	0·87 ($\frac{7}{8}$)	4·94	4·74	4·40 ~~4·46~~	4·20 ~~4·26~~
	$\frac{3}{4}$	0·95 ($\frac{15}{16}$)	5·02	4·82	4·54	4·34
	$\frac{7}{8}$	1·03 (1)	5·10	4·90	4·62	4·42
	1	1·10 ($1\frac{1}{8}$)	5·17	4·97	4·69	4·49
	$1\frac{1}{8}$	1·17 ($1\frac{3}{16}$)	5·24	5·04	4·76	4·56

Type of joint.	Thickness of plate, t.	Diameter of rivet. $d = 1{\cdot}1\sqrt{t}$.	Steel plates and rivets (drilled holes).	
			Inner row.	Outer row. $8{\cdot}4 + d$.
	in.	in.		
D 	$\frac{3}{4}$	0·95 ($\frac{15}{16}$)	4·67	9·35
	$\frac{7}{8}$	1·03 (1)	4·71	9·43
	1	1·10 (1$\frac{1}{8}$)	4·75	9·50
	1$\frac{1}{8}$	1·17 (1$\frac{3}{16}$)	4·78	9·57
	1$\frac{1}{4}$	1·23 (1$\frac{1}{4}$)	4·81	9·63
	1$\frac{3}{8}$	1·30 (1$\frac{5}{16}$)	4·85	9·70
				$2{\cdot}71 + d$.
E	$\frac{1}{2}$	0·78 ($\frac{3}{4}$)	1·75	3·49
	$\frac{5}{8}$	0 87 ($\frac{7}{8}$)	1·79	3·58
	$\frac{3}{4}$	0·95 ($\frac{15}{16}$)	1·83	3·66
	$\frac{7}{8}$	1·03 (1)	1·87	3·74
	1	1·10 (1$\frac{1}{8}$)	1·91	3·81
	1$\frac{1}{8}$	1·17 (1$\frac{3}{16}$)	1·94	3·88

It is found that if the pitch of the rivets along the caulked edge of a plate exceeds six times the diameter of the rivet, the plates are liable to pucker up when caulked; hence in the above table all the pitches that exceed this are crossed out with a horizontal line, and the greatest permissible pitch inserted.

Bearing Pressure.—The ultimate bearing pressure on a boiler rivet must on no account exceed 50 tons per square inch, or the rivet and plate will crush. It is better to keep it below 45 tons per square inch. The bearing area of a rivet is dt, or $(d + c)t$. Let f_b = the bearing pressure.

The total load on a group of n rivets in a strip of plate of width w $\Big\} = nf_b(d + c)t$

the load on a strip of plate of width w $\Big\} = (w - d - c)f_t t$, or $(w - d - c - \mathrm{K})f_t t$

then $(w - d - c)f_t t = nf_b(d + c)t$

$$\text{and } f_b = \frac{(w - d - c)f_t}{n(d + c)} \not> 50 \text{ tons per sq. inch}$$

The bearing pressure has been worked out for the joints

given in the table above, and in those instances in which it is excessive they have been crossed out with a diagonal line, and the greatest permissible pitch has been inserted.

Efficiency of Joints.—

$$\left.\begin{array}{l}\text{The efficiency}\\ \text{of a riveted}\\ \text{joint}\end{array}\right\} = \frac{\text{strength of joint}}{\text{strength of plate}}$$

$$= \frac{\text{effective width of metal between rivet-holes}}{\text{pitch of rivets}}$$

$$= \frac{w - d - c}{w}, \text{ or } \frac{w - d - c - \text{K}}{w}$$

The table on the following page gives the efficiency of joints corresponding to the table of pitches given above. All the values are per cent.

Zigzag Riveting.—In zigzag riveting, if the two rows are placed too near together, the plates tear across in zigzag fashion. If the material of the plate were equally strong with and across the grain, then x_1, the zigzag distance between the two holes, should be $\frac{x}{2}$. The plate is, however, weaker along than across the grain, consequently when it tears from one row to the other it partially follows the grain, and therefore tears more readily. The joint is found to be equally strong in both directions when $x_1 = \frac{2x}{3}$.

FIG. 398.

Riveted Tie-bar Joints.—When riveting a tie bar, a very high efficiency can be obtained by properly arranging the rivets.

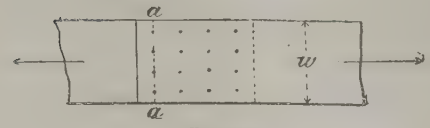


FIG. 399.

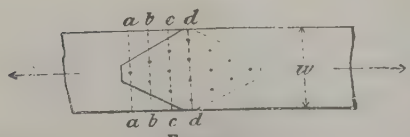
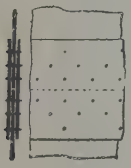


FIG. 400.

The arrangement shown in Fig. 399 is radically wrong for tension joints. The strength of such a joint is, neglecting the clearance in the holes, and damage done by punching—

$(w - 4d)f_t$ at aa

EFFICIENCY OF RIVETED JOINTS.

Type of joint.	Thickness of plate.	Iron plates and rivets.		Steel plates and rivets.	
		Punched.	Drilled.	Punched.	Drilled.
	in.				
A	$\frac{3}{8}$	51	57	48	54
	$\frac{1}{2}$	48	53	45	50
	$\frac{5}{8}$	46	51	43	48
	$\frac{3}{4}$	44	49	41	46
B	$\frac{3}{8}$	(*a*) 65	(*a*) 70	(*a*) 58	(*a*) 64
		67	73	64	70
	$\frac{1}{2}$	65	70	(*a*) 59	(*a*) 64
				62	67
	$\frac{5}{8}$	63	67	(*a*) 59	(*a*) 64
				60	64
	$\frac{3}{4}$	61	66	58	62
	$\frac{7}{8}$	60	64	57	61
	1	59	62	55	59
C	$\frac{1}{2}$	78	82	74	78
	$\frac{5}{8}$	77	81	74	78
	$\frac{3}{4}$	76	79	74	77
	$\frac{7}{8}$	75	78	72	76
	1	74	77	71	74
	$1\frac{1}{8}$	73	76	70	73
D 	$\frac{3}{4}$				89
	$\frac{7}{8}$				89
	1				88
	$1\frac{1}{8}$				87
	$1\frac{1}{4}$				87
	$1\frac{3}{8}$				86
E	$\frac{1}{2}$				72
	$\frac{5}{8}$				70
	$\frac{3}{4}$				69
	$\frac{7}{8}$				67
	1				66
	$1\frac{1}{8}$				65

whereas with the arrangement shown in Fig. 400 the strength is—

$(w - d)f_t t$ at *aa* (tearing only)

$(w - 2d)f_t t + \frac{\pi}{4}d^2 f_s$ at *bb* (tearing and shearing one rivet)

$(w - 3d)f_t t + \frac{3\pi}{4}d^2 f_s$ at *cc* („ „ three rivets)

$(w - 4d)f_t t + \frac{6\pi}{4}d^2 f_s$ at *dd* („ „ six „)

By assuming some dimensions and working out the strength at each place, the weakest section may be found, which will be far greater than that of the joint shown in Fig. 399. The joint in Fig. 400 will be found to be of approximately equal strength at all the sections, hence for simplicity of calculation it may be taken as being—

$$(w - d)f_t t$$

In the above expressions the constants c and K have been omitted; they are not usually taken into account in such joints.

The working bearing pressure on the rivets should not exceed 8 tons per square inch, and where there is much vibration the bearing pressure should not exceed 6 tons per square inch.

Tie bar joints are frequently made with double cover plates; the bearing stress on the rivets in such joints often requires more careful consideration than the shearing stress. When ties are built up of several thicknesses of plate they should be riveted at intervals in order to keep the plates close together, and so prevent rain and moisture from entering.

For this reason the pitch of the rivets in outside work should never exceed $12t$, where t is the thickness of the outside plates of the joint. Attention to this point is also important in the case of all compression members, whether for outside or inside work, because the plates tend to open between the rivets.

Groups of Rivets.—In the Chapter on combined bending and direct stress, it is shown that the stress may be very seriously increased by loading a bar in such a manner that the line of pull or thrust does not coincide with the centre of gravity of the section of the bar. Hence, in order to get the stress evenly distributed over a bar, the centre of gravity of the

group of rivets must lie on the line drawn through the centre of gravity of the cross-section of the bar. And when two bars

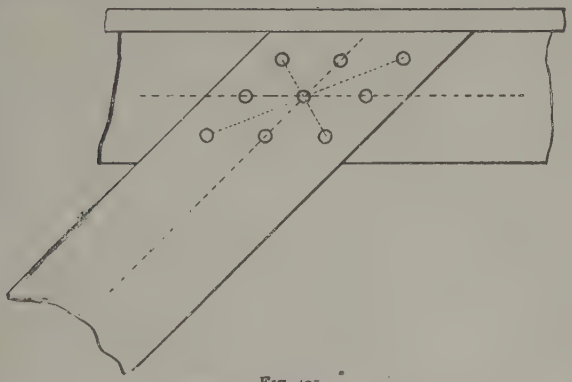


FIG. 401.

not in line are riveted, as in the case of the bracing and the boom of a bridge, the centre of gravity of the group must lie

FIG. 402.

on the intersection of the lines drawn through the centres of gravity of the cross-sections of the two bars.

In other words, the rivets must be arranged symmetrically about the two centre lines (Fig. 401).

Punching Effects.—Although the strength of a punched plate may not be very much less than that of a similar drilled plate, yet it must not be imagined that the effect of the punching is not evident beyond the imaginary ring of $\frac{1}{10}$ inch in thickness. When doing some research work upon this question the author found in some instances that a 1-inch hole punched in a mild steel bar 6 inches wide caused the whole of the fracture on both sides of the hole to be crystalline, whereas the same material gave a clean silky fracture in the unpunched bars. Fig. 402 shows some of the fractures obtained. The punching also very seriously reduces the ductility of the bar; in many instances the reduction of area at fracture in the punched bars was not more than one-tenth as great as in the unpunched bars. These are not isolated cases, but may be taken as the general result of a series of tests on about 150 bars of mild steel of various thicknesses, widths, and diameters of hole.

Strength of Cylinders subjected to Internal Pressure.

Thin Cylinders.—Consider a short cylindrical ring 1 inch in length, subjected to an internal pressure of p lbs. square inch. Then the total pressure tending to burst the cylinder and tear the plates at *aa* and *bb* is pD, where D is the internal diameter in inches. This bursting pressure has to be resisted by the stress in the ring of metal, which is $2f_t t$, where f_t is the tensile stress in the material.

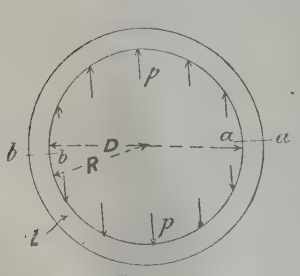


FIG. 403.

Then $p\mathrm{D} = 2f_t t$

or $p\mathrm{R} = f_t t$

When the cylinder is riveted with a joint having an efficiency η, we have—

$$p\mathrm{R} = f_t t \eta$$

In addition to the cylinder tending to burst by tearing the plates longitudinally, there is also a tendency to burst

circumferentially. The total bursting pressure in this direction is $p\pi R^2$, and the resistance of the metal is $2\pi R t f_t$, where f_t is the tensile stress in the material.

$$\text{Then } p\pi R^2 = 2\pi R t f_t$$
$$\text{or } pR = 2tf_t$$

and when riveted—

$$pR = 2tf_t\eta$$

Thus the stress on a circumferential section is one-half as great as on a longitudinal section. On p. 402 a method is given for combining these two stresses.

The above relations only hold when the plates are very thin; with thick-sided vessels the stress is greater than the value obtained by the thin cylinder formula.

Thick Cylinders.

Barlow's Theory.—When the cylinder is exposed to an internal pressure (p), the radii will be increased, due to the stretching of the metal.

When under pressure—

Let r_i be strained to $r_i + n_i r_i = r_i(1 + n_i)$

r_e " " $r_e + n_e r_e = r_e(1 + n_e)$

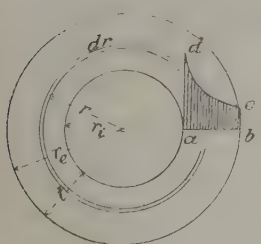


FIG. 404.

where n is a small fraction indicating the elastic strain of the metal, which never exceeds $\frac{1}{1000}$ for safe working stresses (see p. 364).

The sectional area of the cylinder will be the same (to all intents and purposes) before and after the application of the pressure; hence—

$$\pi(r_e^2 - r_i^2) = \pi\{r_e^2(1 + n_e)^2 - r_i^2(1 + n_i)^2\}$$

which on reduction becomes—

$$r_i^2(n_i^2 + 2n_i) = r_e^2(n_e^2 + 2n_e)$$

n being a very small fraction, its square is still smaller and may be neglected, and the expression may be written—

$$\frac{r_i^2}{r_e^2} = \frac{n_e}{n_i}$$

The material being elastic, the stress f will be proportional to the strain n; hence we may write—

$$\frac{r_i^2}{r_o^2} = \frac{f_o}{f_i}, \text{ or } f_i r_i^2 = f_o r_o^2$$

that is, the stress on any thin ring varies inversely as the square of the radius of the ring.

Consider the stress f in any ring of radius r and thickness dr and of unit width.

The total stress on any section of the elementary ring $= f \cdot dr$

$$= \frac{f_i r_i^2}{r^2} dr$$

$$= f_i r_i^2 \cdot r^{-2} dr$$

The total stress on the whole section of one side of the cylinder $= f_i r_i^2 \int_{r_i}^{r_o} r^{-2} dr$

$$= f_i r_i^2 \left(\frac{r_o^{1-2} - r_i^{1-2}}{1-2} \right)$$

$$= \frac{f_i r_i^2 r_o^{-1} - f_i r_i}{-1}$$

(Substituting the value of $f_i r_i^2$ from above)

$$= \frac{f_o r_o - f_i r_i}{-1}$$

$$= f_i r_i - f_o r_o$$

This total stress on the section of the cylinder is due to the total pressure pr_i; hence—

$$pr_i = f_i r_i - f_o r_o$$

Substituting the value of f_o, we have—

$$pr_i = f_i r_i - \frac{f_i r_i^2 r_o}{r_o^2}$$

Dividing by r and reducing, we have—

$$pr_o = f_i r_o - f_i r_i$$

$$p r_o = f_i t$$

For a thin cylinder, we have—

$$pr_i = ft$$

Thus a thick cylinder may be dealt with by the same form of expression as a thin cylinder, taking the pressure to act on the external instead of on the internal radius.

The diagram (Fig. 404, *abcd*) shows the distribution of stress on the section of the cylinder, *ad* representing the stress at the interior, and *bc* at the exterior. The curve *dc* is $fr^2 =$ constant.

Lamés Theory.—All theories of thick cylinders indicate that the stress on the inner skin is greater than that on the outer when the cylinder is exposed to an internal pressure, but they do not all agree as to the exact distribution of the stress.

Consider a thin ring 1 inch long and of internal radius r, of thickness δr.

Let the radial pressure on the inner surface of the ring be p, and on the outer surface $(p - \delta p)$, when the fluid pressure is internal. In addition to these pressures the ring itself is subjected to a hoop stress f, as in the thin cylinder. Each element of the ring is subjected to the stresses as shown in the figure.

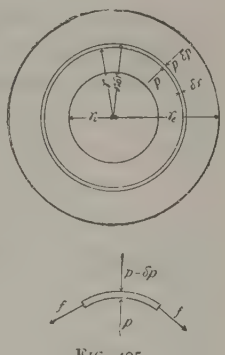


FIG. 405.

The force tending to burst this thin ring $= p \times 2r$ . . (i.)

" " prevent bursting $= \begin{cases} (p - \delta p)2(r + \delta r) \\ +f \times 2\delta r \end{cases}$ (ii.)

These two must be equal—

$$2pr = 2pr + 2p\delta r - 2r\delta p + 2f\delta r$$

$$f = r\frac{\delta p}{\delta r} - p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

We can find another relation between f and p, which will enable us to solve this equation. The radial stress p tends to squeeze the element into a thinner slice, and thereby to cause it to spread transversely; the stress f tends to stretch the element in its own direction, and causes it to contract in

thickness and normal to the plane of the paper. Consider the strain of the element normal to the paper; due to p it is $\frac{p}{n\mathrm{E}}$ (see p. 397), and due to f it is $-\frac{f}{n\mathrm{E}}$, and the total longitudinal strain normal to the paper is $\frac{p}{n\mathrm{E}} - \frac{f}{n\mathrm{E}}$. Both f and p, however, diminish towards the outer skin, and the one stress depends upon the other.

Now, as regards the strain in the direction f both pressures f and p act together, and on the assumption just made—

$$p - f = \text{a constant} = 2a$$

The 2 is inserted for convenience of integration.

From (iii.) we have—

$$r\frac{\delta p}{\delta r} = f + p$$

$$\frac{\delta p}{\delta r} = \frac{2p}{r} - \frac{2a}{r} = \frac{dp}{dr} \text{ in the limit}$$

Integrating we get—

$$p = \frac{b}{r^2} + a$$

$$f = \frac{b}{r^2} - a$$

where a and b are constants.

Let the internal pressure above the atmosphere be p_i, and the external pressure $p_e = 0$, we have the stress on the inner skin—

$$f_i = p_i \frac{r_e^2 + r_i^2}{r_e^2 - r_i^2}$$

Reducing Barlow's formula to the same form of expression, we get—

$$f_i = p_i \frac{r_e^2 + r_i r_e}{r_e^2 - r_i^2}$$

Experimental Determination of the Distribution of Stress in Thick Cylinders.—In order to ascertain whether there was any great difference between the distribution of stress as indicated by Barlow's and by Lamé's theories, the

author, assisted by Messrs. Wales, Day, and Duncan, students, made a series of experiments on two cast-iron cylinders with open ends. Well-fitting plugs were inserted in each end, and the cylinder was filled with paraffin wax; the plugs were then forced in by the 100-ton testing machine, a delicate extensometer, capable of reading to $\frac{1}{100000}$ of an inch, was fitted on

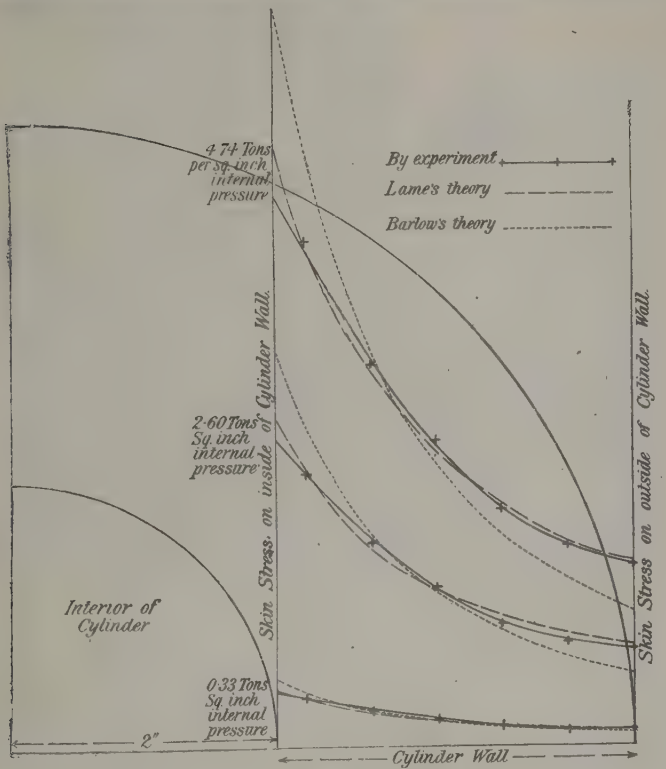

4·74 Tons per sq. inch internal pressure
2·60 Tons Sq. inch internal pressure
0·33 Tons Sq. inch internal pressure
Interior of Cylinder
2"
Skin Stress on inside of Cylinder Wall.
Skin Stress on outside of Cylinder Wall.
By experiment
Lame's theory
Barlow's theory
Cylinder Wall

FIG. 406.

the outside; a series of readings were then taken at various pressures. Two small holes were then drilled diametrically into the cylinder to a depth of 0·5 inch; pointed pins were loosely inserted, and the extensometer was applied to their outer ends, and a second set of observations were taken. The holes were then drilled deeper, and similar observations taken

at various depths. From these observations it is a simple matter to deduce the proportional strain and the stress. The results obtained are shown in Fig. 333*b*, and for purposes of comparison, Lamé's and Barlow's curves are inserted.

Built-up Cylinders.—In order to equalize the stress over the section of a cylinder or a gun, various devices are adopted. In the early days of high pressures, cast-iron guns were cast round chills, so that the metal at the interior was immediately cooled; then when the outside hot metal contracted, it brought the interior metal into compression. Thus the initial stress in a section of the gun was somewhat as shown by the line *ab*, *ag* being compression, and *bh* tension. Then, when subjected to pressure, the curve of stress would have been *dc* as before, but when combined with *ab* the resulting stress on the section is represented by *ef*, thus showing a much more even distribution of stress than before.

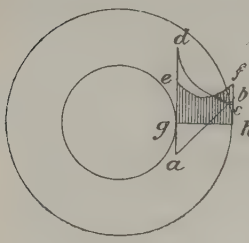


FIG. 407.

This equalizing process is effected in modern guns by either shrinking rings on one another in such a manner that the internal rings are initially in compression and the external rings in tension, or by winding wire round an internal tube to produce the same effect. The exact tension on the wire required to produce the desired effect is regulated by drawing the wire through friction dies mounted on a pivoted arm—in effect, a friction brake.

STRENGTH AND COEFFICIENTS OF ELASTICITY OF MATERIALS IN TONS SQUARE INCH.

Material.	Elastic limit.			Breaking strength.			E. Tension or compression.	G. Shear.	Extension per cent. on 10".	Reduction in area.
	Tension.	Compression.	Shear.	Tension.	Compression.	Shear.				
Wrought iron bars	12–15	—	10–12	21–24	—	17–19	11,000–13,000	5,000 6,000	10–30	15–40
Plates with grain	13–15	—	—	20–22	—	—	—	—	5–10	7–12
,, across ,,	11–13	—	—	18–20	—	—	—	—	2–6	3–7
Best Yorkshire, with grain ...	13–14	12–14	—	20–23	—	17–19	12,000	—	15–30	40–50
Best Yorkshire, across grain	13–14	—	—	19–20	—	—	—	—	10–12	12–20
Steel, 0·1 % C. ...	13–14	—	10–11	21–22	—	16–17	13,000–14,000	5,000 6,000	27–30	45–50
,, 0·2 % C. ...	17–18	16–17	13–14	30–32	—	24–26	,,	,,	20–23	27–32
,, 0·5 % C. ...	20–21	—	16–17	34–35	—	28–29	,,	,,	14–17	17–20
,, 1·0 % C. ...	28–29	22–24	21–23	50–55	—	42–47	14,000	,,	4–5	7–8
Rivet steel ...	15–17	—	12–14	26–28	—	21–22	—	—	20–35	30–50
Steel castings ...	10–11	(not annealed)	annealed)	20–25	—	—	12,000 to 12,500	—	5–12	6–13
,, ,, ...	15–17	(annealed)		30–40	—	—		—	10–20	15–35
Tool steel ...	35–45	40–50	—	40–70	(unhardened)		14,000 to 15,000	—	1–5	1–5
,, ,, ...	69–80	—	—	60–80	(hardened)			—	*nil*	*nil*
Nickel steel Ni 4 %	25–30	—	—	35–40	—	—	—	—	30–35	50–55
,, ,, 11 %	45–65	—	—	90–95	—	—	—	—	10–12	24–27
Manganese steel Mn 1 % ...	15	—	—	30–35	—	—	—	—	28–33	—
,, ,, 2 %	15	—	—	50–55	—	—	—	—	5–7	—
Chrome steel Cr 5 % ...	30–40	—	—	60–75	—	—	—	—	10–15	—
Chrome Vanadium	65–85	water		70–90	(hardened)		—	—	10–15	45–55
,, ,, ...	60–75	oil	oil	65–80	(hardened)		—	—	8–18	45–55
Chrome nickel ...	45–50	—	—	55–60	(annealed)		—	—	12–15	50–55
,, ,, ...	100–105	air	air	105–110	(hardened)		—	—	6–8	30–35
,, ,, ...	115–120	oil	oil	120–130	(hardened)		—	—	5–8	8–10
Cast iron... ...	no	marked limit		7–11	35–60	8–13	6,000 to 10,000	2,500 to 4,000	practically	*nil*
Copper	2–4	10–13 annealed	—	12–15	20–25	11–12	7,000 to 7,500	—	35–40	50–60
,, ...	10–12	hard	drawn	16–20	—	—		—	2–5	40–55
Gun-metal ...	3–4	5–6	2·5–5	9–16	30–50	8–12	5,000–5,500	2,000 2,500	8–15	10–18

Material.	Elastic limit.			Breaking strength.			E. Tension or compression.	G. Shear.	Extension per cent. on 10″.	Reduction in area.
	Tension.	Compression.	Shear.	Tension.	Compression.	Shear.				
Brass	2-4	—	—	7-10	5-6	—	4,000	2,000	10-12	12-15
Delta, bull metal, etc.—										
Cast	5-8	12-14	—	14-20	60-70	—	5,500 to 6,000	—	8-16	10-22
Rolled ...	15-25	16-22	—	27-34	45-60	—		—	17-34	27-50
Phosphor bronze	7-9	—	—	24-26	—	—	6,000	2,500	—	—
Muntz metal ...	20-25	(rolled)	—	25-30	—	—	—	—	10-20	30-40
Aluminium ...	2-7	8-10	—	7-10	—	5-6	3,000 5,000	1,700	4-15	30-70
Duralumin ...	4-5	—	—	22-24	—	—	4,200 4,500	—	10-12	30-38
Oak	—	—	—	4-6	2-5	with grain 0·2	500-700	—	—	—
Soft woods ...	—	—	—	1-3	1-3	0·07 to 0·4	450-500	—	—	—

For Brass, E = 6500, G = 2230, according to "Mats. + Methods" Manual Dec. 1946, p. 1481.

CHAPTER XI.

BEAMS.

The beam illustrated in Fig. 408 is an indiarubber model used for lecture purposes. Before photographing it for this illustration, it was painted black, and some thin paper was stuck on evenly with seccotine. When it was thoroughly set the paper was slightly damped with a sponge, and a weight was placed on the free end, thus causing it to bend; the paper on the upper edge cracked, indicating tension, and that on the lower edge buckled, indicating compression, whereas between the two a strip remained unbroken, thus indicating no longi-

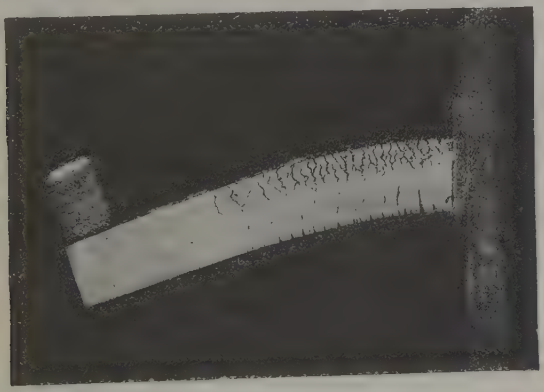


Fig. 408.

tudinal strain or stress. We shall see shortly that such a result is exactly what we should expect from the theory of bending.

General Theory.—The T lever shown in Fig. 409 is hinged at the centre on a pivot or knife-edge, around which the lever can turn. The bracket supporting the pivot simply takes the shear. For the lever to be in equilibrium, the two couples acting on it must be equal and opposite, viz. $Wl = px$.

Replace the T lever by the model shown in Fig. 410. It

is attached to the abutment by two pieces of any convenient material, say indiarubber. The upper one is dovetailed, because it is in tension, and the lower is plain, because it is in

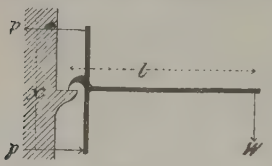


FIG. 409.

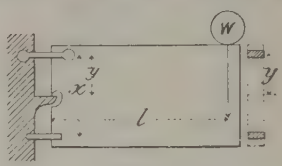


FIG. 410.

compression. Let the sectional area of each block be a; then, as before, we have $Wl = px$. But $p = fa$, where f = the stress in either block in either tension or compression;

$$\text{hence } Wl = fax$$
$$\text{or} = 2fay$$

Or—

$$\left.\begin{matrix}\text{The moment of}\\ \text{the external}\\ \text{forces}\end{matrix}\right\} = \left\{\begin{matrix}\text{the moment of the internal forces, or the}\\ \text{internal moment of resistance of the}\\ \text{beam}\end{matrix}\right.$$

= stress on the area a × (moment of the two areas (a) about the pivot)

Hence the resistance of any section to bending—apart altogether from the strength of the material of the beam—varies directly as the area a and as the distance x, or as the moment ax. Hence the quantity in brackets is termed the "measure of the strength of the section," or the "modulus of the section," and is usually denoted by the letter Z. Hence we have—

$$Wl = M = fZ, \text{ or}$$

$$\left.\begin{matrix}\text{The bending moment}\\ \text{at any section}\end{matrix}\right\} = \left\{\begin{matrix}\text{stress on the}\\ \text{material}\end{matrix}\right\} \times \left\{\begin{matrix}\text{modulus of the}\\ \text{section}\end{matrix}\right.$$

The connection between the T lever, the beam model or Fig. 410, and an actual beam may not be apparent to some readers, so in Fig. 411 we show a rolled joist or I section, having top and bottom flanges, which may be regarded as the two indiarubber blocks of the model, the thin vertical web serves the purpose of a pivot and bracket for taking the shear; then the formula that we have just deduced for the model applies equally well to the joist. We shall have to slightly

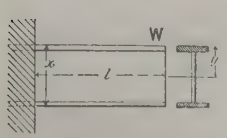


FIG. 411.

modify this statement later on, but the form in which we have stated the case is so near the truth that it is always taken in this way for practical purposes.

Now take a fresh model with four blocks instead of two. When loaded, the outer end will sag as shown by the dotted lines, pivoting about the point resting on the bracket. Then the outer blocks will be stretched and compressed, or strained, more than the inner blocks in the ratio $\frac{x}{x_1}$ or $\frac{y}{y_1}$; *i.e.* the strain is directly proportional to the distance from the point of the pivot.

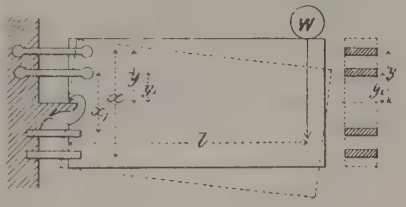


FIG. 412.

The enlarged figure shows this more distinctly, where e, e_1 show the extensions, and c, c_1 show the compressions at the distances y, y_1 from the pivot. From the similar triangles, we have $\frac{e}{e_1} = \frac{y}{y_1}$; also $\frac{c}{c_1} = \frac{y}{y_1}$. But we have previously seen (p. 375) that when a piece of material is strained (*i.e.* stretched or compressed), the stress varies directly as the strain, *provided the elastic limit has not been passed.* Hence, since the strain varies directly as the distance from the pivot, the stress must also vary in the same manner.

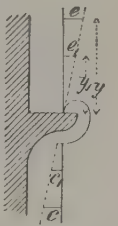


FIG. 413.

Let f = stress in outer blocks, and f_1 = stress in inner blocks; then—

$$\frac{f}{f_1} = \frac{y}{y_1} \text{ or } f_1 = \frac{fy_1}{y}$$

Then, taking moments about the pivot as before, we have—

$$Wl = 2fay + 2f_1ay_1$$

Substituting the value of f_1, we have—

$$Wl = 2fay + \frac{2fay_1^2}{y}$$

$$\text{If } y_1 = \frac{y}{2}, \quad Wl = 2fay + \frac{2fay^2}{4y}$$

$$Wl = 2fay(1 + \tfrac{1}{4})$$

Thus the addition of two inner blocks at one-half the distance of the outer blocks from the pivot has only increased the strength of the beam by $\frac{1}{4}$, or, in other words, the four-block model will only support $1\frac{1}{4}$ times the load (W) that the two-block model will support.

If we had a model with a very large number of blocks, or a beam section supposed to be made up of a large number of layers of area a, a_1, a_2, a_3, etc., and situated at distances y, y_1, y_2, y_3, etc., respectively from the pivot, which we shall now term the *neutral axis*, we should have, as above—

$$Wl = 2fay + \frac{2fa_1y_1^2}{y} + \frac{2fa_2y_2^2}{y} + \frac{2fa_3y_3^2}{y} +, \text{ etc.}$$

$$= \frac{f}{y}2(ay^2 + a_1y_1^2 + a_2y_2^2 + a_3y_3^2 +, \text{ etc.})$$

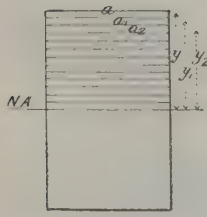


FIG. 414

The quantity in brackets, viz. each little area (a) multiplied by the square of its distance (y^2) from a given line (N.A.), is termed the second moment or moment of inertia of the upper portion of the beam section; and as the two half-sections are similar, twice the quantity in brackets is the moment of inertia (I) of the whole section of the beam. Thus we have—

$$Wl = \frac{fI}{y}, \text{ or}$$

The bending moment at any section

$$= \frac{\left\{\begin{matrix}\text{the stress on the} \\ \text{outermost layer}\end{matrix}\right\} \times \left\{\begin{matrix}\text{second moment, or moment of} \\ \text{inertia of the section}\end{matrix}\right\}}{\text{distance of the outermost layer from the neutral axis}}$$

But we have shown above that the stress varies directly as the distance from the neutral axis; hence the stress on the outermost layer is the maximum stress on any part of the beam section, and we may say—

The bending moment at any section

$$= \frac{\left\{\begin{matrix}\text{the maximum stress} \\ \text{on the section}\end{matrix}\right\} \times \left\{\begin{matrix}\text{second moment, or moment of} \\ \text{inertia of the section}\end{matrix}\right\}}{\text{distance of the outermost layer from the neutral axis}}$$

But we have also seen that $Wl = fZ$, and here we have $Wl = f\frac{I}{y}$;

$$\text{therefore } Z = \frac{I}{y}$$

The quantity $\frac{I}{y}$ is termed the "measure of the strength of the section," or, more briefly, the "modulus of the section;" it is usually designated by the letter Z. Thus we get $Wl = fZ$, or $M = fZ$, or—

$$\left.\begin{matrix}\text{The bending}\\ \text{moment at}\\ \text{any section}\end{matrix}\right\} = \left\{\begin{matrix}\text{the maximum or skin}\\ \text{stress on the section}\end{matrix}\right\} \times \left\{\begin{matrix}\text{the modulus of}\\ \text{the section}\end{matrix}\right.$$

Assumptions of Beam Theory.—To go into the question of all the assumptions made in the beam theory would occupy far too much space. We will briefly consider the most important of them.

First Assumption.—That originally plane sections of a beam remain plane after bending; that is to say, we assume that a solid beam acts in a similar way to our beam model in Fig. 412, in that the strain does increase directly as the distance from the neutral axis. Very delicate experiments clearly show that this assumption is true to within exceedingly narrow limits, *provided the elastic limit of the material is not passed.*

Second Assumption.—That the stress in any layer of a beam varies directly as the distance of that layer from the neutral axis. That the strain does vary in this way we have just seen. Hence the assumption really amounts to assuming that the stress is proportional to the strain. Reference to the elastic curves on p. 364 will show that the elastic line is straight, *i.e.* that the stress does vary as the strain. In most cast materials the line is unquestionably slightly curved, but for low (working) stresses the line is sensibly straight. Hence for *working conditions* of beams we are justified in our assumption. After the elastic limit has been passed, this relation entirely ceases. Hence *the beam theory ceases to hold good as soon as the elastic limit has been passed.*

Third Assumption.—That the modulus of elasticity in tension is equal to the modulus of elasticity in compression. Suppose, in the beam model, we had used soft rubber in the

tension blocks and hard rubber in the compression blocks, *i.e.* that the modulus of elasticity of the tension blocks was less than the modulus of elasticity of the compression blocks; then the stretch on the upper blocks would be greater than the compression on the lower blocks, with the result that the beam would tend to turn about some point other than the pivot, Fig. 415; and the relations given above entirely cease to hold, for the strain and the stress will not vary directly as the distance from the pivot or neutral axis, but directly as the distance from a, which later on we shall see how to calculate.

FIG. 415.

For most materials there is no serious error in making this assumption; but in some materials the error is appreciable, but still not sufficient to be of any practical importance.

Neither of the above assumptions are *perfectly* true; but they are so near the truth that for all practical purposes they may be considered to be perfectly true, but *only so long as the elastic limit of the material is not passed.* In other parts of this book definite experimental proof will be given of the accuracy of the beam theory.

Graphical Method of finding the Modulus of the Section ($Z = \frac{I}{y}$).—The modulus of the section of a beam might be found by splitting the section up into a great many layers and multiplying the area of each by $\frac{y_0^2}{y}$, as shown above. The process, however, would be very tedious.

But in the graphic method, instead of dealing with each strip separately, we graphically find the *magnitude* of the resultant of all the forces, viz.—

$$fa + f_1a_1 + f_2a_2 +, \text{ etc.}$$

acting on each side of the neutral axis, also the *position* or distance apart of these resultants. The product of the two gives us the moment of the forces on each side of the neutral axis, and the sum of the two moments gives us the total amount of resistance for the beam section, viz. fZ.

Imagine a beam section divided up into a great number of thin layers parallel to the neutral axis, and the stress in each layer varying directly as its distance from it. Then if we construct a figure in which the width, and consequently the area,

of each layer is reduced in the ratio of the stress in that layer to the stress in the outermost layer, we shall have the intensity of stress the same in each. Thus, if the original area of the layer be a_1, the reduced area of the layer will be—

$$a_1\left(\frac{f_1}{f}\right) = \text{say } a_1'$$

whence we have $f_1 a_1 = f a_1'$

Then the sum of the forces acting over the half-section, viz.—

$$fa + f_1a_1 + f_2a_2 +, \text{ etc.}$$
$$\text{becomes } fa + fa_1' + fa_2' +, \text{ etc.}$$
$$\text{or } f(a + a_1' + a_2' +, \text{ etc}$$

or f (area of the figure on one side of the neutral axis)
or (the whole force acting on one side of the neutral axis)

Then, since the intensity of stress all over the *figure* is the same, the *position* of the resultant will be at the centroid or centre of gravity of the figure.

Let A_1 = the area of the figure below the neutral axis;
A_2 = „ „ above „ „
d_1 = the distance of the centre of gravity of the lower figure from the neutral axis;
d_2 = the distance of the centre of gravity of the upper figure from the neutral axis.

Then the moment of all the forces acting on one side of the neutral axis $= fA_1 \times d_1$

Then the moment of all the forces acting on *both* sides of the neutral axis $= f(A_1d_1 + A_2d_2)$
$= fZ$ (see p. 354)
or $Z = A_1d_1 + A_2d_2$

We shall shortly show that $A_1 = A_2 = A$ (see next paragraph).

$$\text{Then } Z = A(d_1 + d_2)$$
$$Z = AD$$

where D is the distance between the two centres of gravity. In a section which is symmetrical about the neutral axis $d_1 = d_2 = d$, and $d_1 + d_2 = 2d$.

$$\text{Then } Z = 2Ad$$
$$\text{or } Z = AD$$

The units in which Z is expressed are as follows—

$$Z = \frac{I}{y} = \frac{(\text{length units})^4}{\text{length}} = (\text{length units})^3$$
$$\text{or } Z = AD = \text{area} \times \text{distance}$$
$$= (\text{length units})^2 \times \text{length units}$$
$$= (\text{length units})^3$$

Hence, if a modulus figure be drawn, say, $\frac{1}{n}$ full size, the result obtained must be multiplied by n^3 to get the true value. For example, if a beam section were drawn to a scale of 3 inches = 1 foot, *i.e.* $\frac{1}{4}$ full size, the Z obtained on that scale must be multiplied by $4^3 = 64$.

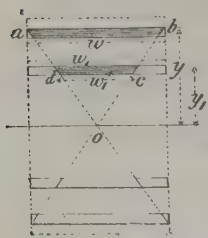


FIG. 416.

We showed above that in order to construct this figure, which we will term a "modulus figure," the width of each strip of the section had to be reduced in the ratio $\frac{f_1}{f}$, which we have previously seen is equal to $\frac{y_1}{y}$. This reduction is easily done thus: Let Fig. 416 represent a section through the indiarubber blocks of the beam model. Join *ao*, *bo*, cutting the inner block in *c* and *d*. Then by similar triangles—

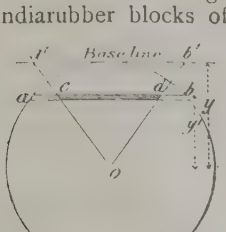


FIG. 417.

$$\frac{cd}{ab} = \frac{y_1}{y} = \frac{f_1}{f}, \text{ or } w_1' = w_1\left(\frac{f_1}{f}\right)$$

In the case of a section in which the strips are not all of the same width, the same construction holds. Project the strip *ab* on to the base-line as shown, viz. *a'b'*. Join *a'o*, *b'o*, cutting the strip in *c* and *d*. By similar triangles we have—

$$\frac{cd}{a'b'} \text{ or } \frac{cd}{ab} = \frac{y_1}{y} = \frac{f_1}{f}, \text{ or } cd = ab\left(\frac{f_1}{f}\right)$$

Several fully worked-out sections will be given later on.

By way of illustration, we will work out the strength of the four-block beam model by this method, and see how it agrees with the expression found above on p. 432.

The area A of the modulus figure on one side of the neutral axis $= a + a_1'$

The distance d of the c. of g. of the modulus figure from the neutral axis (see p. 58) $= \frac{ay + a_1'y_1}{a + a_1'}$

But $Wl = fZ = f \times 2Ad$

or $Wl = 2f(a + a_1') \times \frac{ay + a_1'y_1}{a + a_1'} = 2fay + 2fa_1'y_1$

But $a_1' = a\frac{y_1}{y}$, $\therefore Wl = 2fay + \frac{2fay_1^2}{y}$

which is the same expression as we had on p. 431.

The graphic method of finding Z should only be used when a convenient mathematical expression cannot be obtained.

Position of Neutral Axis.—We have stated above that the neutral axis in a beam section corresponds with the pivot in the beam model; on the one side of the neutral axis the material is in tension, and on the other side in compression, and at the neutral axis, where the stress changes from tension to compression, there is, of course, no stress (except shear, which we will treat later on). In all calculations, whether graphic or otherwise, the first thing to be determined is the position of the neutral axis with regard to the section.

We have already stated on p. 58 that, if a point be so chosen in a body that the sum of the moments of all the gravitational forces acting on the several particles about the one side of any straight line passing through that point, be equal to the sum of the moments on the other side of the line, that point is termed the centre of gravity of the body; or, if the moments on the one side of the line be termed positive (+), and the moments on the other side of the line be termed negative (−), the sum of the moments will be zero. We are about to show that precisely the same definition may be used for stating the position of the neutral axis; or, in other words, we are about to prove that, accepting the assumptions given above, the neutral axis *invariably* passes through the centre of gravity of the section of a beam.

Let the given section be divided up into a large number of strips as shown—

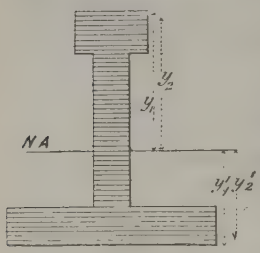


FIG 418.

Let the areas of the strips above the neutral axis be a_1, a_2, a_3, etc.; and below the neutral axis be a_1', a_2', a_3', etc.; and their respective distances above the neutral axis y_1, y_2, y_3, etc.; ditto below y_1', y_2', y_3', etc: and the stresses in the several layers above the neutral axis be f_1, f_2, f_3, etc.; ditto below the neutral axis be f_1', f_2', f_3'.

Then, as the stress in each layer varies directly as its distance from the neutral axis, we have—

$$\frac{f_1}{f_2} = \frac{y_1}{y_2}, \text{ and } f_2 = \frac{f_1 y_2}{y_1}, \text{ also } \frac{f_1}{y_1} = \frac{f_1'}{y_1'}$$

The total stress in all the layers on one side of the neutral axis $= f_1 a_1 + f_2 a_2 +$, etc. $= \frac{f_1}{y_1}(a_1 y_1 + a_2 y_2 +$, etc.$)$

The total stress in all the layers on the other side of the neutral axis $= \frac{f_1'}{y_1'}(a_1' y_1' + a_2' y_2' +$, etc.$)$

But as the tensions and compressions form a couple, the total amount of tension on the one side of the neutral axis must be equal to the total amount of compression on the other side; hence—

$$a y + a_1 y_1 + a_2 y_2 +, \text{ etc.} = a' y' + a_1' y_1' + a_2' y_2' +, \text{ etc.}$$

or, expressed in words, the sum of the moments of all the elemental areas on the one side of the neutral axis is equal to the sum of the moments on the other side of the neutral axis; but this is precisely the definition of a line which passes through the centre of gravity of the section. Hence, *the neutral axis passes through the centre of gravity of the section.*

It should be noticed that not one word has been said in the above proof about the *material* of which the beam is made; all that is taken for granted in the above proof is that the modulus of elasticity in tension is equal to the modulus of elasticity in

compression. The position of the neutral axis has *nothing whatever to do with the relative strengths of the material in tension and compression.* In a reinforced concrete beam the modulus of elasticity of the tension rods differs from that of the concrete, which is in compression. Such beams are dealt with later on.

Unsymmetrical Sections.—In a symmetrical section, the centre of gravity is equidistant from the skin in tension and compression; hence the maximum stress on the material in tension is equal to the maximum stress in compression. Now, some materials, notably cast iron, are from five to six times as strong in compression as in tension; hence, if we use a symmetrical section in cast iron, the material fails on the tension side at from $\frac{1}{5}$ to $\frac{1}{6}$ the load that would be required to make it fail in compression. In order to make the beam equally strong in tension and compression, we make the section of cast-iron beams of such a *form* that the neutral axis is about five or six times[1] as far from the compression flange as from the tension flange, so that the stress in compression shall be five or six times as great as the stress in tension. It should be particularly noted that the reason why the neutral axis is nearer the one flange than the other is entirely due to the *form* of the section, and not to the material; the neutral axis would be in precisely the same place if the material were of wrought iron, or lead, or stone, or timber (provided assumption 3, p. 433, is true). We have shown above on p. 432 that $M = f\frac{I}{y}$, and that $Z = \frac{I}{y}$, where y is the distance of the skin from the neutral axis. In a symmetrical section y is simply *the half depth of the section;* but in the unsymmetrical section y may have two values: the distance of the tension skin from the neutral axis, or the distance of the compression skin from the neutral axis. If the maximum *tensile* stress f_t is required, the y_t must be taken as the distance of the tension skin from the neutral axis; and likewise when the maximum compressive stress f_c is required, the y_c must be measured from the *compression skin.* Thus we have either—

$$M = f_t\frac{I}{y_t} \text{ or } f_c\frac{I}{y_c}$$

[1] We shall show later on that such a great difference as 5 or 6 is undesirable for practical reasons.

and as $\frac{f_t}{y_t} = \frac{f_c}{y_c}$, we get precisely the same value for the bending moment whichever we take. We also have two values of Z, viz. $\frac{I}{y_t} = Z_t$ and $\frac{I}{y_c} = Z_c$;

$$\text{and } M = f_t Z_t \text{ or } f_c Z_c$$

We shall invariably take $f_t Z_t$ when dealing with cast-iron sections, mainly because such sections are always designed in such a manner that they fail in tension.

The construction of the modulus figures for such sections is a simple matter.

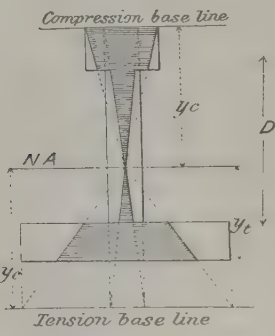


FIG. 419.

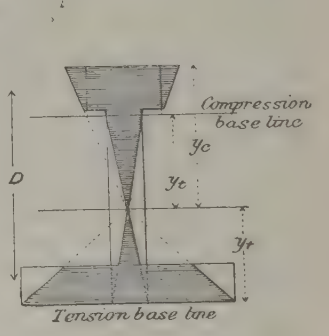


FIG. 420.

Construction for Z_c (Fig. 419).—Find the centre of gravity of the section, and through it draw the neutral axis parallel with the flanges. Draw a compression base-line touching the outside of the compression flange; set off the tension base-line parallel to the neutral axis, at the same distance from it as the compression base-line, viz. y_c; project the parts of the section down to each base-line, and join up to the central point which gives the shaded figure as shown. Find the centre of gravity of each figure (cut out in cardboard and balance). Let D = distance between them; then Z_c = shaded area above *or* below the neutral axis × D.

Construction for Z_t (Fig. 420).—Proceed as above, only the tension base-line is made to touch the outside of the tension flange, and the compression base-line cuts the figure; the parts of the section above the compression base-line have been projected down on to it, and the modulus figure beyond it passes

outside the section; at the base-lines the figure is of the same width as the section. The centre of gravity of the two figures is found as before, also the Z.

The reason for setting the base-lines in this manner will be evident when it is remembered that the stress varies directly as the distance from the neutral axis; hence, the stress on the tension flange f_t is to the stress on the compression flange f_c as y_t is to y_c.

N.B.—The tension base-line touches the tension flange when the figure is being drawn for the tension modulus figure Z_t, and similarly for the compression.

As the tensions and compressions form a couple, the total amount of tension is equal to the total amount of compression, therefore the area of the figure above the neutral axis must be equal to the area of the figure below the neutral axis, whether the section be symmetrical or otherwise; but the *moment* of the tension is not equal to the *moment* of the compression about the neutral axis in unsymmetrical sections. The accuracy of the drawing of modulus figures should be tested by measuring both areas; if they only differ slightly (say not more than 5 per cent.), the mean of the two may be taken; but if the error be greater than this, the figure should be drawn again.

If, in any given instance, the Z_c has been found, and the Z_t is required or *vice versâ*, there is no need to construct the two figures, for—

$$Z_t = Z_c \times \frac{y_c}{y_t}, \text{ or } Z_c = Z_t \times \frac{y_t}{y_c}$$

hence the one can always be obtained from the other.

Most Economical Sections for Cast-iron Beams.—Experiments by Hodgkinson and others show that it is undesirable to adopt so great a difference as 5 or 6 to 1 between the compressive and tensile stresses. This is mainly due to the fact that, if sections be made with such a great difference, the tension flange would be very thick or very wide compared with the compression flange; if a very thick flange were used, as the casting cooled the thin compression flange and web would cool first, and the thick flange afterwards, and set up serious initial cooling stresses in the metal.

The author, when testing large cast-iron girders with very unequal flanges, has seen them break with their own weight before any external load was applied, due to this cause. Very wide flanges are undesirable, because they bend transversely when loaded, as in Fig. 421.

Experiments appear to show that the most economical section for cast iron is obtained when the proportions are roughly those given by the figures in Fig. 421.

"Massing up" Beam Sections.—Thin hollow beam

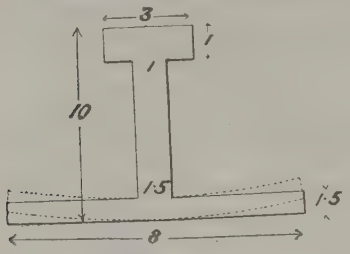


Fig. 421.

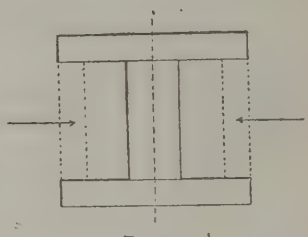
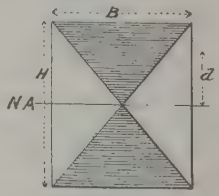


Fig. 422.

sections are usually more convenient to deal with graphically if they are "massed up" about a centre line to form an equivalent solid section. "Massing up" consists of sliding in the sides of the section parallel to the neutral axis until they meet as shown in Fig. 422.

The dotted lines show the original position of the sides, and the full lines the sides after sliding in. The "massing up" process in no way affects the Z, as the distance of each section from the neutral axis remains unaltered; it is done merely for convenience in drawing the modulus figure. In the table of sections several instances are given.

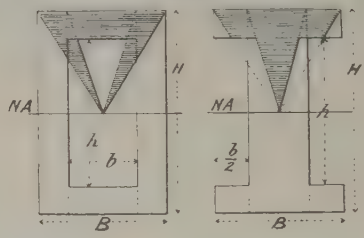
Section.	Examples of Modulus Figures.
Rectangular.	 Fig. 423.
Square.	B = H = S (the side of the square)

Those who have frequently to solve problems involving the strength and other properties of rolled sections will do well to get the book of sections issued by The British Standards Committee.

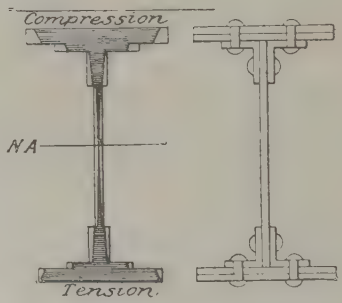
Modulus of the section Z.	Remarks.
$\frac{BH^2}{6}$	The moment of inertia (see p. 80) $= \frac{BH^3}{12}$ $y = \frac{H}{2}$ $Z = \frac{\frac{BH^3}{12}}{\frac{H}{2}} = \frac{BH^2}{6}$
	Also by graphic method—
(Square) $\frac{S^3}{6}$	The area $A = \frac{BH}{4}$ $d = \frac{2}{3} \times \frac{H}{2} = \frac{H}{3}$ $Z = 2Ad = 2 \times \frac{BH}{4} \times \frac{H}{3} = \frac{BH^2}{6}$

Section.

Examples of Modulus Figures.



Hollow rectangles and girder sections.



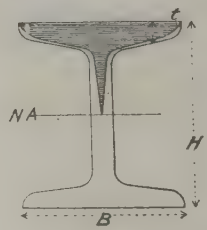


FIG. 424.

One corrugation of a trough flooring.

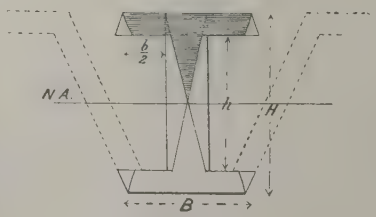


FIG. 425.

Modulus of the section Z.

$$\frac{BH^3 - bh^3}{6H}$$

Moment of inertia for outer rectangle $= \frac{BH^3}{12}$

„ „ inner „ $= \frac{bh^3}{12}$

„ „ hollow „ $= \frac{BH^3 - bh^3}{12}$

$$y = \frac{H}{2}$$

$$Z = \frac{\frac{BH^3 - bh^3}{12}}{\frac{H}{2}} = \frac{BH^3 - bh^3}{6H}$$

Approximately, when the web is thin, as in a rolled joist—

$$BtH$$

where t is the mean thickness of the flange.

This might have been obtained direct from the Z for the solid section, thus—

Z for outer section $= \frac{BH^2}{6}$

Z „ inner „ $= \frac{bh^2}{6} \times \frac{h}{H} = \frac{bh^3}{6H}$

Z „ hollow „ $= \frac{BH^3}{6} - \frac{bh^3}{6H} = \frac{BH^3 - bh^3}{6H}$

The Z for the inner section was multiplied by the ratio $\frac{h}{H}$, because the stress on the interior of the flange is less than the stress on the exterior in the ratio of their distances from the neutral axis.

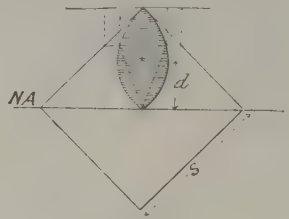
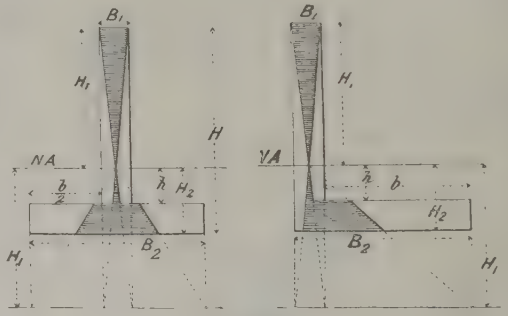
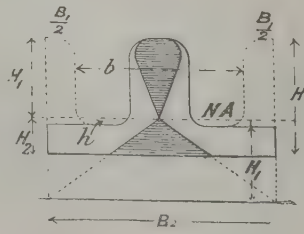
The approximate methods neglect the strength of the web, and assume the stress evenly distributed over the two flanges.

For rolled joists BtH is rather nearer the truth than BtH_0, where H_0 is measured to the middle of the flanges, and is more readily obtained. For almost all practical purposes the approximate method is sufficiently accurate.

N.B.—The safe loads given in makers' lists for their rolled joists are usually too high. The author has tested some hundreds in the testing-machine on both long and short spans, and has rarely found that the strength was more than 75 per cent. of that stated in the list.

In corrugated floorings or built-up sections, if there are rivets in the tension flanges, the area of the rivet-holes should be deducted from the Bt. Thus, if there are n rivets of diameter d in any one cross-section, the Z will be—

$$(B - nd)tH \text{ (approx.)}$$

Section.	Examples of Modulus Figures.
Square on edge.	 FIG. 426.
Tees, angles, and ⌊⌋ sections.	 FIG. 427.
	 FIG. 428.

This figure becomes a T or L when massed up about a vertical line.

Modulus of the section Z.

$0{\cdot}118S^3$

The moment of inertia (see p. 88) $= \frac{S^4}{12}$

$$y = \frac{S}{\sqrt{2}}$$

$$Z = \frac{\frac{S^4}{12}}{\frac{S}{\sqrt{2}}} = \frac{\sqrt{2}S^3}{12} = 0{\cdot}118S^3$$

$$\frac{B_1H_1^3 + B_2H_2^3 - bh^3}{3H_1}$$

$H_1 = 0{\cdot}7H$ approx.
$H_2 = 0{\cdot}3H$ „

The moment of inertia for the part above the N.A. $= \frac{B_1H_1^3}{3}$

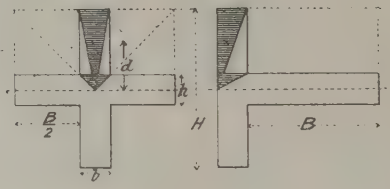
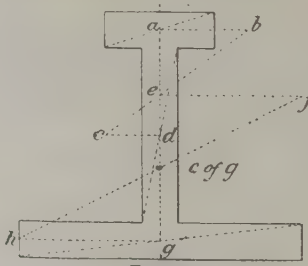
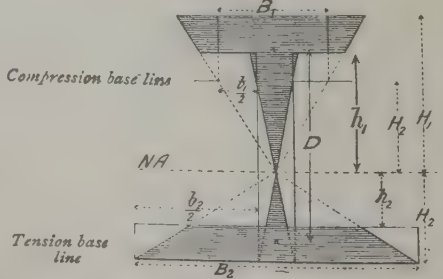
Ditto below $= \frac{B_2H_2^3 - bh^3}{3}$

Ditto for whole section $= \frac{B_1H_1^3 + B_2H_2^3 - bh^3}{3}$

Z for stress at top $= \frac{B_1H_1^3 + B_2H_2^3 - bh^3}{3H_1}$

If the position of the centre of gravity be calculated for the form of section usually used, it will be found to be approximately $0{\cdot}3H$ from the bottom.

Rolled sections, of course, have not square corners as shown in the above sketches, but the error involved is not material if a mean thickness be taken.

Section.	Examples of Modulus Figures.
T section on edge and cruciform sections.	 Fig. 429.
Unequal flanged sections.	 Fig. 430.  Fig. 431.

Modulus of the section Z.

$$\frac{bH^3 + Bh^3}{6H}$$

Moment of inertia for vertical part $= \frac{bH^3}{12}$

,, ,, horizontal ,, $= \frac{Bh^3}{12}$

,, ,, whole section $= \frac{bH^3 + Bh^3}{12}$.

Z for ,, $= \frac{bH^3 + Bh^3}{6H}$

Or this result may be obtained direct from the moduli of the two parts of the section. Thus—

Approx. Z $\frac{bH^2}{6}$

Z for vertical part $= \frac{bH^2}{6}$

,, horizontal ,, $= \frac{Bh^2}{6} \times \frac{h}{H}$ (see p. 445)

,, whole section $= \frac{bH^2}{6} + \frac{Bh^3}{6H} = \frac{bH^3 + Bh^3}{6H}$

It should be observed in the figure how very little the horizontal part of the section adds to the strength. The approximate Z neglects this part.

The strength of the T section when bent in this manner is very much less than when bent as shown in the previous figure.

$$\frac{B_1H_1^3 + B_2H_2^2 - b_1h_1^3 - b_2h_2^3}{3H_2}$$

Moment of inertia for upper part $= \frac{B_1H_1^3}{3} - \frac{b_1h_1^3}{3}$

Ditto lower part $= \frac{B_2H_2^3}{3} - \frac{b_2h_2^3}{3}$

Moment of inertia for whole section $= \frac{B_1H_1^3 + B_2H_2^3 - b_1h_1^3 - b_2h_2^3}{3} = I$

$Z = \frac{I}{H_2}$ for the stress on the tension flange

Section.

Examples of Modulus Figures.

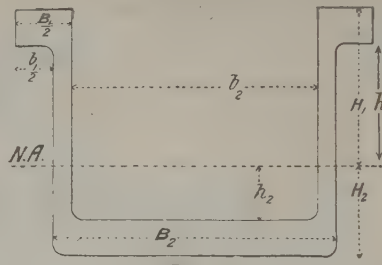


FIG. 432.

This figure becomes the same as the last when massed about a centre line.

Triangle.

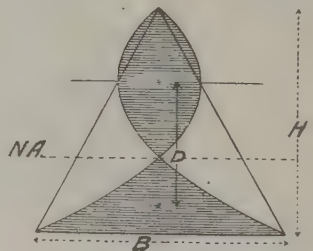


FIG. 433.

Trapezium.

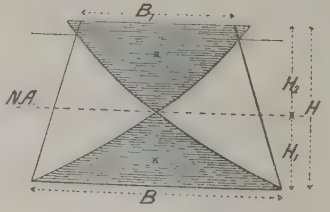


FIG. 434.

Modulus of the section Z.

The construction given in Fig. 430 is a convenient method of finding the centre of gravity of such sections.

Where ab = area of web, cd = area of top flange, ef = area of bottom flange, $gh = ab + cd$. The method is fully described on p. 63.

For stress at base— $\frac{BH^2}{12}$

For stress at apex— $\frac{BH^2}{24}$

The moment of inertia for a triangle about its c. of g. $= \frac{BH^3}{36}$ (see p. 82)

y for stress at base $= \frac{H}{3}$, at apex $= \frac{2H}{3}$

Z for stress at base $= \frac{\frac{BH^3}{36}}{\frac{H}{3}} = \frac{BH^2}{12}$

" apex $= \frac{BH^2}{24}$

For stress at wide side— $\frac{BH^2}{12}\left(\frac{n^2+4n+1}{2n+1}\right)$

For stress at narrow side— $\frac{BH^2}{12}\left(\frac{n^2+4n+1}{n+2}\right)$

Approximate value for Z— $\frac{BH^2}{12}(n+1)$ $n = \frac{B_1}{B}$

Moment of inertia $= \frac{BH^3}{36}\left(\frac{n^2+4n+1}{n+1}\right)$ (see p. 86)

y for stress at wide side $= H_1 = \frac{H}{3}\left(\frac{2n+1}{n+1}\right)$

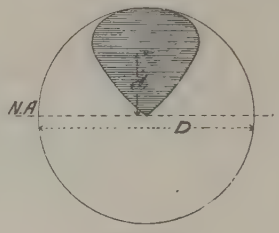
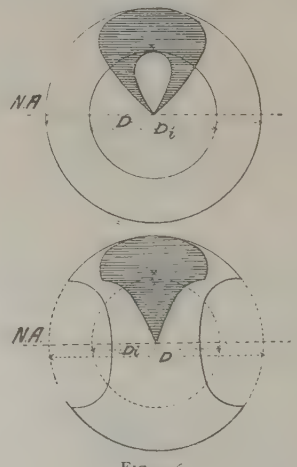
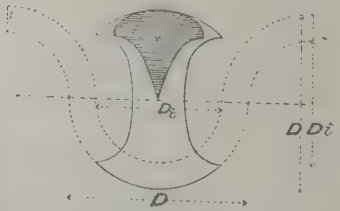
Z for stress at wide side $= \frac{\frac{BH^3}{36}\left(\frac{n^2+4n+1}{n+1}\right)}{\frac{H}{3}\left(\frac{2n+1}{n+1}\right)}$

$= \frac{BH^2}{12}\left(\frac{n^2+4n+1}{2n+1}\right)$

y for stress at narrow side $= H_2 = \frac{H}{3}\left(\frac{n+2}{n+1}\right)$

and dividing as above, we get the value given in the column.

The approximate method has been described on p. 86. It must not be used if the one side is more than twice the length of the other. For error involved, see also p. 87.

Section.	Examples of Modulus Figures.
Circle.	 FIG. 435.
Hollow circle and corrugated section.	 FIG. 436.  FIG. 437.

Modulus of the section Z.

The moment of inertia of a circle about a diameter (see p. 88) $= \frac{\pi D^4}{64}$

$$y = \frac{D}{2}$$

$$Z = \frac{\frac{\pi D^4}{64}}{\frac{D}{2}} = \frac{\pi D^3}{32}$$

$\frac{\pi D^3}{32}$ or $\frac{D^3}{10 \cdot 2}$

The moment of inertia of a hollow circle about a diameter (see p. 88) $= \frac{\pi(D^4 - D_i^4)}{64}$

$$y = \frac{D}{2}$$

$$Z = \frac{\frac{\pi(D^4 - D_i^4)}{64}}{\frac{D}{2}}$$

$$Z = \frac{\pi(D^4 - D_i^4)}{32D}$$

$\frac{\pi(D^4 - D_i^4)}{32D}$

This may be obtained direct from the Z thus—

Z for outer circle $= \frac{\pi D^3}{32}$

„ inner „ $= \frac{\pi D_i^3}{32} \times \frac{D_i}{D}$ (see p. 445)

„ hollow „ $= \frac{\pi(D^4 - D_i^4)}{32D}$

For corrugated sections in which the corrugations are not perfectly circular, the error involved is very slight if the diameters D and D_i are measured vertically. The expression given is for one corrugation. It need hardly be pointed out that the corrugations must not be placed as in Fig. 438. The strength, then, is simply that of a rectangular section of height H = thickness of plate.

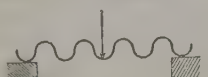
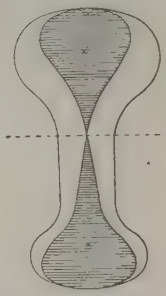
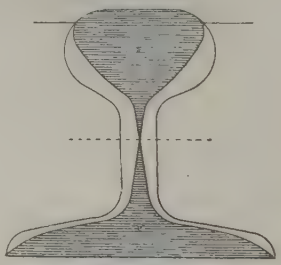
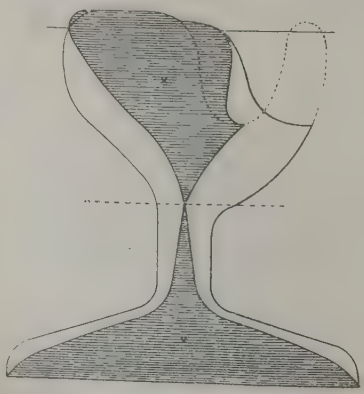


FIG. 438.

Irregular sections.	Examples of Modulus Figures.
Bull-headed rail.	 Fig. 439.
Flat-bottomed rail.	 Fig. 440.
Tram rail (distorted).	 Fig. 441.

Irregular sections.

Examples of Modulus Figures.

Bulb section.

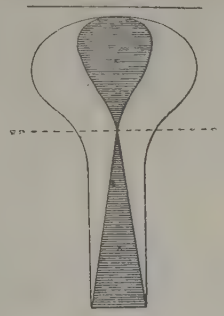
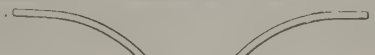
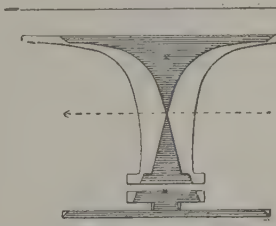


FIG. 442.

Hobson's patent flooring.



One section.



Four sections massed up.

FIG. 443.

Fireproof flooring.

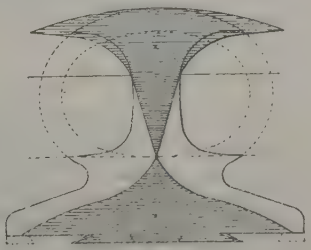


FIG. 444.

Irregular sections.

Examples of Modulus Figures.

Fireproof flooring.

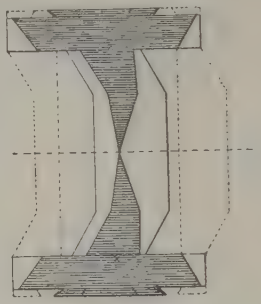


FIG. 445.

Table of hydraulic press.

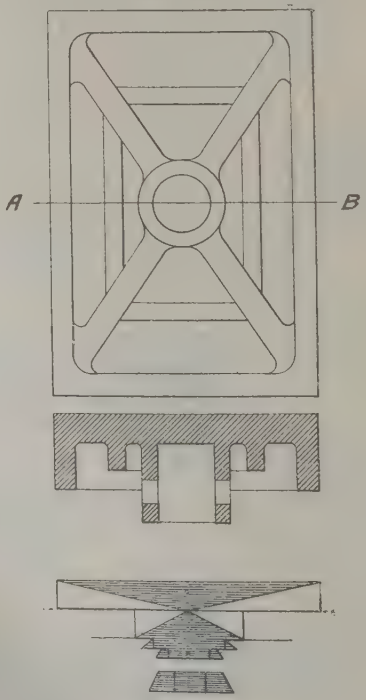


FIG. 446.

Shear on Beam Sections.—In the Fig. 447 the rectangular element *abcd* on the unstrained beam becomes *a'b'c'd'* when the beam is bent, and the element has undergone a shear. The total shear force on any vertical section = W, and, assuming for the present that the shear stress is evenly distributed over the whole section, the mean shear stress $= \frac{W}{A}$, where A = the area of the section; or we may write it $\frac{W}{B \cdot H}$. But we have shown (p. 390) that the shear stress along any two parallel sides of a rectangular element is equal to the shear stress along the other two parallel sides, hence the shear stress along *cd* is also equal to $\frac{W}{A}$.

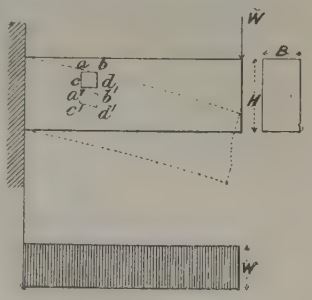


FIG. 447.

The shear on vertical planes tends to make the various parts of the beam slide downwards as shown in Fig. 448, *a*, but the shear on the horizontal planes tends to make the parts of the beam slide as in Fig. 448, *b*. This action may be illustrated by bending some thin strips of wood, when it will be found that they slide over one another in the manner shown. Solid timber beams often fail in this manner when tested.

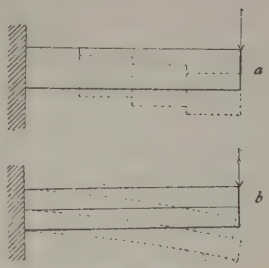


FIG. 448.

In the paragraph above we assumed that the shear stress was evenly distributed over the section; this, however, is far from being the case, for the shearing force at any part of a beam section is the algebraic sum of the shearing forces acting on either side of that part of the section (see p. 479). We will now work out one or two cases to show the distribution of the shear on a beam

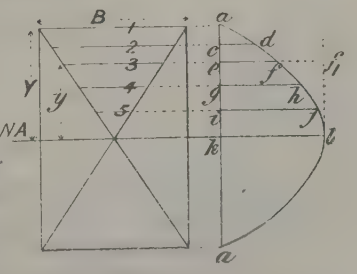


FIG. 449.

section by a graphical method, and afterwards find an analytical expression for the same.

In Fig. 449 the distribution of stress is shown by the width of the modulus figure. Divide the figure up as shown into strips, and construct a figure at the side on the base-line *aa*, the ordinates of which represent the shear at that part of the section, *i.e.* the sum of the forces acting to either side of it, thus—

The shear at 1 is zero
" " 2 is proportional to the area of the strip between 1 and 2 = *cd* on a given scale.
" " 3 is proportional to the area of the strip between 1 and 3 = *ef* on a given scale.
" " 4 is proportional to the area of the strip between 1 and 4 = *gh* on a given scale.
" " 5 is proportional to the area of the strip between 1 and 5 = *ij* on a given scale.
" " 6 is proportional to the area of the strip between 1 and 6 = *kl* on a given scale.

Let the width of the modulus figure at any point distant y from the neutral axis $= b$; then—

$$\text{the shear at } y = \frac{BY}{2} - \frac{by}{2}$$

$$\text{But } \frac{b}{y} = \frac{B}{Y}, \text{ and } b = \frac{By}{Y}$$

$$\text{the shear at } y = \frac{BY}{2} - \frac{By^2}{2Y} = kl - \frac{B}{2Y}(y^2)$$

$$\text{in the figure } kl = \frac{BY}{2} \qquad ff_1 = \frac{B}{2Y}(y^2)$$

Thus the shear curve is a parabola, as the ordinates ff_1, etc., vary as y^2; hence the maximum ordinate $kl = \frac{3}{2}$ (mean ordinate) (see p. 30), or the maximum shear on the section is $\frac{3}{2}$ of the mean shear.

In Figs. 450, 451 similar curves are constructed for a circular and for an **I** section.

It will be observed that in the **I** section nearly all the shear is taken by the web; hence it is usual, in designing plate girders of this section, to assume that the whole of the shear is taken by the web. The outer line in Fig. 451 shows the total shear and the inner figure the intensity of shear at the different layers. The shear at any section (Fig. 452) $ab = \frac{W}{2}$, and the intensity of

shear on the above assumption $= \frac{W}{2A_w}$, where A_w = the sectional area of the web, or the intensity of shear $= \frac{W}{2ht}$. But the intensity of shear stress on aa_1 = the intensity of shear stress

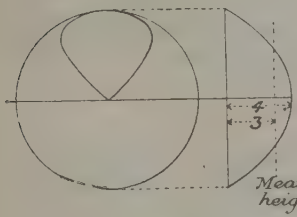


FIG. 450.

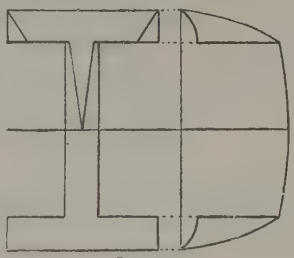


FIG. 451.

on ab, hence the intensity of the shear stress between the web and flange is also $= \frac{W}{2ht}$. We shall make use of this when working out the requisite spacing for the rivets in the angles between the flanges and the web of a plate girder.

In all the above cases it should be noticed that the shear stress is a maximum at the neutral plane, and the total shear

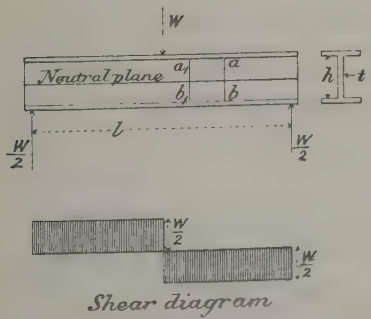


FIG. 452.

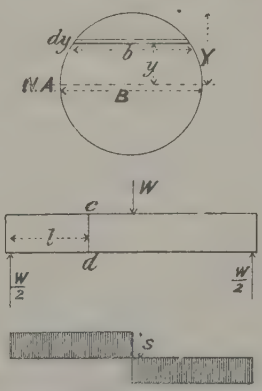


FIG. 453.

there is equal to the total direct tension or compression acting above or below it.

We will now get out an expression for the shear at any part of a beam section.

We have shown a circular section, but the argument will be seen to apply equally to any section.

Let b = breadth of the section at a distance y from the neutral axis;

F = stress on the skin of the beam distant Y from the neutral axis;

f = stress at the plane b distant y from the neutral axis;

M = bending moment on the section cd;

I = moment of inertia of the section;

S = shear force on section cd.

$$\left.\begin{array}{l}\text{The area of the strip distant } y \\ \text{from the neutral axis}\end{array}\right\} = b \,.\, dy$$

$$\text{the total force acting on the strip} = f \,.\, b \,.\, dy$$

$$\text{But } \frac{f}{F} = \frac{y}{Y}, \text{ or } f = \frac{Fy}{Y}$$

Substituting the value of f in the above, we have—

$$\frac{F}{Y} b \,.\, y \,.\, dy$$

$$\text{But } M = \frac{FI}{Y}, \text{ or } \frac{F}{Y} = \frac{M}{I} \text{ (see p. 432)}$$

Substituting the value of $\frac{F}{Y}$ in the above, we have—

$$\text{the total force acting on the strip} = \frac{M}{I} b \,.\, y \,.\, dy$$

$$\text{But } M = Sl \text{ (see p. 482)}$$

Substituting in the equation above, we have—

$$\frac{Sl}{I} b \,.\, y \,.\, dy$$

Dividing by the area of the plane, viz. $b \,.\, l$, we get—

$$\text{The intensity of shearing stress on that plane} = \frac{Sl}{Ibl} \int_y^Y b \,.\, y \,.\, dy$$

$$\left.\begin{array}{l}\text{the maximum value at the neutral axis,} \\ \text{when } b = B, \text{ and } y = 0\end{array}\right\} = \frac{S}{IB} \int_0^Y b \,.\, y \,.\, dy$$

$$\text{the mean intensity of shear stress} = \frac{S}{A}$$

where A = the area of the section.

$$\left.\begin{array}{l}\text{The ratio of the maximum}\\ \text{intensity to the mean}\end{array}\right\} = \frac{\dfrac{S}{IB}\displaystyle\int_0^Y b \,.\, y \,.\, dy}{\dfrac{S}{A}}$$

$$K = \frac{A}{IB}\int_0^Y b \,.\, y \,.\, dy$$

The value of K is easily obtained by this expression for geometrical figures, but for such sections as tram-rails a graphic solution must be resorted to.

The value of $\displaystyle\int_0^Y b \,.\, y \,.\, dy$ is the sum of the moments of all the small areas $b \,.\, dy$ about the N.A., between the limits of $y = 0$, *i.e.* starting from the N.A., and $y = Y$, which is the moment of the shaded area A_1 about the N.A., viz. $A_1 Y_c$, where Y_c is the distance of the centre of gravity of the shaded area from the N.A.

FIG. 454.

Since the neutral axis passes through the centre of gravity of the section, the above quantity will be the same whether the moments be taken above or below the N.A.

Deflection due to Shear.—The shear in beam sections increases the deflection over and above that due to the bending moment. The shear effect is negligible in solid beams of ordinary proportions, but in the case of beams having narrow webs, especially when the length of span is small compared to the depth of the section, the shear deflection may be 20 or 30 per cent. of the total.

Consider a short cantilever of length l, loaded at the free end. The deflection due to shear is x. For the present the deflection due to the bending moment is neglected.

$$\text{Work done by W in deflecting the beam by shear} = \frac{Wx}{2}$$

Let the force required to deflect the strip of area bdh through the distance x be dW, let the shear stress in the strip be f_s then—

$$\frac{x}{l} = \frac{f_s}{G} = \frac{dW}{G b dh} \text{ (see page 376)}$$

$$x = \frac{lf}{G} \qquad dW = f_s b dh$$

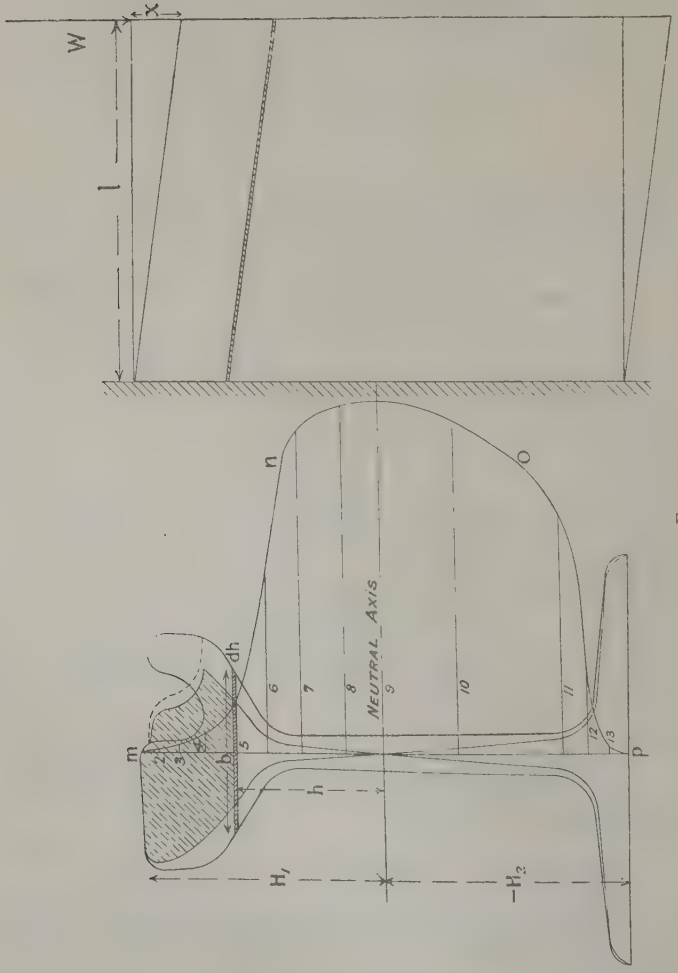


FIG. 455.

Work done in deflecting the strip by shear $\left.\right\} = \frac{x\,d\mathrm{W}}{2} = \frac{f_s^2 l b\,dh}{2\mathrm{G}}$

Total work done in deflecting the beam section by shear $\left.\right\} = \frac{l}{2\mathrm{G}}\int_{-\mathrm{H}_2}^{\mathrm{H}_1} f_s^2 b\,dh = \frac{\mathrm{W}x}{2}$

$$x = \frac{l}{\mathrm{WG}}\int_{-\mathrm{H}_2}^{\mathrm{H}_1} f_s^2 b\,dh$$

Let a be the area of the modulus figure between h and H_1. Let f_1 be the skin stress due to bending. The total longitudinal force acting on the portion of the beam section between h and H_1 is af_1, and the shear area over which this force is distributed is bl, hence $f_s = \frac{af_1}{bl}$, since the shear is constant throughout the length.

$$\text{and } x = \frac{l}{\mathrm{WG}}\int_{-\mathrm{H}_2}^{\mathrm{H}_1} \frac{a^2 f_1^2}{bl^2}\,dh$$

$$x = \frac{f_1^2}{\mathrm{WG}l}\int_{-\mathrm{H}_2}^{\mathrm{H}_1} \frac{a^2\,dh}{b} = \frac{\mathrm{A}_0 f_1^2}{\mathrm{WG}l}$$

where $\mathrm{A}_0 = \int_{-\mathrm{H}_2}^{\mathrm{H}_1} \frac{a^2\,dh}{b}$, which is the area of the figure *mnop* between the limits H_1 and $-\mathrm{H}_2$. Substituting the value of f_1 in terms of the bending moment M and the modulus of the section Z_1

$$x = \frac{\mathrm{A}_0\mathrm{M}^2}{\mathrm{WGZ}_1^2 l} = \frac{5\mathrm{A}_0\mathrm{M}^2}{2\mathrm{WEZ}_1^2 l}$$

$x = \frac{5\mathrm{A}_0\mathrm{W}l}{2\mathrm{EZ}_1^2}$ for a cantilever, and the total deflection at the free end of a cantilever with an end load, due to bending and shear, is—

$$\Delta = \delta + x = \frac{\mathrm{W}l^3}{3\mathrm{E}\mathrm{I}} + \frac{5\mathrm{A}_0\mathrm{W}l}{2\mathrm{EZ}_1^2}$$ (see page 510)

and for a beam of length $\mathrm{L} = 2l$ supporting a central load $\mathrm{W}_1 = 2\mathrm{W}$

$$\Delta = \frac{\mathrm{W}_1\mathrm{L}^3}{48\mathrm{EI}} + \frac{5\mathrm{A}_0\mathrm{W}_1\mathrm{L}}{8\mathrm{EZ}_1^2}$$

$$\text{and } \mathrm{E} = \frac{\mathrm{W}_1\mathrm{L}}{8\Delta}\left(\frac{\mathrm{L}^2}{6\mathrm{I}} + \frac{5\mathrm{A}_0}{\mathrm{Z}_1^2}\right)$$

In the case of beams of such sections as rectangles and circles the deflection due to shear is very small and is usually

negligible, especially when the ratio of length to depth is great.

In the case of plate web sections, rolled joists, braced girders, etc., the deflection due to the shear is by no means negligible. In text books on bridge work it is often stated that the central deflection of a girder is always greater than that calculated by the usual bending formula, on account of the "give" in the riveted joints between the web and the flanges. It is probable that a structure may take a "permanent set" due to this cause after its first loading, but after this has once occurred, it is unreasonable to suppose that the riveted joints materially affect the deflection; indeed, if one calculates the deflection due to both bending and shear, it will be found to agree well with the observed deflection. Bridge engineers often use the ordinary deflection formula for bending, and take a lower modulus of elasticity (about 9000 to 10,000 tons square inch) to allow for the shear.

EXPERIMENTS ON THE DEFLECTION OF I SECTIONS.

Section.	Span L.	Depth of section H.	E.	
			From δ.	From Δ.
Rolled joist	28″	6″	7,120	12,300
,, ,,	36″	6″	8,760	12,300
,, ,,	42″	6″	9,290	12,400
,, ,,	56″	6″	10,750	12,700
,, ,,	60″	6″	11,200	12,800
Riveted girder	60′	5′	10,200	12,200
,, ,,	60′	6′	9,000	12,200
,, ,,	75′	4′	11,000	12,600
,, ,,	160′	12′	8,900	12,000

It will be seen that the value of E, as derived from the expression for Δ, is tolerably constant, and what would be expected from steel or iron girders, whereas the value derived from δ is very irregular.

The application of the theory given above often presents difficulties, therefore, in order to make it quite clear, the full working out of a hard steel tramway rail is given below. The section of the rail was drawn full size, but the horizontal width of the diagram, *i.e.* $\frac{a^2}{b}$ was drawn $\frac{1}{4}$ of full size, hence the actual

area of *mnop* was afterwards multiplied by 4. The original drawing has been reduced to 0·405 of full size to suit the size of page.

a in square inches. Between		Total.	a^2	b in inches.	$\frac{a^2}{b}$
m and 2	0·38	0·38	0·14	2·20	0·07
2 „ 3	0·60	0·98	0·96	2·35	0·41
3 „ 4	0·60	1·58	2·50	3·03	0·82
4 „ 5	0·64	2·22	4·93	2·25	2·19
5 „ 6	0·35	2·57	6·60	0·72	9·17
6 „ 7	0·12	2·69	7·24	0·42	17·2
7 „ 8	0·10	2·79	7·78	0·42	18·5
8 „ 9	0·03	2·82	7·95	0·42	18·9
9 „ 10	−0·09	2·73	7·45	0·42	17·7
10 „ 11	−0·42	2·31	5·34	0·42	12·7
11 „ 12	−0·23	2·08	4·33	1·00	4·33
12 „ 13	−0·82	1·26	1·59	5·30	0·30
13 „ *p*	−1 26	0	0	—	0

Depth of section		6·5 ins.
Modulus of section (Z). In this case the two moduli are the same		14·8
Moment of inertia of section (I)		48·1
A_0 (area *mnop*)		73·9 sq. ins.
Span (l)	60 ins.	28 ins.
Load at which the deflection is measured	6 tons	12 tons
Mean deflection Δ for the above loads	0·046 in.	0·013 in.
E from $\frac{Wl^3}{48\delta I}$ tons sq. in.	12,200	8790
E from $\frac{Wl}{8\Delta}\left(\frac{l^2}{I}+\frac{5A_0}{Z_1^2}\right)$ „ „	13,900	14,200
E from a tension specimen cut from the head of rail	14,050 tons sq. in.	

In the case of a rectangular section of breadth B and depth H the value of A_0 is $\frac{BH^3}{30}$, by substitution in the expression for Δ for a centrally loaded beam of rectangular section, the deflection due to shear $x = \frac{3Wl}{4EBH}$ and the ratio

$$\frac{\text{Deflection due to shear}}{\text{Deflection due to bending}} = \frac{3H^2}{l^2}.$$

Taking $\frac{H}{l}$ as $\frac{1}{12}$, the shear deflection is about 2 per cent. of the bending deflection.

Discrepancies between Experiment and Theory.—Far too much is usually made of the slight discrepancies between experiments and the theory of beams; it has mainly arisen through an improper application of the beam formula, and to the use of very imperfect appliances for measuring the elastic deflection of beams.

The discrepancies may be dealt with under three heads—

(1) Discrepancies below the elastic limit
(2) „ at „ „
(3) „ after „ „

(1) The discrepancies below the elastic limit are partly due to the fact that the modulus of elasticity (E_c) in compression is not always the same as in tension (E_t).

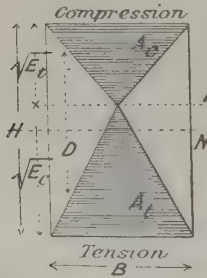


FIG. 456.

For example, suppose a piece of material to be tested by pure tension for E_t, and the same piece of material to be afterwards tested as a beam for E_b (modulus of elasticity from a bending test); then, by the usual beam theory, the two results should be identical, but in the case of cast materials it will probably be found that the bending test will give the higher result; for if, as is often the case, the E_c is greater than the E_t, the compression area A_c of the modulus figure will be smaller than the tension area A_t, for the tensions and compressions form a couple, and $A_cE_c = A_tE_t$. The modulus of the section will now be $A_t \times D$, or $\frac{BH}{4} \times \frac{\sqrt{E_c}}{\sqrt{E_c} + \sqrt{E_t}} \times D$; thus the Z is increased in the ratio $\frac{2\sqrt{E_c}}{\sqrt{E_c} + \sqrt{E_t}}$. If the E_c be 10 per cent.

greater than E_t, the E_b found from the beam will be about 2·5 per cent. greater than the E_t found by pure traction.

This difference in the elasticity will certainly account for considerable discrepancies, and will nearly always tend to make the E_b greater than E_t. There is also another discrepancy which has a similar tendency, viz. that some materials do not *perfectly* obey Hooke's law; the strain increases slightly more rapidly than the stress (see p. 364). This tends to increase the size of the modulus figures, as shown exaggerated in dotted lines, and thereby to increase the value of Z, which again tends to increase its strength and stiffness, and consequently make the E_b greater than E_t.

On the other hand, the deflection due to the shear (see p. 463) is usually neglected in calculating the value E_b, which consequently tends to make the deflection greater than calculated, and reduces the value of E_b. And, again, experimenters often measure the deflection between the bottom of the beam and the supports as shown (Fig. 458). The supports slightly indent the beam when loaded, and they moreover spring slightly, both of which tend to make the deflection greater than it should be, and consequently reduce the value of E_b.

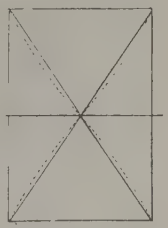


FIG. 457.

The discrepancies, however, between theory and experiment in the case of beams which are not loaded beyond the elastic limit are very, very small, far smaller than the errors usually made in estimating the loads on beams.

(2) The discrepancies at the elastic limit are more imaginary than real. A beam is usually assumed to pass the elastic limit when the rate of increase of the deflection per unit increase of load increases rapidly, *i.e.* when the slope of the tangent to the load-deflection diagram increases rapidly, or when a marked permanent set is produced, but the load at which this occurs is far beyond the true elastic limit of the material. In the case of a tension bar the stress is evenly distributed over any cross-section, hence the whole section of the bar passes the elastic limit at the same instant; but in the case of a beam the stress is not evenly distributed, consequently only a very thin skin of the metal

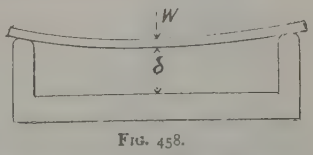


FIG. 458.

passes the elastic limit at first, while the rest of the section remains elastic, hence there cannot possibly be a sudden increase in the strain (deflection) such as is experienced in tension. When the load is removed, the elastic portion of the section restores the beam to very nearly its original form, and thus prevents any marked permanent set. Further, in the case of a tension specimen, the sudden stretch at the elastic limit occurs over the whole length of the bar, but in a beam only over a very small part of the length, viz. just where the bending moment is a maximum, hence the load at which the sudden stretch occurs is much less definitely marked in a beam than in a tension bar.

In the case of a beam of, say, mild steel, the distribution of stress in a section just after passing the elastic limit is approximately that shown in the shaded modulus figure of Fig. 459, whereas if the material had remained perfectly elastic, it would have been that indicated by the triangles *aob*. By the methods described in the next chapter, the deflection after the elastic limit can be calculated, and thereby it can be readily shown that the rate of increase in the deflection for stresses far above the elastic limit is very gradual, hence it is practically impossible to detect the true elastic limit of a piece of material from an ordinary bending test. Results of tests will be found in the Appendix.

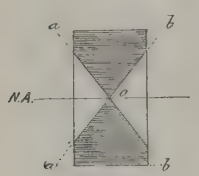


FIG. 459.

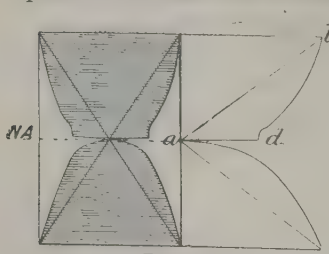


FIG. 460.

(3) The discrepancies after the elastic limit have occurred. The word "discrepancy" should not be used in this connection at all, for if there is one principle above all others that is laid down in the beam theory, it is that the material is taken to be perfectly elastic, *i.e.* that it has not passed the elastic limit, and yet one is constantly hearing of the "error in the beam theory," because it does not hold under conditions in which the theory expressly states that it will not hold. But, for the sake of those who wish to account for the apparent error, they can do it approximately in the following way. The beam theory assumes the stress to be proportional to the distance from the neutral axis, or to vary as shown by the

line *ab*; under such conditions we get the usual modulus figure. When, however, the beam is loaded beyond the elastic limit, the distribution of stress in the section is shown by the line *adb*, hence the width of the modulus figure must be increased in the ratio of the widths of the two curves as shown, and the Z thus corrected is the shaded area × D as before.[1] This in many instances will entirely account for the so-called error. Similar figures corrected in this manner are shown below, from which it will be seen that the difference is much greater in the circle than in the rolled joist, and, for obvious reasons, it will be seen that the difference is greatest in those sections in which much material is concentrated about the neutral axis.

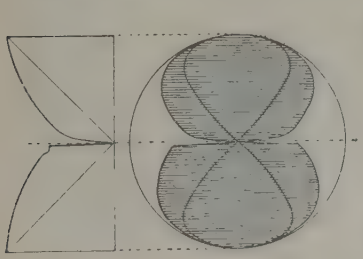


FIG. 461.

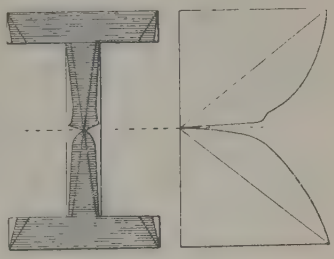


FIG. 462.

But before leaving this subject the author would warn readers against such reasoning as this. The actual breaking strength of a beam is very much higher than the breaking strength calculated by the beam formula, hence much greater stresses may be allowed on beams than in the same material in tension and compression. Such reasoning is utterly misleading, for the apparent error only occurs *after* the elastic limit has been passed.

Reinforced Concrete Beams.—The tensile strength of concrete is from 80 to 250 pounds per square inch, but the compressive strength is from 1500 to 5000 pounds per square inch. Hence a concrete beam of symmetrical section will always fail in tension long before the compressive stress reaches its

[1] Readers should refer to *Proceedings I.C.E.*, vol. cxlix. p. 313. It should also be remembered that when one speaks of the tensile strength of a piece of material, one always refers to the *nominal* tensile strength, not to the *real*; the difference, of course, is due to the reduction of the section as the test proceeds. Now, no such reduction in the Z occurs in the beam, hence we must multiply this corrected Z by the ratio of the real to the nominal tensile stress at the maximum load.

ultimate value. In order to strengthen the tension side of the beam, iron or steel rods are embedded in the concrete, it is then known as a reinforced beam. The position of the neutral axis of the section can be obtained thus—

Let E = the modulus of elasticity of the rods.
E_c = the " " " concrete.
$r = \frac{E}{E_c}$ = from 10 to 12.
f = the tensile stress in the rods.
f_c = the maximum compressive stress in the concrete.
a = the combined sectional area of the rods in square inches.

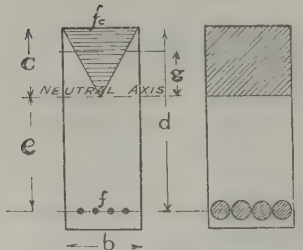


Fig. 463. Fig. 464.

The tension in the concrete may be neglected owing to the fact that it generally cracks at quite a low stress. It is assumed that originally plane sections of the beam remain plane after loading, hence the strain varies directly as the distance from the neutral axis, or

$$\frac{\frac{f_c}{E_c}}{\frac{f}{E}} = \frac{c}{e}, \text{ and } \frac{f_c}{f} = \frac{cE_c}{eE}$$

The total compressive force acting on the concrete is equal to the total tensile force acting on the rods, then for a section of breadth b,

$$f_c\frac{bc}{2} = fa = \frac{f_c eEa}{cE_c}$$

$$\frac{bc^2}{2} = \frac{eEa}{E_c} \quad \text{or} \quad bc \times \frac{c}{2} = a\frac{E}{E_c} \times e$$

which may be written $\frac{bc^2}{2} = (d - c)ra$, from which c can be obtained.

The quantity $bc \times \frac{c}{2}$ is the moment of the concrete area about the neutral axis, and $a\frac{E}{E_c} \times e$ is the moment of the rod

area, increased in the ratio r of the two moduli of elasticity, also about the neutral axis. Hence if the sectional area of the rods be increased in the ratio of the modulus of elasticity of the rods to that of the concrete the neutral axis passes through the centre of gravity, or the centroid, of the section when thus corrected as shown in Fig. 464.

The moment of resistance of the section (fZ) is, when considering the stress in the concrete,

$$f_c\left(\frac{bc}{2}\right)(g+e) \quad \text{or} \quad f_c\left(\frac{bc}{2}\right)\left(\tfrac{2}{3}c+e\right)$$

and when considering the stress in the rods—

$$fa(g+e) \quad \text{or} \quad fa\left(\tfrac{2}{3}c+e\right)$$

The working stress in the concrete f_c is usually taken from 400 lbs. to 600 lbs. per square inch, and f from 10,000 lbs. to 16,000 lbs. per square inch.

For the greatest economy in the reinforcements, the moment of resistance of the concrete should be equal to that of the rods, but if they are not equal in any given case, the lower value should be taken.

The modulus of the section for the concrete side is—

$$Z_c = \left(\frac{bc}{2}\right)\left(\tfrac{2}{3}c+e\right)$$

and the corresponding moment of inertia—

$$I = \left(\frac{bc^2}{2}\right)\left(\tfrac{2}{3}c+e\right)$$

Similarly, in the case of the reinforced side of the section—

$$I = aer\left(\tfrac{2}{3}c+e\right)$$

Example.—Total depth of section 18 inches, breadth 6 inches, four rods $\frac{1}{2}$ inch diameter, distance of centres from bottom edge $1\frac{1}{2}$ inch, the compressive stress in the concrete 400 lbs. per square inch, $r = 12$. Find the moment of resistance of the section, the stress in the rods, and the moment of inertia of the section.

In this case $d = 16{\cdot}5$ inches, $a = 0{\cdot}785$ square inch.

$$\frac{6 \times c^2}{2} = (16{\cdot}5 - c)12 \times 0{\cdot}785$$

$c = 5{\cdot}8$ inches. $\qquad e = 10{\cdot}7$ inches.

Moment of resistance for the concrete—

$$400\left(\frac{6 \times 5{\cdot}8}{2}\right)\left(\frac{2 \times 5{\cdot}8}{3} + 10{\cdot}7\right) = 101{,}400 \text{ inch-lbs.}$$

For the stress in the rods—

$$f \times 0{\cdot}785\left(\frac{2 \times 5{\cdot}8}{3} + 10{\cdot}7\right) = 101{,}400 \text{ inch-lbs.}$$

$$f = 8860 \text{ lbs. per square inch}$$

The moment of inertia—

$$\left(\frac{6 \times 5{\cdot}8^2}{2}\right)\left(\frac{2 \times 5{\cdot}8}{3} + 10{\cdot}7\right) = 1470 \text{ inch}^4\text{-units}$$

or

$$(0{\cdot}785 \times 10{\cdot}7 \times 12)\left(\frac{2 \times 5{\cdot}8}{2} + 10{\cdot}7\right) = 1470 \text{ inch}^4\text{-units}$$

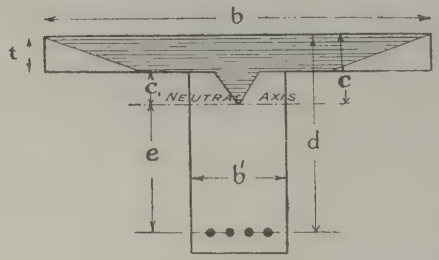


Fig. 465.

In reinforced concrete floors, the upper portion is a part of the floor; at intervals reinforced beams are arranged as shown.

Example.—Total depth = 24 inches, thickness of floor (t) = 4 inches, breadth (b) = 22 inches, breadth of web (b') = 7 inches. Six $\frac{3}{4}''$ rods, the centres of which are $1{\cdot}5''$ from the tension skin. Compressive stress in concrete (f_c) 400 lbs. per sq. inch. $r = 12$. Find the moment of resistance, the stress in the rods, and the moment of inertia.

Here $d = 22{\cdot}5$ inches $\qquad a = 2{\cdot}65$ square inches.

Position of neutral axis—

$$22 \times 4(c - 2) + 7\frac{(c - 4)^2}{2} = 2{\cdot}65 \times 12(22{\cdot}5 - c)$$

$$c = 7{\cdot}15'' \qquad c' = 3{\cdot}15'' \qquad e = 15{\cdot}35''$$

Moment of resistance of concrete—

$$\left\{\frac{22 \times 7{\cdot}15}{2}\left(\tfrac{2}{3} \times 7{\cdot}15 + 15{\cdot}34\right) - \left(\frac{22-7}{2}\right)3{\cdot}15 \times \frac{3{\cdot}15}{7{\cdot}15}\left(\tfrac{2}{3} \times 3{\cdot}15 + 15{\cdot}35\right)\right\}400(1400)400 = 560{,}000 \text{ in.-lbs.}$$

Stress on rods—

$$560{,}000 = f \times 2{\cdot}65 \times 20{\cdot}1$$
$$f = 10{,}500 \text{ lbs. square inch.}$$

Moment of inertia—

$$1400 \times 7{\cdot}15 = 10{,}000 \text{ inch}^4\text{-units}$$
$$\text{or } 2{\cdot}65 \times 15{\cdot}35 \times 12 \times 20{\cdot}1 = 9810 \quad \text{,,} \quad \text{,,}$$

In order to get the two values to agree exactly it would be necessary to express c to three places of decimals.

For further details of the design of reinforced concrete beams, books specially devoted to the subject should be consulted.

CHAPTER XII.

BENDING MOMENTS AND SHEAR FORCES.

Bending Moments.—*When two*[1] *equal and opposite couples are applied at opposite ends of a bar in such a manner as to tend to rotate it in opposite directions, the bar is said to be subjected to a bending moment.*

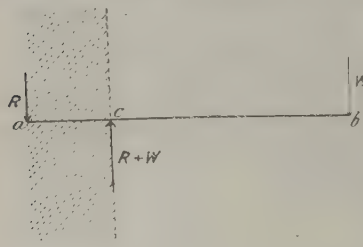


FIG. 466.

Thus, in Fig. 466, the bar *ab* is subjected to the two equal and opposite couples R . *ac* and W . *bc*, which tend to make the two parts of the bar rotate in opposite directions round the point *c*; or, in other words, they tend to *bend* the bar, hence the term "bending moment." Likewise in Fig. 467 the couples are $R_1 ac$ and $R_2 bc$, which have the same effect as the couples in Fig. 466. The bar in Fig. 466 is termed a "cantilever." The couple R . *ac* is due to the resistance of the wall into which it is built.

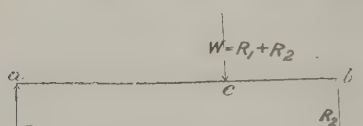


FIG. 467.

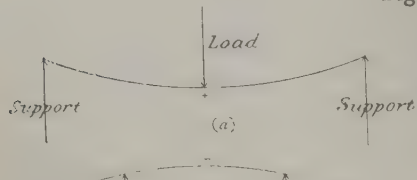


FIG. 468.

The bar in Fig. 467 is termed a "beam."

When a cantilever or beam is subjected to a bending moment which tends to bend it

[1] If there be more than two couples, they can always be reduced to two

concave upwards, as in Fig. 468 (*a*), the bending moment will be termed positive (+), and when it tends to bend it the reverse way, as in Fig. 468 (*b*), it will be termed negative (−).

Bending-moment Diagrams.—In order to show the variation of the bending moments at various parts of a beam, we frequently make use of bending-moment diagrams. The bending moment at the point *c* in Fig. 469 is W . *bc*; set down from *c* the ordinate $cc' = W . bc$ on some given scale. The bending moment at $d = W . bd$; set down from *d*, the ordinate $dd' = W . bd$ on the same scale; and so on for any number of points: then, as the bending moment at any point increases directly as the distance of that point from W, the points *b*, *d'*, *c'*, etc., will lie on a straight line. Join up these points as shown, then the depth of the diagram below any point in the beam represents on the given scale the bending moment at that point. This diagram is termed a "bending-moment diagram."

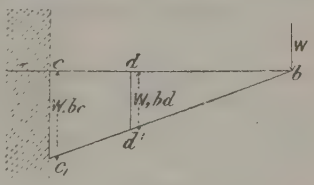


FIG. 469.

In precisely the same manner the diagram in Fig. 470 is obtained. The ordinate dd_1 represents on a given scale the bending moment R_1ad, likewise cc_1 the bending moment R_1ac or R_2bc, also ee_1 the bending moment R_2be.

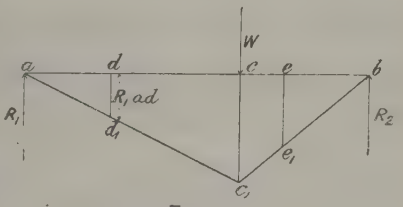


FIG. 470.

The reactions R_1 and R_2 are easily found by the principles of moments thus. Taking moments about the point *b*, we have—

$$R_1ab = Wbc \qquad R_1 = \frac{W . bc}{ab} \qquad R_2 = W - R_1$$

In the cantilever in Fig. 469, let W = 800 lbs., $bc = 6{\cdot}75$ feet, $bd = 4{\cdot}5$ feet.

The bending moment at $c = W . bc$
$= 800$ (lbs.) $\times$ $6{\cdot}75$ (feet)
$= 5400$ (lbs.-feet)

Let 1 inch on the bending-moment diagram = 12,000 (lbs.-feet), or a scale of 12,000 (lbs.-feet) per inch, or $\frac{12000 \text{ lbs.-feet}}{1 \text{ (inch)}}$.

$$\text{Then the ordinate } cc_1 = \frac{5400 \text{ (lbs.-feet)}}{\frac{12000 \text{ (lbs.-feet)}}{1 \text{ (inch)}}} = 0{\cdot}45 \text{ (inch)}$$

Measuring the ordinate dd_1, we find it to be 0·3 inch.

$$\text{Then } 0{\cdot}3 \text{ (inch)} \times \frac{12000 \text{ (lbs.-feet)}}{1 \text{ (inch)}} = \begin{cases} 3600 \text{ (lbs.-feet) bending} \\ \text{moment at } d \end{cases}$$

In this instance the bending moment could have been obtained as readily by direct calculation; but in the great majority of cases, the calculation of the bending moment is long and tedious, and can be very readily found from a diagram.

In the beam (Fig. 470), let W = 1200 lbs., $ac = 5$ feet, $bc = 3$ feet, $ad = 2$ feet.

$$R_1 = \frac{W \,.\, bc}{ab} = \frac{1200 \text{ (lbs.)} \times 3 \text{ (feet)}}{8 \text{ (feet)}} = 450 \text{ lbs.}$$

$$\text{the bending moment at } c = 450 \text{ (lbs.)} \times 5 \text{ (feet)}$$
$$= 2250 \text{ (lbs.-feet)}$$

Let 1 inch on the bending-moment diagram = 4000 lbs.-feet), or a scale of 4000 (lbs.-feet) per inch, or $\frac{4000 \text{ (lbs.-feet)}}{1 \text{ (inch)}}$.

$$\text{Then the ordinate } cc_1 = \frac{2250 \text{ (lbs.-feet)}}{\frac{4000 \text{ (lbs.-feet)}}{1 \text{ (inch)}}} = 0{\cdot}56 \text{ (inch)}$$

Measuring the ordinate dd_1, we find it to be 0·225 (inch).

$$\text{Then } \frac{0{\cdot}225 \text{ (inch)} \times 4000 \text{ (lbs.-feet)}}{1 \text{ (inch)}} = \begin{cases} 900 \text{ (lbs.-feet) bending} \\ \text{moment at } d \end{cases}$$

General Case of Bending Moments.—*The bending moment at any section of a beam is the algebraic sum of all the moments of the external forces about the section acting either to the left or to the right of the section.*

Thus the bending moment at the section f in Fig. 471 is, taking moments to the left of f—

$$R_1af - W_1cf - W_2df$$

or, taking moments to the right of f—

$$R_2bf - W_3ef$$

That the same result is obtained in both cases is easily verified by taking a numerical example.

Let $W_1 = 30$ lbs., $W_2 = 50$ lbs., $W_3 = 40$ lbs.; $ac = 2$ feet, $cd = 2{\cdot}5$ feet, $df = 1{\cdot}8$ feet, $fe = 2{\cdot}2$ feet, $eb = 3$ feet.

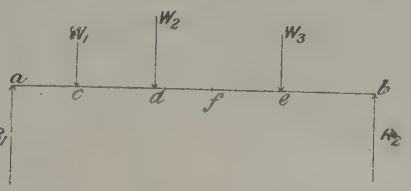


FIG. 471.

We must first calculate the values of R_1 and R_2. Taking moments about b, we have—

$$R_1 ab = W_1 cb + W_2 db + W_3 eb$$

$$R_1 = \frac{W_1 cb + W_2 db + W_3 eb}{ab}$$

$$R_1 = \frac{30(\text{lbs.}) \times 9{\cdot}5(\text{feet}) + 50(\text{lbs.}) \times 7(\text{feet}) + 40(\text{lbs.}) \times 3(\text{feet})}{11{\cdot}5 \text{ (feet)}}$$

$$= \frac{755 \text{ (lbs.-feet)}}{11{\cdot}5 \text{ (feet)}} = 65{\cdot}65 \text{ lbs.}$$

$$R_2 = W_1 + W_2 + W_3 - R_1$$

$$R_2 = 30 \text{ (lbs.)} + 50 \text{ (lbs.)} + 40 \text{ (lbs.)} - 65{\cdot}65 \text{ (lbs.)} = 54{\cdot}35 \text{ (lbs.)}$$

The bending moment at f, taking moments to the left of f,

$$= R_1 af - W_1 cf - W_2 df$$

$$= 65{\cdot}65 \text{ (lbs.)} \times 6{\cdot}3 \text{ (feet)} - 30 \text{ (lbs.)} \times 4{\cdot}3 \text{ (feet)} - 50 \text{ (lbs.)} \times 1{\cdot}8 \text{ (feet)} = 194{\cdot}6 \text{ (lbs.-feet)}$$

The bending moment at f, taking moments to the right of f,

$$= R_2 bf - W_3 ef$$

$$= 54{\cdot}35 \text{ (lbs.)} \times 5{\cdot}2 \text{ (feet)} - 40 \text{ (lbs.)} \times 2{\cdot}2 \text{ (feet)}$$

$$= 194{\cdot}6 \text{ (lbs.-feet)}$$

Thus the bending moment at f is the same whether we take moments to the right or to the left of the point f. The calculation of it by both ways gives an excellent check on the accuracy of the working, but generally we shall choose that side of the section that involves the least amount of calculation. Thus, in the case above, we should have taken moments to the right of the section, for that only involves the calculation of two moments, whereas if we had taken it to the left it would have involved three moments.

The above method becomes very tedious when dealing with many loads. For such cases we shall adopt graphical methods.

Shearing Forces.—When couples are applied to a beam in the way described above, the beam is not only subjected to a bending moment, but also to a shearing action. In a long beam or cantilever, the bending is by far the most important, but a short stumpy beam or cantilever will nearly always fail by shear.

Let the cantilever in Fig. 472 be loaded until it fails. It

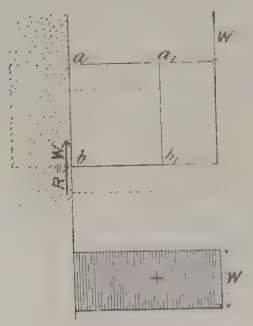


FIG. 472.

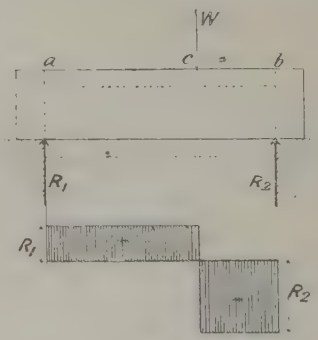


FIG. 473.

will bend down slightly at the outer end, but that we may neglect for the present. The failure will be due to the outer part shearing or sliding off bodily from the built-in part of the cantilever, as shown in dotted lines.

The shear on all vertical sections, such as ab or $a'b'$, is of the same value, and equal to W.

In the case of the beam in Fig. 473, the middle part will shear or slide down relatively to the two ends, as shown in dotted lines. The shear on all vertical sections between b and c is of the same value, and equal to R_2, and on all vertical sections between a and c is equal to R_1.

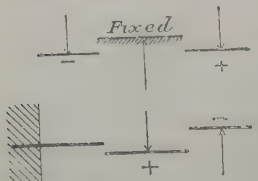


FIG. 474.

We have spoken above of positive and negative bending moments. We shall also find it convenient to speak of positive and negative shears.

When the sheared part slides in a { clockwise / contra-clockwise } direction relatively to the fixed part, we term it a { positive (+) shear / negative (−) shear }

Shear Diagrams.—In order to show clearly the amount of shear at various sections of a beam, we frequently make use of shear diagrams. In cases in which the shear is partly positive and partly negative, we shall invariably place the *positive* part of the shear diagram *above* the base-line, and the *negative* part *below* the base-line. Attention to this point will save endless trouble.

In Fig. 472, the shear is positive and constant at all vertical sections, and equal to W. This is very simply represented graphically by constructing a diagram immediately under the beam or cantilever of the same length, and whose depth is equal to W on some given scale, then the depth of this diagram at every point represents on the same scale the shear at that point. Usually the shear diagram will not be of uniform depth. The construction for various cases will be shortly considered. It will be found that its use greatly facilitates all calculations of the shear in girders, beams, etc.

In Fig. 473, the shear at all sections between *a* and *c* is constant and equal to R_1. It is also positive (+), because the slide takes place in a clockwise direction; and, again, the shear at all sections between *b* and *c* is constant and equal to R_2, but it is of negative (−) sign, because the slide takes place in a contra-clockwise direction; hence the shear diagram between *a* and *c* will be above the base-line, and that between *b* and *c* below the line, as shown in the diagram. The shear changes sign immediately under the load, and the resultant shear at that section is $R_1 - R_2$.

General Case of Shear.—*The shear at any section of a beam or cantilever is the algebraic sum of all the forces acting to the right or to the left of that section.*

One example will serve to make this clear.

In Fig. 475 three forces are shown acting on the cantilever fixed at *d*, two acting downwards, and one acting upwards.

The shear at any section between *a* and *d* = +W due to W
" " " *b* " *d* = −W_1 " W_1
" " " *c* " *d* = +W_2 " W_2

Construct the diagrams separately for each shear as shown, then combine by superposing the − diagram on the + diagram. The unshaded portion shows where the − shear neutralizes the

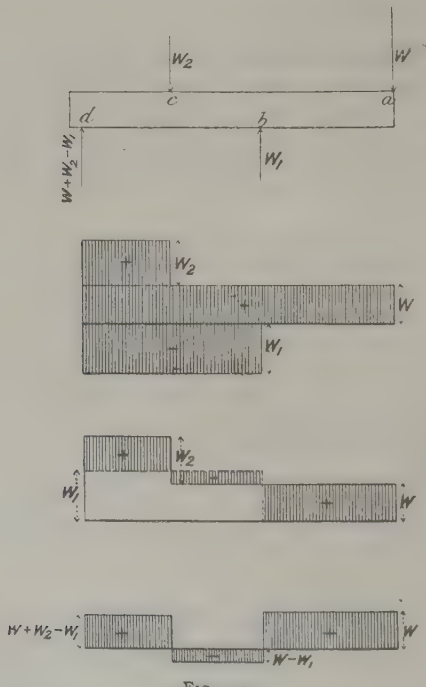


FIG. 475.

+ shear; then bringing the + portions above the base-line and the − below, we get the final figure.

Resultant shear at any section—		
Between	To the right.	To the left.
a and *b*	W	$-W_1 + W_2 - (W + W_2 - W_1)$ $= -W$
b and *c*	$W - W_1$	$W_2 - (W + W_2 - W_1)$ $= -(W - W_1)$
c and *a*	$W_2 - W_1 + W$ or $W + W_2 - W_1$	$-(W + W_2 - W_1)$

In the table above are given the algebraic sum of the forces to the right and to the left of various sections. On comparing them with the results obtained from the diagram, they will be found to be identical. In the case of the shear between the sections *b* and *c*, the diagram shows the shear as negative. The table, in reality, does the same, because W_1 in this case is greater than W. It should be noticed that when the shear is taken to the left of a section, the sign of the shear is just the reverse of what it is when taken to the right of the section.

Connection between Bending-moment and Shear Diagrams.—In the construction of shear diagrams, we make their depth at any section equal, on some given scale, to the shear at the section, *i.e.* to the algebraic sum of the forces to the right or left of that section, and the length of the diagrams equal to the distance from that section.

Let any given beam be loaded thus: Loads W_1, W_2, W_3, $-W_4$, $-W_5$ at distances l_1, l_2, l_3, l_4, l_5 respectively from any given section *a*, as shown in Fig. 476.

The bending moment at *a* is $= W_2l_2 + W_3l_3 - W_4l_4$ or $-W_1l_1 + W_5l_5$.

But W_2l_2 is the area of the shear diagram due to W_2 between W_2 and the section *a*, likewise W_3l_3 is the area between W_3 and *a*, also $-W_4l_4$ is the area between $-W_4$ and *a*. The positive areas are partly neutralized by the negative areas. The parts not neutralized are shown shaded.

The shaded area $= W_2l_2 + W_3l_3 - W_4l_4$, but we have shown above that this quantity is equal to the bending moment at *a*. In the same manner, it can be shown that the shaded area of the shear diagram to the left of the section *a* is equal to

$-W_1l_1 + W_5l_5$, *i.e.* to the bending moment at *a*. Hence we get this relation—

The bending moment at any section of a freely supported beam is equal to the area of the shear diagram up to that point. The bending moment is therefore a maximum where the shear changes sign.

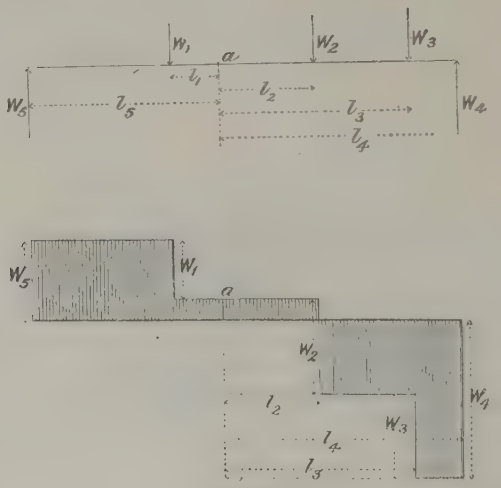


FIG 476.

Due attention must, of course, be paid to positive and negative areas in the shear diagram.

To make this quite clear, we will work out a numerical example.

In the figure, let $W_1 = 50$ lbs. $l_1 = 1$ foot
$W_2 = 80$ lbs. $l_2 = 2$ feet
$W_3 = 70$ lbs. $l_3 = 4$ feet
By moments we find $W_4 = 132{\cdot}2$ lbs. $l_4 = 5$ feet
$W_5 = 67{\cdot}8$ lbs. $l_5 = 4$ feet

The figure is drawn to the following scales—

Length 1 inch = 4 feet
load 1 inch = 160 lbs.

hence 1 square inch on the shear diagram = 4 (feet) × 160 (lbs.)
= 640 (lbs.-feet)

The area of the negative part of the shear diagram below the base-line is $-0{\cdot}401$ sq. inch, and the positive part above the base-line is $0{\cdot}056$ sq. inch; thus the area of the shear diagram up to the section a is $-0{\cdot}401 + 0{\cdot}056 = 0{\cdot}345$ sq. inch. But 1 sq. inch on the shear diagram = 640 (lbs.-feet) bending moment, thus the bending moment at the section $a = 0{\cdot}345 \times 640 = 221$ (lbs.-feet). The area of the shear diagram to the left of $a = 0{\cdot}345$ sq. inch, *i.e.* the same as the area to the right of the section. As a check on the above, we will calculate the bending moment at a by the direct method, thus—

$$\begin{aligned}\text{The bending moment at } a &= W_2l_2 + W_3l_3 - W_4l_4 \\ &= 80 \text{ (lbs.)} \times 2 \text{ (feet)} + 70 \text{ (lbs.)} \times \\ &\quad 4 \text{ (feet)} - 132{\cdot}2 \text{ (lbs.)} \times 5 \text{ (feet)} \\ &= 221 \text{ (lbs.-feet)}\end{aligned}$$

which is the same result as we obtained above from the shear diagram.

This interesting connection between the two diagrams can be shown to hold in all cases from load to slope diagrams. If a beam supports a distributed load, it can be represented at every part of the beam by means of a diagram whose height is proportional to the load at each point; then the amount of load between the abutment and any given point is proportional to the area of the load diagram over that portion of the beam. But we have shown that the shear at any section is the algebraic sum of all the forces acting either to the right or to the left of that section, whence the shear at that section is equal to the reaction *minus* the area of the load diagram between the section in question and the said reaction. We have already shown the connection between the shear and bending moment diagrams, and we shall shortly show that the slope between a tangent to the bent beam at any point and any other point is proportional to the area of the bending-moment diagram enclosed by normals to the bent beam drawn through those points.

Cantilever with single load at free end.

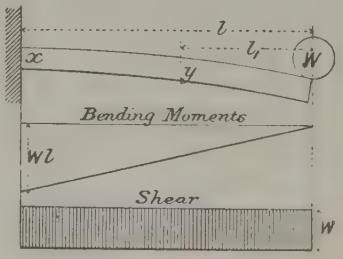


FIG. 477.

Cantilever with two loads.

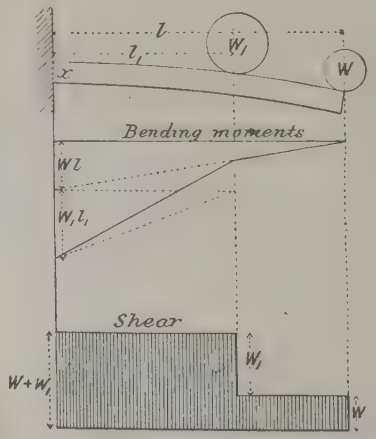


FIG. 478.

Bending moment M in lbs.-inches	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.	Remarks.
$M_x = Wl$ $= W$ (lbs.) $\times l$ (in.) $M_y = Wl_1$ $M = d \,.\, mn$ at any section where d = depth of bending-moment diagram in inches	$\frac{Wl}{mn}$ $\frac{Wl_1}{mn}$	The only moment acting to the right of x is Wl, which is therefore the bending moment at x. Likewise at y. The complete statement of the units for the depth of the bending-moment diagram is as follows :— m(lbs.) = 1 inch on diagram, or $\frac{m\text{(lbs.)}}{1\text{(inch)}}$ n(in.) = 1 " " $\frac{W\text{ (lbs.)}}{\frac{m\text{ (lbs.)}}{1\text{ inch}}} \times \frac{l_1\text{(inches)}}{n\text{ (inches)}} = \frac{Wl}{mn}$ (inches).
$M_2 = Wl + W_1l_1$ $M = d \,.\, mn$	$\frac{Wl + W_1l_1}{mn}$	This is a simple case of combining two such bending-moment diagrams as we had above. The lower one is tilted up from the diagram shown in dotted lines.

Cantilever with an evenly distributed load of *w* lbs. per *inch* run.

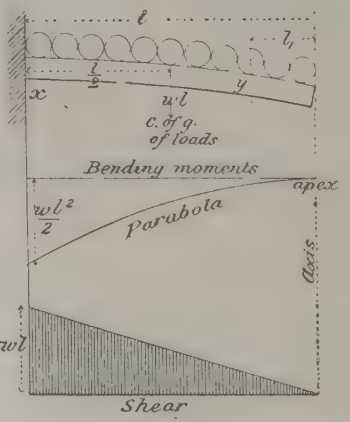


FIG. 479.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.	Remarks.
$M_x = \frac{wl^2}{2}$ $\frac{w\text{ (lbs.)}}{\text{inches}} \times \frac{l^2\text{ (inches)}^2}{2\text{ (constant)}}$ $= wl^2 \frac{\text{(lbs.-inches)}}{\text{constant}}$	$\frac{Wl}{2mn}$	In statics any system of forces may always be replaced by their resultant, which in this case is situated at the centre of gravity of the loads; and as the distribution of the loading is uniform, the resultant acts at a distance $\frac{l}{2}$ from x. The total load on the beam is wl, or W; hence the bending moment at $x = wl \times \frac{l}{2} = \frac{wl^2}{2}$. At any other section, $y = wl_1 \times \frac{l_1}{2} = \frac{wl_1^2}{2}$. Thus the bending moment at any section varies as the square of the distance of the section from the free end of the beam, therefore the bending moment diagram is a parabola. As the beam is fully covered with loads, the sum of the forces to the right of any section varies directly as the length of the beam to the right of the section; therefore the shearing force at any section varies directly as the distance of that section from the free end of the beam, and the depth of the shearing-force diagram varies in like manner, and is therefore triangular, with the apex at the free end as shown, and the depth at any point distant l_θ from the free end is wl_θ, *i.e.* the sum of the loads to the right of l_θ, and the area of the shear diagram up to that point is $\frac{wl_\theta \times l_\theta}{2} = \frac{wl_\theta^2}{2}$, *i.e.* the bending moment at that point.
Let $W = wl$ $M_x = \frac{Wl}{2}$ $M = d \,.\, mn$		

Cantilever irregularly loaded.

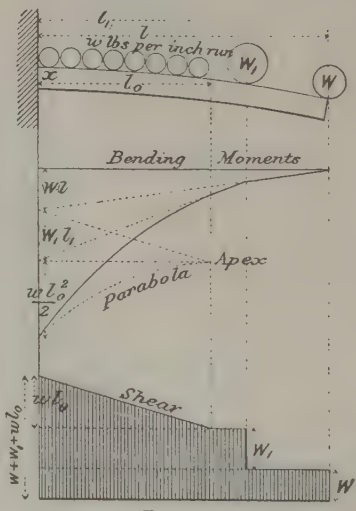


FIG. 480.

Beam supported at both ends, with a central load.

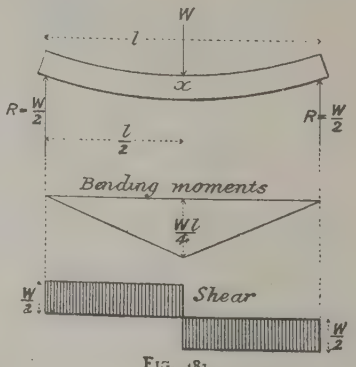


FIG. 481.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.	Remarks.
$M_x = Wl + W_1 l_1 + \frac{wl_0^2}{2}$ $M = d \cdot mn$	$\frac{Wl + W_1 l_1 + \frac{wl_0^2}{2}}{mn}$	This is simply a case of the combination of the diagrams in Figs. 478 and 479. However complex the loading may be, this method can always be adopted, although the graphic method to be described later on is generally more convenient for many loads.
$M_x = \frac{Wl}{4}$ $M = d \cdot mn$	$\frac{Wl}{4mn}$	Each support or abutment takes one-half the weight $= \frac{W}{2}$. The only moment to the right or left of the section x is $\frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4}$. At any other section the bending moment varies directly as the distance from the abutment; hence the diagram is triangular in form as shown. The only force to the right or left of x is $\frac{W}{2}$; hence the shear diagram is of constant depth as shown, only positive on one side of the section x, and negative on the other side.

Beam supported at both ends, with one load not in the middle of the span.

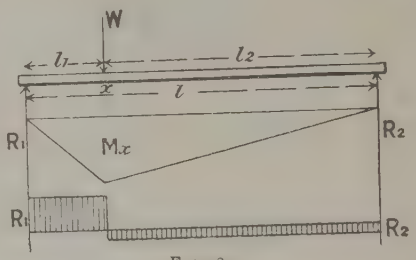


FIG. 482.

Beam supported at both ends, with two symmetrically placed loads.

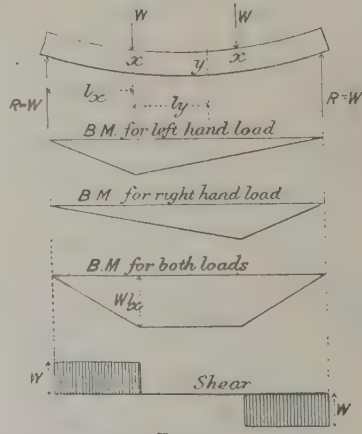


FIG. 483.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.	Remarks.
$M_x = \frac{W}{l}(l_1 l_2)$ $M = d \,.\, mn$	$\frac{W l_1 l_2}{lmn}$	Taking moments about one support, we have $R_1 l = W l_2$, or $R_1 = \frac{W l_2}{l}$. The bending moment at x— $M_x = R_1 l_1 = \frac{W}{l}(l_1 l_2)$
$M_x = W l_x$ $M_y = W l_x$ $M = d \,.\, mn$	$\frac{W l_x}{mn}$	The beam being symmetrically loaded, each abutment takes one weight = W = R. The only moment to the right of the right-hand section x is W . l_x; likewise with the left-hand section. At any other section y between the loads, and distant l_y from one of them, we have, taking moments to the left of y, $R(l_x + l_y) - W \,.\, l_y = R \,.\, l_x + R \,.\, l_y - R \,.\, l_y = R \,.\, l_x$ or $W \,.\, l_x$, *i.e.* the bending moment is constant between the two loads. The sum of the forces to the right or left of $y = W - R = 0$, and to the right of the right-hand section the sum of the forces = R = W at every section.

Beam supported at both ends, load evenly distributed, *w* lbs. per *inch* run.

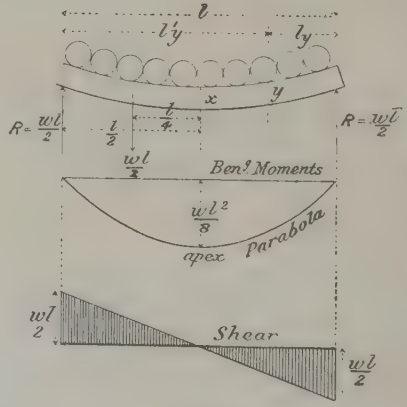


FIG. 484.

Bending moment M in lbs.-inches.

Depth of bending-moment diagram in inches.

Scale of W, m lbs. = 1 inch.

Scale of l, $\frac{1}{n}$ full size

Remarks.

As in the case of the uniformly loaded cantilever, we must replace the system of forces by their resultant.

The load being symmetrically placed, the abutments each take one-half the load $= \frac{wl}{2}$.

Then, taking moments to the left of x, we have—

$$\frac{wl}{2} \times \frac{l}{2} - \frac{wl}{2} \times \frac{l}{4} = \frac{wl^2}{8}$$

$M_x = \frac{wl^2}{8}$

Let $W = wl$

$M_x = \frac{Wl}{8}$

The $\frac{wl}{2}$ shown midway between x and the abutment is the resultant of the loads on half the beam, acting at the centre of gravity of the load, viz. $\frac{l}{4}$ from x, or the abutment.

$\frac{Wl}{8mn}$

The bending moment at any other section y, distant l_y from the abutment, is: taking moments to the right of y—

$$\frac{wl}{2} \times l_y - wl_y \times \frac{l_y}{2} = \frac{wl_y}{2}(l - l_y)$$

$$= \frac{w}{2}(l_y \times l_y') = \frac{W}{l}\left(\frac{l_y l_y'}{2}\right)$$

where $l_y' = l - l_y$.

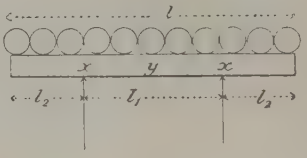
Thus the bending moment at any section is proportional to the product of the segments into which the section divides the beam. Hence the bending-moment diagram is a parabola, with its axis vertical and under the middle of the beam as shown.

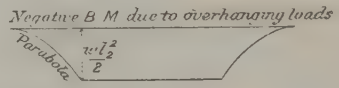
The forces acting to the right of the section $x = \frac{wl}{2} - \frac{wl}{2} = 0$; *i.e.* the shear at the middle section is zero.

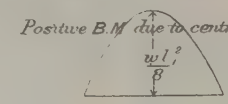
At the section $y = wl_y - \frac{wl}{2} = w\left(l_y - \frac{l}{2}\right)$. Hence the shear varies inversely as the distance from the abutment, and at the abutment. where $l_y = 0$, it is $= \frac{wl}{2}$.

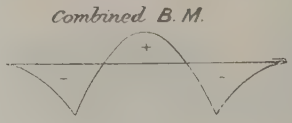
N.B.—Be very careful to reduce the distributed load to pounds per *inch* run if the dimensions of the beam are in *inches*.

Beam supported at two points equidistant from the ends, and a load of w lbs. per inch run evenly distributed.









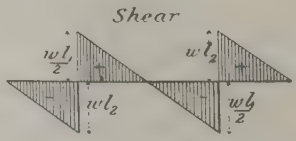


FIG. 485.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.
$M_x = -\frac{wl_2^2}{2}$	$\frac{M_x}{mn}$
$M_y = \frac{wl_1^2}{8} - \frac{wl_2^2}{2}$ $= \frac{w}{2}\left(\frac{l_1^2}{4} - l_2^2\right)$	$\frac{M_y}{mn}$

Remarks.

The bending-moment diagram for the loads on the overhanging ends is a combination of Figs. 479 and 483, and the diagram for the load on the central span is simply Fig. 484. Here we see the importance of signs for bending moments.

The beam will be subject to the smallest bending moment when $M_x = M_y$; or when

$$\frac{wl_2^2}{2} = \frac{wl_1^2}{8} - \frac{wl_2^2}{2}$$

$$wl_2^2 = \frac{wl_1^2}{8} \qquad \frac{l_1^2}{l_2^2} = 8$$

$$l_1 = 2{\cdot}83l_2$$

But $l_1 + 2l_2 = l$

substituting the value of l_1 above $\Big\}$ $= 2{\cdot}83l_2 + 2l_2 = l$

$$\text{or } l = 4{\cdot}83l_2$$

or say $l_2 = \frac{1}{5}l$ for the conditions of maximum strength of the beam.

The shear diagram will be seen to be a combination of Figs. 479 and 484.

Beam supported at each end and irregularly loaded.

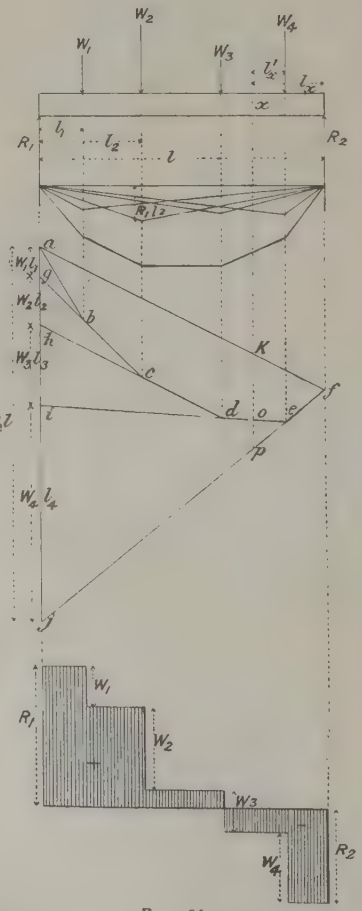


FIG. 486.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W_1, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.
	$\frac{W_1 l_1}{mn}$
$M = d \,.\, mn$	or
	$\frac{W_2 l_2}{mn}$
	etc.

REMARKS.

The method shown in the upper figure is simply that of drawing in the triangular bending-moment diagram for each load treated separately, as in Fig. 481, then adding the ordinates of each to form the final diagram by stepping off with a pair of dividers.

In the lower diagram, the heights *ag*, *gh*, etc., are set off on the vertical drawn through the abutment = $W_1 l_1$, $W_2 l_2$, etc., as shown. The sum of these, of course, = $R_2 l$. From the starting-point *a* draw a sloping line *ab*, cutting the vertical through W_1 in the point *b*. Join *gb* and produce to *c*, join *hc* and produce to *d*, and so on, till the point *f* is reached; join *fa*, which completes the bending-moment diagram *abcdef*, the depth of which in inches multiplied by *mn* gives the bending moment. The proof of the construction is as follows: The bending moment at any point *x* is $R_2 l_x - W_4 l'_x$.

On the bending-moment diagram $\frac{Kp}{aj} = \frac{l_x}{l}$; or

$$Kp = \frac{aj \times l_x}{l} = \frac{R_2 l \times l_x}{l} = R_2 l_x$$

$$Op = \frac{ij \times l'_x}{l_4} = \frac{W_4 l_4 \times l'_x}{l_4} = W_4 l'_x$$

and the depth of the bending-moment diagram $= KO = Kp - Op = R_2 l_x - W_4 l'_x$

It will be observed that this construction does not involve the calculation of R_1 and R_2. For the shear diagram R_2 can be obtained thus: Measure off *aj* in inches; then $\frac{aj \times mn}{l} = R_2$, where *l* is the actual length of the beam in inches.

Beam supported at the ends and irregularly loaded

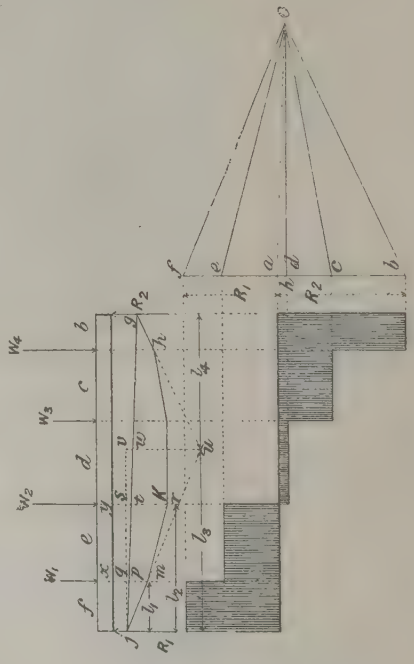


FIG. 487.

Bending moment M in lbs.-inches.

REMARKS.

Make the height of the load lines on the beam proportional to the loads, viz. $\frac{W_1}{m}$, $\frac{W_2}{m}$, etc., inches. Drop perpendiculars through each as shown. On a vertical fb set off $fe = \frac{W_1}{m}$, $ed = \frac{W_2}{m}$, etc. Choose any convenient point O distant Oh from the vertical. Oh is termed the "polar distance." Join fO, eO, etc. From any point j on the line passing through R_1 draw a line jm parallel to fO; from m draw mK parallel to eO, and so on, till the line through R_2 is reached in g. Join gj, and draw Oa on the vector polygon parallel to this last line; then the reaction $R_1 = fa$, and $R_2 = ba$. Then the vertical depth of the bending-moment diagram at any given section is proportional to the bending moment at that section.

Proof.—The two triangles jpm and Oaf are similar, for jm is parallel to fO, and jp to aO, and pm to fa; also jq is drawn at right angles to the base mp. Hence—

$$\frac{\text{height of } \triangle\, jpm}{\text{height of } \triangle\, Oaf} = \frac{\text{base of } \triangle\, jpm}{\text{base of } \triangle\, Oaf}$$

$$\text{or } \frac{jq}{mp} = \frac{Oh}{af}$$

$$\therefore \frac{l_1}{D_x} = \frac{Oh}{R_1}$$

$M = m \,.\, n(D \times Oh)$

For $jq = l_1$ and $af = R_1$; and let $mp = D_x$, *i.e.* the depth of the bending-moment diagram at the section x, or $R_1 l_1 = D_x Oh = M_x$ = the bending moment at x.

By similar reasoning, we have—

$$R_1 l_2 = rt \times Oh$$

$$\text{also } W_1(l_2 - l_1) = r\text{K} \times Oh$$

$$\text{the bending moment at } y = M_y = R_1 l_2 - W_1(l_2 - l_1)$$

$$= Oh(rt - r\text{K})$$

$$= Oh(\text{K}t)$$

$$= Oh \,.\, D_y$$

where D_y = the depth of the bending-moment diagram at the section y.

Thus the bending moment at any section is equal to the depth of the diagram at that section multiplied by the polar distance, both taken to the proper scales, which we will now determine. The diagram is drawn so that—

1 inch on the load scale = m lbs.
1 ,, ,, length ,, = n inches.

Hence the measurements taken from the diagram in inches must be multiplied by mn.

The bending moment expressed in *lbs.-inches* at any section $= M = m \,.\, n \,.\, (D \,.\, Oh)$

Beam supported at two points with overhanging ends and irregularly loaded.

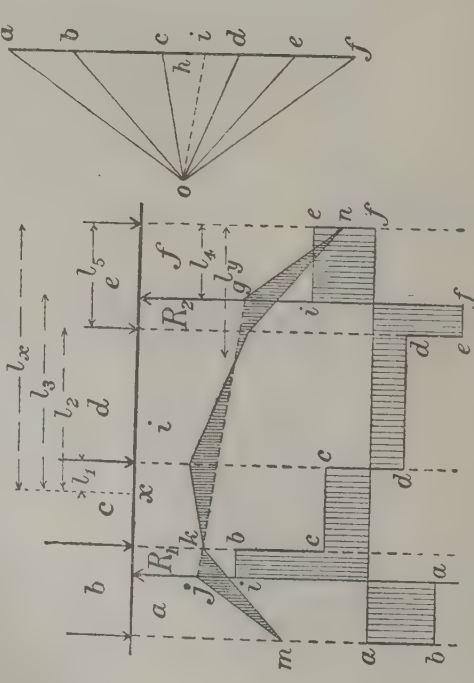


FIG. 488.

Bending moment M in lbs.-inches.

where D is the depth of the diagram in inches at that section, and Oh is the polar distance in inches.

In Chap. IV. we showed that the resultant of such a system of parallel forces as we have on the beam passes through the meet of the first and last links of the link polygon, viz. through u, where jm cuts gh. Then, as the whole system of loads may be replaced by the resultant, we have $R_1 l_3 = R_2 l_4$. But we have shown above that the triangles jvw and Oaf are similar; hence $\frac{jv}{Oh} = \frac{uw}{af}$. But $jv = l_3$, therefore $af \times l_3 = Oh \times uw = R_1 l_3$, or $af = R_1$. Similarly it may be shown that $ab = R_2$.

$M = m \cdot n(D \times Oh)$

The loads are set down to the proper scale on the vector polygon as in the last case. A pole O is chosen as before. The vertical load lines are dotted in order to keep them distinct from the reaction lines which are shown in full. Starting from the point j on the reaction line R_1, a line jm is drawn parallel to aO on the vector polygon, from m a line mk (in the space b) is drawn parallel to bO, and so on till the point u is reached, from u a line is drawn parallel to fO to meet the reaction line R_2 in the point g. Join jg. From the pole O draw a line Oi parallel to jg.

Then ai gives the reaction R_1, and if the reaction R_2. The bending moment diagram is shown shaded. The points where the bending moment is zero are known as the points of contrary flexure.

The construction of the shear diagram will be evident when it is remembered that the shear at any section is the algebraic sum of the forces to the right or to the left of the section.

The bending moment at any section of a beam loaded in this manner can be readily calculated. The reactions must first be found by taking moments about one of the points of support. The bending moment at any section x distant l_x from the load ef is

$$M_x = \overline{cd} l_1 + \overline{de} l_2 - R_2 l_3 + \overline{ef} l_x$$

and the distance l_y of the point of contrary flexure from the load ef is obtained thus

$$\overline{ef} l_y - R_2(l_y - l_4) + \overline{de}(l_y - l_5) = 0$$

$$l_y = \frac{\overline{de} l_5 - R_2 l_4}{\overline{ef} + \overline{de} - R_2}$$

Beam supported at each end and loaded with an evenly distributed load of w lbs. per inch run over a part of its length.

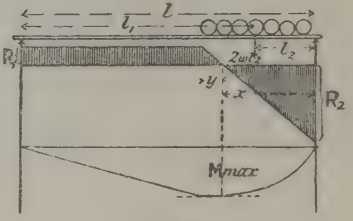


FIG. 489.

Beam supported at each end with a distributed load which varies directly as the distance from one end.

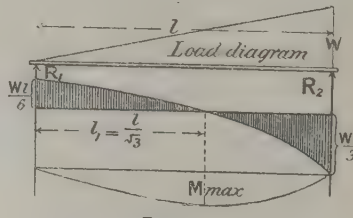


FIG. 490

Bending moment M in lbs.-inches.

Remarks.

Remembering that the bending moment at any section is equal to the area of the shear diagram up to that section, the maximum bending moment will occur at the section where the shear changes sign.

$$M_{max.} = \frac{2wl_2^2l_1^2}{l^2}$$

$$R_1 = \frac{2wl_2^2}{l} \qquad R_2 = \frac{2wl_2l_1}{l}$$

$$x = \frac{2l_2R_2}{R_2 + R_1} = \frac{2l_2R_2}{2wl_2} = \frac{2l_2l_1}{l}$$

$$M_{max.} = \frac{R_2x}{2} = \frac{2wl_2^2l_1^2}{l^2}$$

Let w_0 be the intensity of loading at any point distant l_0 from the apex of the load diagram.

$$\text{The shear at this point} = R_1 - \int_0^{l_0} w_0 dl = R_1 - \frac{W}{l}\int_0^{l_0} l_0 dl$$

$$= \frac{Wl}{6} - \frac{Wl_0^2}{2l}$$

Therefore the shear diagram is parabolic.

The shear is zero when—

$$\frac{Wl}{6} = \frac{Wl_0^2}{2l}$$

$$\text{or when } l_0 = \frac{l}{\sqrt{3}}$$

$$M_{max.} = \frac{Wl^2}{9\sqrt{3}}$$

The maximum bending moment occurs at the section where the shear changes sign, and is equal to the area of the shear curve; hence—

$$M_{max.} = \tfrac{2}{3}\frac{Wl}{6} \times \frac{l}{\sqrt{3}} = \frac{Wl^2}{9\sqrt{3}}$$

For another method of arriving at this result, see p. 185.

Beam built in at both ends and centrally loaded.

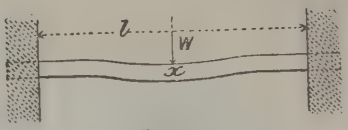


FIG. 491.

Ditto with evenly distributed load.

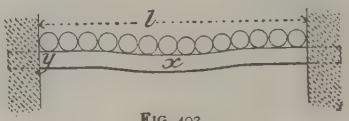


FIG. 492.

Cantilever propped at the outer end with evenly distributed load.

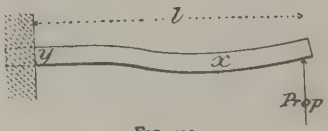


FIG. 493.

Beam built in at both ends, the load applied on one of the ends, which slides parallel to the fixed end.

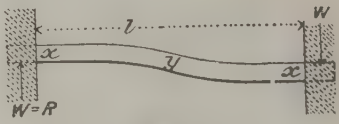


FIG. 494.

Bending moment M in lbs.-inches.	
$M_x = \frac{Wl}{8}$ $M_y = \frac{Wl}{8}$	
$M_x = \frac{wl^2}{24}$ $M_y = \frac{wl^2}{12}$	The determination of these bending moments depends on the elastic properties of the beams, which are fully discussed in Chap. XIII.
$M_x = \frac{9wl^2}{128}$ $M_y = \frac{wl^2}{8}$	In all these cases the beam is shown built in at both ends. The beams are assumed to be free endwise, and guided so that the ends shall remain horizontal as the beam is bent. If they were rigidly held at both ends, the problem would be much more complex.
$M_x = \frac{Wl}{2}$ $M_y = 0$	

CHAPTER XIII.

DEFLECTION OF BEAMS.

Beam bent to the Arc of a Circle.—Let an elastic beam be bent to the arc of a circle, the radius of the neutral axis being ρ. The length of the neutral axis will not alter by the bending. The distance of the skin from the neutral axis $= y$.

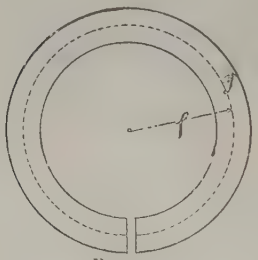


FIG. 495.

The original length of the outer skin $= 2\pi\rho$

the length of the outer skin after bending $= 2\pi(\rho + y)$

the strain of the skin due to bending $= 2\pi(\rho + y) - 2\pi\rho = 2\pi y$

But we have (see p. 373) the following relation :—

$$\frac{\text{strain}}{\text{original length}} = \frac{\text{stress}}{\text{modulus of elasticity}}$$

$$\text{or } \frac{2\pi y}{2\pi\rho} = \frac{y}{\rho} = \frac{f}{E}$$

But we also have—

$$f = \frac{M}{Z}$$

Substituting this value in the above equation—

$$\frac{y}{\rho} = \frac{M}{EZ}$$

$$\text{whence } M = \frac{EZy}{\rho}; \text{ or } M = \frac{EI}{\rho}$$

Central Deflection of a Beam bent to the Arc of a Circle.—From the figure we have—

$$\rho^2 = (\rho - \delta)^2 + \left(\frac{L}{2}\right)^2$$

whence $2\rho\delta - \delta^2 = \frac{L^2}{4}$

The elastic deflection (δ) of a beam is rarely more than $\frac{1}{600}$ of the span (L); hence the δ^2 will not exceed $\frac{L}{360,000}$, which is quite negligible;

hence $2\rho\delta = \frac{L^2}{4}$ $\qquad \delta = \frac{L^2}{8\rho}$

But $\rho = \frac{EI}{M}$

hence $\delta = \frac{ML^2}{8EI}$

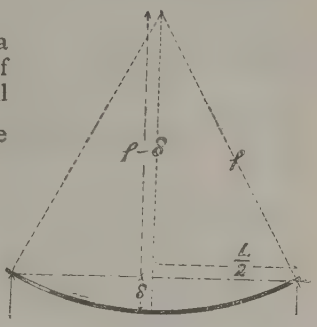


FIG. 496.

We shall shortly give another method for arriving at this result.

General Statement regarding Deflection. — In speaking of the deflection of a cantilever or beam, we always mean the deflection measured from a line drawn tangential to that part of the bent cantilever or beam which remains parallel to its unstrained position. The deflection δ will be seen by referring to the figures shown.

The point t at which the tangent touches the beam we shall term the "tangent point." When dealing with beams, we shall find it convenient to speak of the deflection at the support, *i.e.* the height of the support above the tangent.

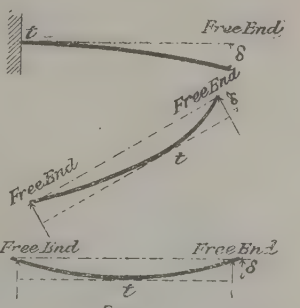


FIG. 497.

Deflection of a Cantilever.—Let the upper diagram (Fig. 498) represent the distribution of bending moment acting on the cantilever, the dark line the bent cantilever, and the straight dotted line the unstrained position of the cantilever. Consider any very small portion yy, distant l_y from the free end of the cantilever. We will suppose the length $\overline{yy}$ so small that the radius of curvature ρ_y is the same at both points, y, y. Let the angle subtending yy be θ_y (circular measure); then the angle

between the two tangents ya, yb will also be θ_y. Then the deflection at the extremity of these tangents due to the bending between y, y is—

$$\delta_y = \theta_y \cdot l_y$$

But $\theta_y = \dfrac{\overline{yy}}{\rho_y}$

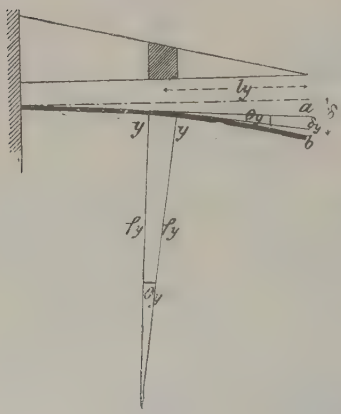


FIG. 498.

and from p. 424, we have—

$$\rho_y = \frac{EI}{M_y}$$

where M_y is the mean bending moment between the points y, y.

Then by substitution, we have—

$$\theta_y = \frac{M_y \cdot \overline{yy}}{EI}$$

where θ_y is the "slope" between the two tangents to the bent beam at y y;

But $M_y\overline{yy}$ = area (shown shaded) of the bending-moment diagram between y, y

$$= A_y$$

$$\text{hence } \theta_y = \frac{A_y}{EI}$$

$$\text{and } \delta_y = \frac{A_y l_y}{EI}$$

That is, the deflection at the free end of the cantilever due to the bending between the points y, y is numerically equal to the moment of the portion of the bending-moment diagram over yy about the free end of the cantilever divided by EI. The total deflection at the free end is—

$$\delta = \Sigma(\delta_y + \delta_x +, \text{ etc.})$$

$$\delta = \frac{1}{EI}\Sigma A_y l_y + A_x l_x +, \text{ etc.}$$

where the suffix x refers to any other very small portion of the cantilever xx.

Thus the total deflection at the free end of the cantilever is

numerically equal to the sum of the moments of each little element of area of the bending-moment diagram about the free end of the cantilever divided by EI. But, instead of dealing with the moment of each little element of area, we may take the moment of the whole bending-moment diagram about the free end, *i.e.* the area of the diagram × the distance of its centre of gravity from the free end;

$$\text{or } \delta = \frac{AL_c}{EI}$$

where A = the area of the bending-moment diagram;
L_c = the distance of the centre of gravity of the bending-moment diagram from the free end.

To readers familiar with the integral calculus, it will be seen that the length that we have termed *yy* above, is in calculus nomenclature dl in the limit, and the deflection at the free end due to the bending over the elementary length dl is—

$$\delta_y = \frac{M_y \cdot l_y \cdot dl}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

and the total deflection between points distant L and o from the free end is—

$$\delta = \frac{1}{EI} \int_{l=o}^{l=L} Ml \cdot dl \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

where M = the bending moment at the point distant l from the free end.

Another calculus method commonly used is as follows. The slope of the beam between P and Q (the distance PQ is supposed to be infinitely small) is denoted by $\frac{dy}{dx}$. This ratio is constant if the beam is straight, but in bent beams the slope varies from point to point, and the change of slope in a given length dx is—

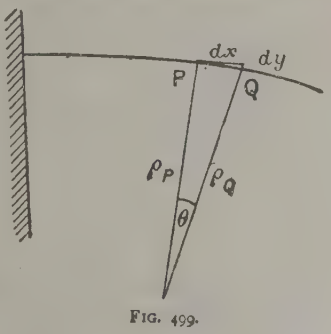


FIG. 499.

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2}.$$

When θ is very small dx becomes equal to the arc subtending the angle θ, and $\rho_P = \rho_Q = \rho$, then $\frac{dy}{dx} = \frac{PQ}{\rho}$, in the limit PQ = dx, and the change in slope in the length dx is

$$\frac{d\left(\frac{dx}{\rho}\right)}{dx} = \frac{1}{\rho} = \frac{M}{EI} = \frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

This expression will be utilized shortly for finding the deflection in certain cases of loaded beams.

CASE I.—*Cantilever with load* W *on free end. Length* L. *Method* (i).

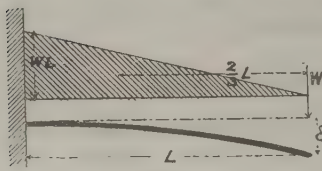


FIG 500.

$$A = WL \times \frac{L}{2} = \frac{WL^2}{2}$$

$$L_c = \tfrac{2}{3}L$$

$$\delta = \frac{WL^2}{2} \times \tfrac{2}{3}L \times \frac{1}{EI}$$

$$\delta = \frac{WL^3}{3EI}$$

Method (ii.).—By integration

$$M = Wl$$

$$\text{hence } \delta = \frac{W}{EI}\int_{l=0}^{l=L} l^2 \, . \, dl = \frac{WL^3}{3EI}$$

Method (iii.).—Consider a section of the beam distant x from the abutment.

The bending moment $M = W(L - x) = EI\frac{d^2y}{dx^2}$

$$L - x = \frac{EI}{W}\frac{d^2y}{dx^2}$$

Integrating $$Lx - \frac{x^2}{2} + C = \frac{EI}{W}\frac{dy}{dx}$$

where C is the integration constant. When $x = 0$ the slope $\frac{dy}{dx}$ is also zero, hence C = 0.

Integrating again—

$$\frac{Lx^2}{2} - \frac{x^3}{6} + K = \frac{EI}{W}y$$

When $x = 0$, the deflection y is zero, hence $K = 0$ and

$$y = \frac{W}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$$

which gives the deflection of the beam at any point distant x from the abutment. An exactly similar expression can be obtained by method (i.).

The deflection δ at the free end of the beam where $x = L$ is—

$$\delta = \frac{WL^3}{3EI}$$

In this particular case the result could be obtained much more readily by methods (i.) and (ii.).

CASE II.—*Cantilever with load* W *evenly distributed or* w *per unit length. Length* L.

Method (i.)—

$$A = \tfrac{1}{3}\left(\frac{wL^2}{2} \times L\right)$$

$$L_c = \tfrac{3}{4}L$$

$$\delta = \tfrac{1}{3}\left(\frac{wL^2}{2} \times L\right) \times \tfrac{3}{4}L \times \frac{1}{EI}$$

$$\delta = \frac{wL^4}{8EI} = \frac{WL^3}{8EI}$$

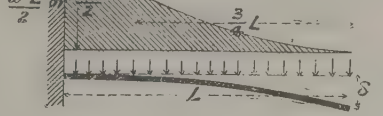


FIG. 501.

or by integration—

Method (ii.) $M = \frac{wl^2}{2}$

$$\text{hence } \delta = \frac{w}{EI}\int_{l=0}^{l=L} \frac{l^3}{2}dl = \frac{wL^4}{8EI}$$

$$= \frac{WL^3}{8EI}$$

Method (iii.)—Consider a section distant x from the abutment.

$$\text{The bending moment } M = \frac{w(L-x)^2}{2} = EI\frac{d^2y}{dx^2}$$

$$L^2 + x^2 - 2Lx = \frac{2EI}{w}\frac{d^2y}{dx^2}$$

Integrating $$L^2x + \frac{x^3}{3} - \frac{2Lx^2}{2} + C = \frac{2EI}{w}\frac{dy}{dx}$$

For the reason given in Case I, $C = 0$

Integrating again $$\frac{L^2x^2}{2} + \frac{x^4}{12} - \frac{Lx^3}{3} + K = \frac{2EI}{w}y$$

as explained above $K = 0$,

$$y = \frac{w}{24EI}(6L^2x^2 + x^4 - 4Lx^3)$$

which gives the deflection of the beam at any point distant x from the abutment. This is an instance in which the value of y is found more readily by method (iii.) than by (i.) or (ii.). The deflection δ at the end of the beam where $x = L$ is—

$$\delta = \frac{wL^4}{8EI} = \frac{WL^3}{8EI}$$

Case III.—*Cantilever with load* W *not at the free end.*

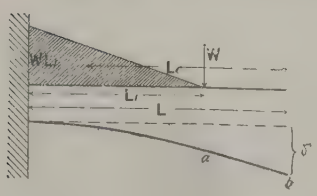


Fig. 502.

$$A = WL_1 \times \frac{L_1}{2} = \frac{WL_1^2}{2}$$

$$L_c = L - \frac{L_1}{3}$$

$$\delta = \frac{WL_1^2}{2EI}\left(L - \frac{L_1}{3}\right)$$

N.B.—The portion *ab* is straight.

CASE IV.—*Cantilever with two loads* W_1, W_2, *neither of them at the free end.*

$$\delta = \frac{W_1 L_1^2}{2EI}\left(L - \frac{L_1}{3}\right) + \frac{W_2 L_2^2}{2EI}\left(L - \frac{L_2}{3}\right)$$

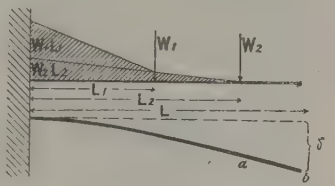


FIG. 503.

CASE V.—*Cantilever with load unevenly distributed. Length* L.

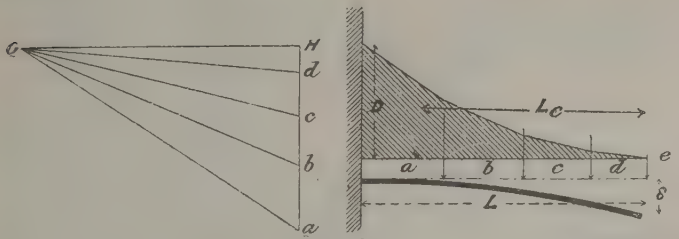


FIG. 504.

Let the bending-moment diagram shown above the cantilever be obtained by the method shown on p. 487.

Then if 1 inch = m lbs. on the load scale;
1 inch = n inches on the length scale;
D = depth of the bending-moment diagram measured in inches;
$\overline{OH}$ = the polar distance in inches;
M = the bending-moment in inch-lbs.;

$$M = m \,.\, n \,.\, D \,.\, \overline{OH};$$

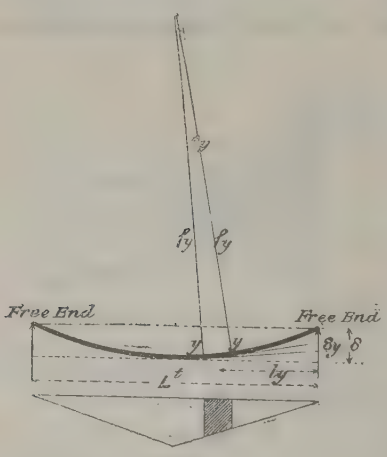
hence 1 inch depth on the bending-moment diagram represents $\frac{M}{D} = m \,.\, n \,.\, \overline{OH}$ inch-lbs.; and 1 square inch of the bending-moment diagram represents $m \,.\, n^2 \,.\, \overline{OH}$ inch-inch-lbs.; hence—

$$\delta = \frac{\left(\begin{matrix}\text{area of bending-moment} \\ \text{diagram in square inches}\end{matrix}\right) m \,.\, n^2 \,.\, \overline{OH} \times L_c}{EI}$$

The deflection δ found thus will be somewhere between the deflection for a single end load and for an evenly distributed load; generally by inspection it can be seen whether it will approach the one or the other condition. Such a calculation is useful in preventing great errors from creeping in.

In irregularly loaded beams and cantilevers, the deflection cannot conveniently be arrived at by an integration.

Deflection of a Beam freely supported.—Let the lower diagram represent the distribution of bending moment on the beam. The dark line represents the bent beam, and the straight dotted line the unstrained position of the beam. By the same process of reasoning as in the case of the cantilever, it is readily shown that the deflection of the free end or the support is the sum of the moments of each little area of the bending-moment diagram between the tangent point and the free end about the free end; or, as before, instead of dealing with the moment of each little area, we may take the moment of the whole area of the bending-moment diagram between the free end and the tangent point, about the free end, *i.e.* the area of the bending-moment diagram between the tangent point and the free end × distance of the centre of gravity of this area from the free end. Then, as before—



$$\delta = \frac{AL_c}{EI}$$

where A and L_c have the slightly modified meanings mentioned above.

CASE VI.—*Beam loaded with central load* W. *Length* L.

$$A = \frac{WL}{4} \times \frac{L}{4}$$

$$L_c = \frac{L}{3}$$

$$\delta = \frac{WL}{4} \times \frac{L}{4} \times \frac{L}{3} \times \frac{1}{EI}$$

$$\delta = \frac{WL^3}{48EI}$$

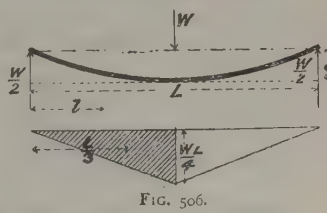


FIG. 506.

Or by integration, at any point distant l from the support—

$$M = \frac{W}{2} l$$

$$\delta = \frac{W}{EI} \int_0^l \frac{l^2 \cdot dl}{2} = \frac{W}{EI}\left(\frac{l^3}{6}\right)$$

When $l = \frac{L}{2}$, we have—

$$\delta = \frac{WL^3}{2^3 \times 6EI} = \frac{WL^3}{48EI}$$

CASE VII.—*Beam loaded with an evenly distributed load* w *per unit length. Length* L.

$$A = \frac{2}{3}\left(\frac{wL^2}{8} \times \frac{L}{2}\right) = \frac{wL^3}{24}$$

$$L_c = \frac{5L}{16}$$

$$\delta = \frac{wL^3}{24} \times \frac{5L}{16} \times \frac{1}{EI}$$

$$\delta = \frac{5wL^4}{384EI} = \frac{5WL^3}{384EI}$$

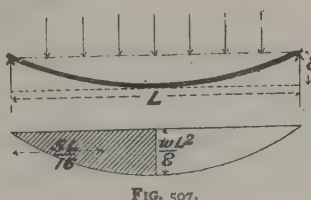


FIG. 507.

Or by integration, at any point distant l from the support (see p. 512) the bending moment is—

$$M = \frac{w}{2}(lL - l^2) = \frac{wL^2}{8} - \frac{wx^2}{2}$$

where x is the distance measured from the middle of the beam and y the vertical height of the point above the tangent at the middle of the beam. Then by method (iii.)—

$$EI\frac{d^2y}{dx^2} = \frac{wL^2}{8} - \frac{wx^2}{2}$$

$$EI\frac{dy}{dx} = \frac{wxL^2}{8} - \frac{wx^3}{6} + C \qquad (C = 0)$$

$$EIy = \frac{wL^2x^2}{16} - \frac{wx^4}{24} + K \qquad (K = 0)$$

$$y = \frac{w}{384EI}(24L^2x^2 - 16x^4)$$

$$\delta = \frac{5wL^4}{384EI} = \frac{5WL^3}{384EI} \quad \text{when } x = \frac{L}{2}$$

CASE VIII.—*Beam loaded with two equal weights symmetrically placed.*—By taking the moments of the triangular area *abc* and the rectangle *bced*, the deflection becomes—

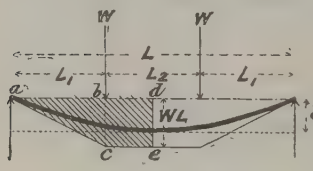


FIG. 508.

$$\delta = \frac{WL_1}{24EI}(12L_1L_2 + 3L_2^2 + 8L_1^2)$$

and when $L_1 = L_2 = \frac{L}{3}$, this expression becomes—

$$\delta = \frac{23WL^3}{648EI} = \frac{WL^3}{28EI} \text{ (nearly)}$$

or if W_0 be the *total* load—

$$\delta = \frac{W_0L^3}{56EI} \text{ (nearly)}$$

CASE IX.—*Beam loaded with one eccentric concentrated load.*

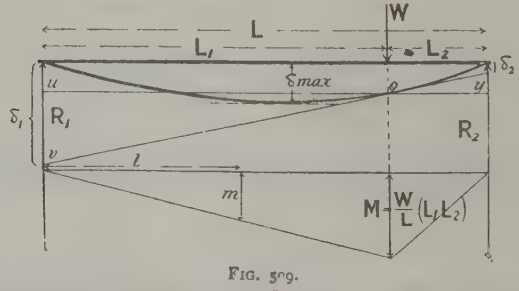


FIG. 509.

It should be noted that the point of maximum deflection does not coincide with that of the maximum bending moment.

We have shown that the point of maximum bending moment in a beam is the point where the shear changes sign, and we

shall also show that the bent form of a beam is obtained by constructing a second bending-moment diagram obtained by taking the original bending-moment diagram as a load diagram.

Hence, if we construct a second shear diagram, still treating the original bending-moment diagram as a load diagram, we shall find the point at which the second shear changes sign, or where the second bending moment, *i.e.* the deflection, is a maximum. This is how we propose to find the point of maximum deflection in the present instance.

Referring to p. 483, we have—

$$\left.\begin{matrix}\text{The shear at a section}\\ \text{distant } l \text{ from } R_1\end{matrix}\right\} = R_1 - \frac{Ml^2}{2L_1}$$

$$R_1 = \frac{ML_1}{2L}\left(L_2 + \frac{L_1}{3}\right) + \frac{ML_2^2}{3L}$$

The shear changes sign and the deflection is a maximum when—

$$\frac{ML_1}{2L}\left(L_2 + \frac{L_1}{3}\right) + \frac{ML_2^2}{3L} = \frac{Ml^2}{2L_1}$$

$$\text{or when } l = \sqrt{\frac{L_1^2}{L}\left(L_2 + \frac{L_1}{3}\right) + \frac{2L_1L_2^2}{3L}} = L\sqrt{\frac{2n - n^2}{3}}$$

where $L_1 = nL$ and $L_2 = L - L_1$, *i.e.* the shorter of the two segments; and the deflection at this point is—

$$\delta_{max.} = \frac{ml^2}{3EI} = \frac{Wl^3L_2}{3EIL} = \frac{WL^3}{3EI}(1 - n)\left(\frac{2n - n^2}{3}\right)^{\frac{3}{2}}$$

The deflection δ under the load itself can be found thus—

Let the tangent to the beam at this point o be vx; then we have—

$$\delta_1 = \frac{R_1L_1^3}{3EI}, \text{ and } \delta_2 = \frac{R_2L_2^3}{3EI}$$

But these are the deflections measured from the tangent vx.

Let S = slope of the tangent vx; then $uv = SL_1$, and $xy = SL_2$, and the actual deflection under the load, or the vertical distance of o from the original position of the beam, is—

$$\delta = \delta_1 - uv, \text{ or } \delta = \delta_2 + xy$$

$$\text{whence } \frac{R_1L_1^3}{3EI} - SL_1 = \frac{R_2L_2^3}{3EI} + SL_2$$

$$S = \frac{1}{3EI}\left(\frac{R_1L_1^3 - R_2L_2^3}{L_1 + L_2}\right)$$

$$\text{and } \delta = \frac{R_2L_2^3}{3EI} + \frac{L_2}{3EI}\left(\frac{R_1L_1^3 - R_2L_2^3}{L_1 + L_2}\right)$$

which reduces to—

$$\delta = \frac{WL_1^2L_2^2}{3EI(L_1 + L_2)} = \frac{WL_1^2L_2^2}{3EIL}$$

CASE X.—*Beam hinged at one end, free at the other, propped in the middle. Load at the free end.*

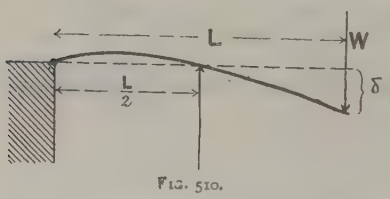


FIG. 510.

The load on the hinge is also equal to W, since the prop is central, and the load on the prop is 2W; hence we may treat it as a beam supported at each end and centrally loaded with a load 2W. Hence the central deflection would be $\frac{2WL^3}{48EI}$ if the two ends were kept level; but the deflection at the free end is twice this amount; or—

$$\delta = \frac{4WL^3}{48EI} = \frac{WL^3}{12EI}$$

CASE XI.—*General case of a beam whose section varies from point to point.* (See also p. 270.)

(i.) Let the depth of the section be constant, and let the breadth of the section vary directly as the bending moment; then the stress will be constant. We have—

$$M = \frac{fI}{y} = \frac{EI}{\rho}, \text{ or } \frac{f}{Ey} = \frac{1}{\rho}$$

But, since f, y, and E are constant in any given case, ρ is also constant, whence the beam bends to the arc of a circle.

(ii.) Let both the depth of the section and the stress vary; then $\frac{f}{y} = \frac{E}{\rho}$ if y varies directly as f, $\frac{f}{y}$ will be constant, and the beam will again bend to the arc of a circle.

From p. 507, we have—

$$\delta = \frac{ML^2}{8EI} = \frac{WL^3}{32EI}$$

$$\delta = \frac{3WL^3}{8EBt^3}$$

Let the plate be cut up into strips, and bring the two long edges of each together, making a plate with pointed ends of the same form as plate 1 on the plan; pack all the strips as shown into a symmetrical heap. Looked at sideways, we see a plate railway spring.

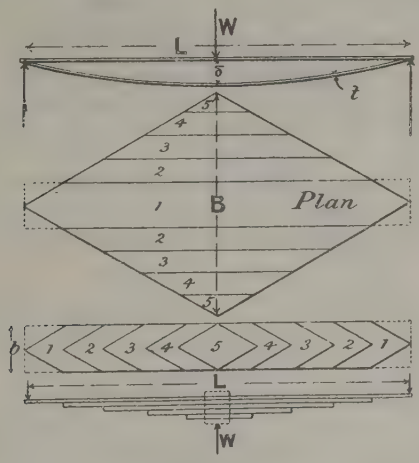


FIG. 511

Let there be n plates, in this case 5, each of breadth b; then $B = nb$. Substitute in the expression above—

$$\delta = \frac{3WL^3}{8Enbt^3}$$

If a railway-plate spring be tested for deflection, it may not, probably will not, quite agree with the calculated value on account of the friction between the plates or leaves. The result of a test is shown in Fig. 512. When a spring is very rusty it deflects less, and when unloading more than the formula gives, but when clean and well oiled it much more closely agrees with

the formula, as shown by the dotted lines. If friction could be entirely eliminated, probably experiment and theory would agree.

In calculating the deflection of such springs, E should be taken at about 26,000,000, which is rather below the value for the steel plates themselves. Probably the deflection due to shear is partly responsible for the low modulus of elasticity, and

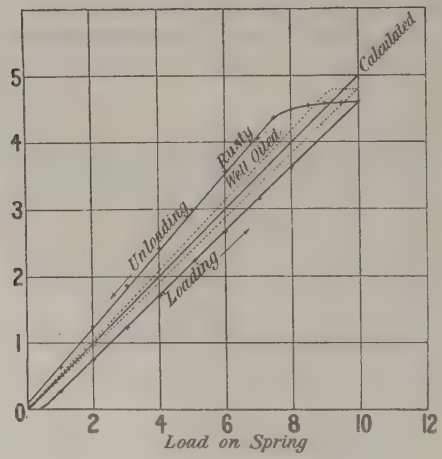


FIG. 512.

from the fact that the small central plate (No. 5) is always omitted in springs.

CASE XII. *Beam unevenly loaded.*—Let the beam be loaded as shown. Construct the bending-moment diagram shown below the beam by the method given on p. 498. Then the bending moment at any section is $M = m \cdot n \cdot D \cdot \overline{OH}$ inch-lbs., using the same notation as on p. 499. Then 1 inch on the vertical scale of the bending-moment diagram $= \frac{M}{D} = m \cdot n \cdot \overline{OH}$ inch-lbs., and 1 inch on the horizontal scale $= n$ inches. Hence one square inch on the diagram $= m \cdot n^2 \overline{OH}$ inch-lbs. Then $A = a \cdot m \cdot n^2 \overline{OH}$, where a = the shaded area measured in square inches.

The centre of gravity must be found by one of the methods

described in Chap. III. Then $L_e = n \cdot l_e$, where l_e is measured in inches, and the deflection—

$$\delta = \frac{AL_e}{EI} = \frac{a \cdot m \cdot n^3 \overline{OH} \cdot l_e}{EI}$$

It is evident that the height of the supports above the tangent is the same at both ends. Hence the moment of the areas about the supports on either side of the tangent point must be the same. The point of maximum deflection must be found in this way by a series of trials and errors, which is very clumsy.

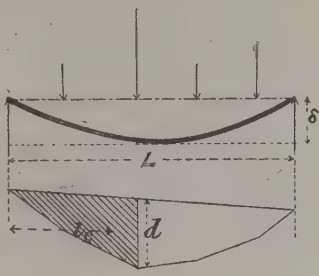


FIG. 513.

The deflection may be more conveniently found by a somewhat different process, as in Fig. 514.

We showed above that the deflection is numerically equal to the moment of each little element of area of the bending-moment diagram about the free end ÷ EI. The moment of

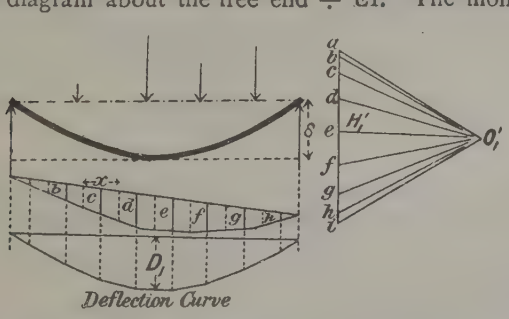


FIG. 514.

each portion of the bending-moment diagram may be found readily by a link-and-vector polygon, similar to that employed for the bending-moment diagram itself.

Treat the bending-moment diagram as a load diagram; split it up into narrow strips of width x, as shown by the dotted lines; draw the middle ordinate of each, as shown in full lines: then any given ordinate × by x is the area of the strip. Set down these ordinates on a vertical line as shown; choose a

pole O', and complete both polygons as in previous examples. The link polygon thus constructed gives the *form* of the bent beam; this is then reproduced to a horizontal base-line, and gives the bent beam shown in dark lines above. The only point remaining to be determined is the scale of the deflection curve.

We have 1 inch on the load scale of the first bending-moment diagram $\Big\}$ = m lbs.

also 1 inch on the length of the bending-moment diagram $\Big\}$ = n inches

and the bending moment at any point $M = m \,.\, n \,.\, D \,.\, \overline{OH}$

where D is the depth of the bending-moment diagram at the point in inches, and $\overline{OH}$ is the polar distance, also expressed in inches. Hence 1 inch depth of the bending-moment diagram represents $\frac{M}{D} = mn \,.\, \overline{OH}$ inch-lbs., and 1 square inch of the bending-moment diagram represents $mn^2\overline{OH}$ inch-inch-lbs. Hence the area xD represents $x\text{D}mn^2\overline{OH}$ inch-inch-lbs.; but as this area is represented on the second vector polygon by D, its scale is $xmn^2\overline{OH}$; hence—

$$\delta = \frac{xmn^2\overline{OH}n\text{D}_1\overline{O_1H_1}}{EI}$$

$$= \frac{xmn^3\text{D}_1\overline{OH}\overline{O_1H_1}}{EI}$$

If it be found convenient to reduce the vertical ordinates of the bending-moment diagram when constructing the deflection vector polygon by say $\frac{1}{r}$, then the above expression must be multiplied by r.

The following table of deflection constants k will be found very useful for calculating the deflection at any section, if the load W is expressed in tons, E must be expressed in tons per square inch. The length L and the moment of inertia I are both to be expressed in inch units.

Example.—A beam 20 feet long supports a load of 3 tons at a point 4 feet from one support: find the deflection at 12 feet from the same support. I = 138 inch^4-units. E = 12,000 tons per square inch.

The position of the load is $\frac{4}{20} = 0{\cdot}2$.

TABLE FOR CALCULATING THE DEFLECTION OF BEAMS FREELY SUPPORTED AT EACH END AND OF CONSTANT CROSS SECTION.

$$\text{DEFLECTION} = \frac{kWL^3}{EI}$$

Position of load	Position at which the deflection is measured.									Position of load
	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	
	Values of k.									
0·1	0·0026	0·0047	0·0060	0·0064	0·0061	0·0055	0·0045	0·0032	0·0017	0·1
0·2	0·0047	0·0085	0·0110	0·0120	0·0118	0·0107	0·0087	0·0063	0·0032	0·2
0·3	0·0060	0·0110	0·0147	0·0165	0·0165	0·0150	0·0124	0·0087	0·0045	0·3
0·4	0·0064	0·0120	0·0165	0·0192	0·0196	0·0180	0·0150	0·0107	0·0055	0·4
0·5	0·0061	0·0118	0·0165	0·0196	0·0208	0·0196	0·0165	0·0118	0·0061	0·5
0·6	0·0055	0·0107	0·0150	0·0180	0·0196	0·0192	0·0165	0·0120	0·0064	0·6
0·7	0·0045	0·0087	0·0124	0·0150	0·0165	0·0165	0·0147	0·0110	0·0060	0·7
0·8	0·0032	0·0063	0·0087	0·0107	0·0118	0·0120	0·0110	0·0085	0·0047	0·8
0·9	0·0017	0·0032	0·0045	0·0055	0·0064	0·0064	0·0060	0·0047	0·0026	0·9
	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	

The position at which the deflection is measured $= \frac{12}{20} = 0{\cdot}6$. Referring to the table, $k = 0{\cdot}0107$.

$$\text{Then } \delta = \frac{0{\cdot}0107 \times 3 \times 240^3}{12{,}000 \times 138} = 0{\cdot}24 \text{ inch.}$$

When a beam supports a number of loads, the deflection due to each must be calculated and the results added. When the loads are not on the even spaces given in the table, the constant can be obtained approximately by interpolation or by plotting.

A very convenient diagram for calculating the deflection of beams has been constructed by Mr. Livingstone; it is published by "The Electrician" Printing and Publishing Co., Fleet Street, London.

Another type of diagram for the same purpose was published in *Engineering*, January 13, 1913, p. 143.

Example.—A beam 20 feet long, freely supported at each end, was loaded as follows:—

Load in tons (W).	Distance from end of beam.	Position of load.	K	KW
3	3′ 6″	0·175	0·0095	0·0285
5	7′ 6″	0·375	0·0177	0·0885
2	11′ 8″	0·59	0·0194	0·0388
4	15′ 2″	0·76	0·0139	0·0556
				0·2114

Find the deflection under the 2 tons load. E = 12,000 tons per sq. inch. I = 630.

$$\delta = \frac{0{\cdot}2114 \times 240^3}{12{\cdot}000 \times 630} = 0{\cdot}39 \text{ inch.}$$

By a graphical process $\delta = 0{\cdot}40$ inch.

Deflection of Built-in Beams.—When a beam is built in at one end only, it bends down with a convex curvature

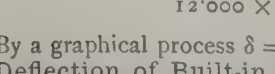


FIG. 515.

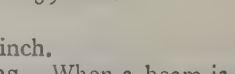


FIG. 516.

upwards (Fig. 515); but when it is supported at both ends, it bends with a convex curvature *downwards* (Fig. 516);

and when a beam is built in at both ends (Fig. 517), we get a combined curvature, thus—

FIG. 517.

FIG. 518.

FIG 519.

Then considering the one kind of curvature as positive and the other kind as negative, the curvature will be zero at the points xx (Fig. 518), at which it changes sign; such are termed "points of contrary flexure." As the beam undergoes no bending at these points, the bending moment is zero. Thus the beam may be regarded as a short central beam with free ends resting on short cantilevers, as shown in Fig. 519.

Hence, in order to determine the strength and deflection of built-in beams, we must calculate first the positions of the points x, x. It is evident that they occur at the points at which the upward slope of the beam is equal to the downward slope of the cantilever.

We showed above that the slope of a beam or cantilever at any point is given by the expression—

$$\text{Slope} = \frac{A}{EI}$$

CASE XIII. *Beam built in at both ends, with central load.*

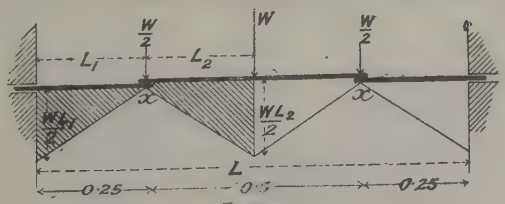


FIG. 520.

$$\text{A for cantilever} = \frac{WL_1}{2} \times \frac{L_1}{2} = \frac{WL_1^2}{4}$$

$$\text{A for beam} = \frac{WL_2}{2} \times \frac{L_2}{2} = \frac{WL_2^2}{4}$$

Hence, as the slope is the same at the point where the beam joins the cantilever, we have—

$$\frac{WL_1^2}{4} = \frac{WL_2^2}{4}, \text{ or } L_1 = L_2 = \frac{L}{4}$$

Maximum bending moment in middle of central span—

$$\frac{WL_2}{2} = \frac{WL}{8}$$

Maximum bending moment on cantilever spans—

$$\frac{WL_1}{2} = \frac{WL}{8}$$

Deflection of central span—

$$\delta_2 = \frac{W(2L_2)^3}{48EI} = \frac{W\left(\frac{L}{2}\right)^3}{48EI} = \frac{WL^3}{384EI}$$

Deflection of cantilever—

$$\delta_2 = \frac{\frac{W}{2}L_1^3}{3EI} = \frac{\frac{W}{2}\left(\frac{L}{4}\right)^3}{3EI} = \frac{WL^3}{384EI}$$

Total deflection in middle of central span—

$$\delta = \delta_2 + \delta = \frac{WL^3}{192EI}$$

This problem may be treated by another method, which, in some instances, is simpler to apply than the one just given.

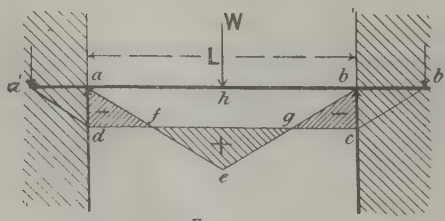


FIG. 521.

When a beam is built in at both ends, the ends are necessarily level, or their slope is zero; hence the summation of the slope taken over the whole beam is zero, if downward slopes be

given the opposite sign to that of upward slopes. Since the slope between any two sections of a beam is proportional to the area of the bending-moment diagram between those sections, the net area of the bending-moment diagram for a built-in beam must also be zero.

A built-in beam may be regarded as a free-ended beam having overhanging ends, $d'a$, $b'b$, which are loaded in such a manner that the negative or pier moments are just sufficient to bring those portions of the beam which are over the supports to a level position. Then, since the net area must be zero, we have the areas—

$$feg - adf - gcb = 0$$

But in order that this condition may be satisfied, the area of the pier-moment diagram *adcb* must be equal to the area of the bending-moment diagram *aeb* for a freely supported beam, or—

$$ad = \frac{he}{2}$$

Whence the bending moment at the middle and ends is $\frac{wL}{8}$, and the distance between the points of contrary flexure $fg = \frac{L}{2}$; all the other quantities are the same as those found by the previous method.

It will be seen that *dc* is simply the mean height-line of the bending-moment diagram for the free-ended beam.

Thus when the ends are built in, the maximum bending moment is reduced to one-half, and the deflection to one-quarter, of what it would have been with free ends.

Case XIV.—*Beam built in at both ends, with a uniformly distributed load.*

A for cantilever—

$$\left(\frac{wL_2L_1}{2} + \frac{wL_1^2}{6}\right)L_1$$

A for beam—

$$\frac{wL_2^2}{2} \times \tfrac{2}{3}L_2 = \frac{wL_2^3}{3}$$

These must be equal, as explained above—

$$\frac{wL_2^3}{3} = \frac{wL_1^2L_2}{2} + \frac{wL_2^3}{6}$$

Let $L_1 = nL_2$.

$$\text{Then } \frac{L_2^3}{3} = \frac{n^2L_2^3}{2} + \frac{n^3L_2^3}{6}$$

$$2 = 3n^2 + n^3$$

which on solving gives us $n = 0{\cdot}732$.

We also have—

$$L_1 + L_2 = \frac{L}{2}$$

$$\text{or } 1{\cdot}732L_2 = \frac{L}{2}$$

$$L_2 = 0{\cdot}289L$$

$$\text{and } L_1 = 0{\cdot}732 \times 0{\cdot}289L = 0{\cdot}211L$$

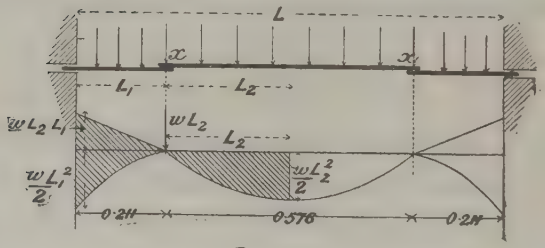


FIG. 522.

Maximum bending moment in middle of central span—

$$\frac{wL_2^2}{2} = \frac{w \times 0{\cdot}289^2L^2}{2} = \frac{wL^2}{24}$$

Maximum bending moment on cantilever spans—

$$wL_2L_1 + \frac{wL_1^2}{2} = w \times 0{\cdot}289L \times 0{\cdot}211L + \frac{w \times 0{\cdot}211^2L^2}{2}$$

$$= \frac{wL^2}{12}$$

Deflection of central span—

$$\delta_2 = \frac{5w(0{\cdot}578L)^4}{384EI} = \frac{wL^4}{689EI}$$

Deflection of cantilevers due to distributed load—

$$\delta_1 = \frac{w(0{\cdot}211L)^4}{8EI} = \frac{wL^4}{4038EI}$$

Deflection due to half-load on central part—

$$\delta_0 = \frac{wL_2 \times L_1^3}{3EI} = \frac{w \times 0{\cdot}289L \times 0{\cdot}211^3L^3}{3EI}$$
$$= \frac{wL^4}{1105EI}$$

Total deflection in middle of central span—

$$\delta = \delta_2 + \delta_1 + \delta_0 = \frac{wL^4}{384EI}$$

This problem may also be treated in a similar manner to the last case. The area of the parabolic bending-moment diagram $axbxc = \frac{2}{3}\overline{bf} \cdot \overline{ac}$, and the mean height $ae = \frac{2}{3}\overline{bf}$; whence ae, the bending moment at the ends, is—

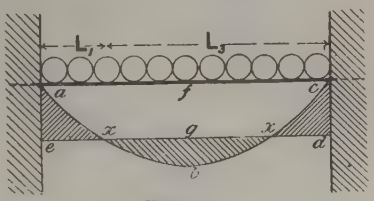


FIG. 523.

$$\frac{2}{3}\frac{wL^2}{8} = \frac{wL^2}{12}$$

and bg, the bending moment in the middle, is—

$$\frac{1}{3}\frac{wL^2}{8} = \frac{wL^2}{24}$$

and for the distance xx, we have—

$$\frac{w}{2}(L_1L_2) = \frac{wL^2}{12}$$
$$L_1(L - L_1) = \frac{L^2}{6}$$
$$L_1 = 0{\cdot}211L.$$

These calculations will be sufficient to show that identical results are obtained by both methods. Thus, when the ends are built in and free to slide sideways, the maximum bending moment on a uniformly loaded beam is reduced to $\frac{8}{12} = \frac{2}{3}$, and the deflection to $\frac{1}{5}$ of what it would have been with free ends.

CASE XV.—*Beam built in at both ends, with an irregularly distributed load.*

Since the ends of the beam are guided horizontally the slope of the ends is zero, hence the net area of the bending

moment diagram is also zero. The area of the bending moment diagram A for a freely supported beam is therefore equal to that of the pier moment diagram.

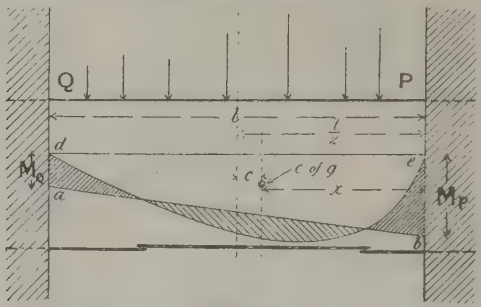


FIG. 524.

$$A = \frac{M_Q + M_P}{2} \times l$$

$$\text{and } M_Q = \frac{2A}{l} - M_P \qquad M_P = \frac{2A}{l} - M_Q$$

$$x = \frac{l}{3}\left(\frac{M_P + 2M_Q}{M_Q + M_P}\right) \quad \text{(See p. 60.)}$$

Substituting the value of M_Q and reducing—

$$M_P = \frac{2A}{l^2}(2l - 3x)$$

$$\text{also } M_Q = \frac{2A}{l^2}(3x - l)$$

where x = The distance of the centre of gravity of the bending moment diagram for a freely supported beam from the nearest abutment (the centre of gravity of the pier moment diagram is at the same distance from the abutment).

c = The distance of the centre of gravity of the bending moment diagram from the middle of the beam.

A = The area of the bending moment diagram for a freely supported beam.

If the beam be regarded as a cantilever fixed at one end, say Q, and free at the other. The moment of the external system of loading between P and Q causes it to bend downwards, but the pier moment causes it to bend upwards, and

since the deflection at P is zero under the two systems of loading it is evident that the moment of the bending-moment diagram due to the external loads between P and Q is equal to the moment of the pier bending-moment diagram, and since the areas of the two diagrams are equal the distance of the centre of gravity of each is at the same distance from Q.

When the load is symmetrically disposed $c = o$, and the bending moment at the ends of the built-in beam is simply the mean bending moment for a freely supported beam, under the same system of loading. And the maximum bending moment in the middle of the built-in beam is the maximum bending moment for the freely supported beam *minus* the mean bending moment. The reader should test the accuracy of this statement for the cases already given.

Beams supported at more than Two Points.—When a beam rests on three or more supports, it is termed a continuous beam. We shall only treat a few of the simplest cases in order to show the principle involved.

Case XVI. *Beam resting on three supports, load evenly distributed.*—The proportion of the load carried by each support entirely depends upon their relative heights. If the central support or prop be so low that it only just touches the beam, the end supports will take the whole of the load. Likewise, if it be so high that the ends of the beam only just touch the end supports, the central support will take the whole of the load.

The deflection of an elastic beam is strictly proportional to the load. Hence from the deflection we can readily find the load.

$$\left.\begin{matrix}\text{The deflection in the middle}\\ \text{when not propped}\end{matrix}\right\} = \delta = \frac{5WL^3}{384EI}$$

Let W_1 be the load on the central prop.

$$\text{Then the } \textit{upward} \text{ deflection due to } W_1 = \delta_1 = \frac{W_1L^3}{48EI}$$

If the top of the three supports be in one straight line, the upward deflection due to W_1 must be equal to the downward deflection due to W, the distributed load ; then we have—

$$\frac{5WL^3}{384EI} = \frac{W_1L^3}{48EI}$$

$$\text{whence } W_1 = \tfrac{5}{8}W$$

Thus the central support or prop takes $\frac{5}{8}$ of the whole load; and as the load is evenly distributed, each of the end supports takes one-half of the remainder, viz. $\frac{3}{16}$ of the load.

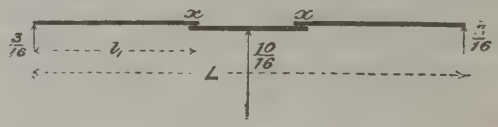


Fig. 525.

The bending moment at any point x distant l_1 from the end support is—

$$M_x = \tfrac{3}{16}wLl_1 - wl_1 \times \frac{l_1}{2}$$

$$= wl_1\left(\tfrac{3}{16}L - \frac{l_1}{2}\right) = \frac{wl_1}{16}(3L - 8l_1)$$

The points of contrary flexure occur at the points where the bending moment is zero, *i.e.* when—

$$\frac{wl_1}{16}(3L - 8l_1) = 0 \qquad \text{or when } 3L = 8l_1 \qquad \text{or } l_1 = \tfrac{3}{8}L$$

Thus the length of the middle span is $\frac{L}{4}$. It is readily shown, by the methods used in previous paragraphs, that the maximum bending moment occurs over the middle prop, and is there $\frac{wL^2}{32}$, or $\frac{1}{4}$ as great as when not propped.

When the three supports are not level. Let the load on the prop be—

$$nwL = nW$$

Then the upward deflection due to the prop is—

$$\frac{nWL^3}{48EI}$$

and the difference of level between the central support and the end supports is—

$$\frac{5}{384}\frac{WL^3}{EI} - \frac{nWL^3}{48EI} = \frac{WL^3}{384EI}(5 - 8n)$$

When the result is negative it indicates that the central support is higher than the end supports; if $n = 1$ the whole load is taken by the prop, and its height is $\frac{WL^3}{128EI}$ above the end supports.

When the load is evenly distributed over the three supports $n = \frac{1}{3}$ the prop is then below the end supports by $\frac{7WL^3}{1152EI} = \frac{WL^3}{165EI}$ nearly.

Where there are two props symmetrically placed at a distance x from the middle of the beam, the downward deflection at these points when freely supported at each end is (see page 516)—

$$\delta - y = \frac{5wL^4}{384EI} - \frac{w}{384EI}(24L^2x^2 - 16x^4)$$

If the spans are equal $x = \frac{L}{6}$ and

$$\delta - y = \frac{4{\cdot}346wL^4}{384EI} = \frac{WL^3}{88{\cdot}4EI}$$

The upward deflection due to the two props is—

$$\frac{0{\cdot}0309PL^3}{EI}$$

where P is the load on each prop; the constant is taken from the table on page 523.

When all the supports are level—

$$\frac{WL^3}{88{\cdot}4EI} = \frac{0{\cdot}0309PL^3}{EI}$$

$$P = \frac{W}{2{\cdot}73} = 0{\cdot}37W$$

And the load on each end support is $0{\cdot}13W$.

Case XVII. *Beam with the load unevenly distributed, with an uncentral prop.*—Construct the bending moment and deflection curves for the beam when supported at the ends only (Fig. 526).

Then, retaining the same scales, construct similar curves for the beam when supported by the prop only (Fig. 527). If, due to the uneven distribution of the load, the beam does not balance on its prop, we must find what force must be applied

at one end of the beam in order to balance it. The unbalanced moment is shown by xy (Fig. 527). In order to find the force required at z to balance this, join xz and yz, and from the pole of the vector polygon draw lines parallel to them; then the intercept $x_1y_1 = W_1$ on the vertical load line gives the required force *acting upwards* (in this case).

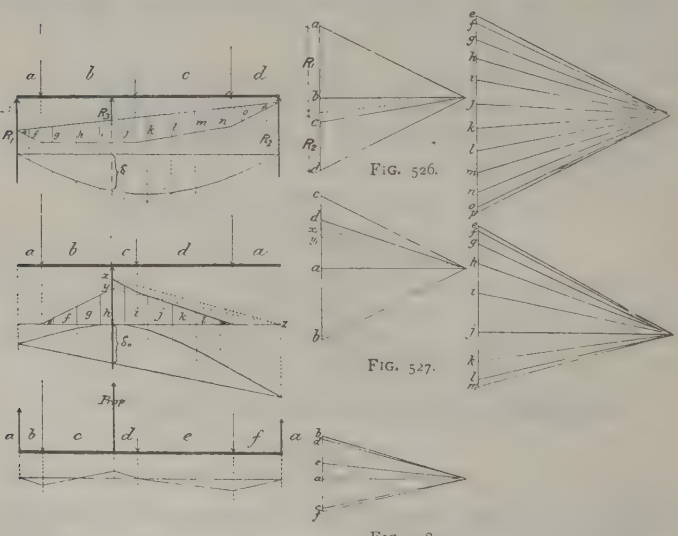


FIG. 526.

FIG. 527.

FIG. 528.

FIG. 529.

In Fig. 528 set off δ and δ_0 on a vertical. If too small to be conveniently dealt with, increase by the method shown to *jf*, *ej*, and construct the rectangle *efgh*. If the prop be lowered so that the beam only just touches it, the whole load will come on the end supports; the proportion on each is obtained from R_1 and R_2 in Fig. 526. Divide *gh* in *i* in this proportion.

As the prop is pushed up, the two ends keep on the end

supports until the deflection becomes $\delta + \delta_0$; at that instant the reaction R_1 becomes zero just as the beam end is about to lift off the support, but the other reaction R_2 supports the unbalanced force W_1. This is shown in the diagram by $ee_1 = W_1$ to same scale as R_1 and F_2.

Join ie and ge_1; then, if the three supports be level, the prop will be at the height j. Draw a horizontal from j to meet ge_1 in g_0; erect a perpendicular. Then the proportion of the load taken by the prop is $\frac{f_0g_0}{f_0h_0}$, by the support R_1 is $\frac{i_0h_0}{f_0h_0}$, by the support R_2 is $\frac{g_0i_0}{f_0h_0}$.

Likewise, if the prop be raised to a height corresponding to f_1, the proportions will be as above, with the altered suffixes to the letters.

In Fig. 528, we have the final bending-moment diagram for the propped beam when all the supports are level; comparing it with Fig. 526, it will be seen how greatly a prop assists in reducing the bending moment.

It should be noted that in the above constructions there is no need to trouble about the scale of the deflections when the supports are level, but it is necessary when the prop is raised or lowered above or below the end supports.

This method, which the author believes to be new, is equally applicable to continuous beams of any number of spans, but space will not allow of any further cases being given.

Stiffness of Beams.—The ratio $\frac{\text{deflection}}{\text{span}}$ is termed the "stiffness" of a beam. This ratio varies from about $\frac{1}{2000}$ in the best English practice for bridge work; it is often as great as $\frac{1}{600}$ for small girders and rolled joists.

By comparing the formulas given above for the deflection, it will be seen that it may be expressed thus—

$$\delta = \frac{ML^2}{nEI}$$

where M is the bending moment and n is a constant depending on the method of loading.

In the above equation we may substitute fZ for M and Zy for I; then—

$$\delta = \frac{fZL^2}{nEZy} = \frac{fL^2}{nEy}$$

Hence for a stiffness of $\frac{1}{2000}$, we have—

$$\frac{\delta}{L} = \frac{1}{2000} = \frac{fL}{nEy}$$

$$\text{or } 2000\,fL = nEy$$

Let $f = 15{,}000$ lbs. square inch;
$E = 30{,}000{,}000$ „ „

$$\text{Then } ny = L$$

$$\text{But } y = \frac{d}{2}$$

where d = depth of section (for symmetrical sections); then—

$$nd = 2L$$

$$\text{and } \frac{d}{L} = \frac{2}{n}$$

VALUES OF n.

	Beam.	Cantilever.
(*a*) Central load	12	—
End load	—	3
(*b*) Evenly distributed load	9·6	4
(*c*) Two equal symmetrically placed loads dividing beam into three equal parts	9·3	—
(*d*) Irregular loading (approx.)	11	3·5

VALUES OF $\frac{L}{d}$

	Stiffness. $\frac{1}{2000}$	$\frac{1}{1000}$	$\frac{1}{500}$
Beam, central load	6	12	24
Cantilever, end load	1·5	3	6
Beam, evenly distributed load	4·8	9·6	19·2
Cantilever, „ „	2	4	8
Beam, two symmetrically placed loads, as in Fig. 423	4·65	9·3	18·6
Beam, irregular loading (approx.)	5·5	11	22
Cantilever, „ „	1·75	3·5	7

This table shows the relation that must be observed between the span and the depth of the section for a given stiffness.

The stress can be found direct from the deflection of a given beam if the modulus of elasticity be known; as this does not vary much for any given material, a fairly accurate estimate of the stress can be made. We have above—

$$\delta = \frac{2fL^2}{nEd}$$

$$\text{hence } f = \frac{n E d \delta}{2 L^2}$$

The system of loading being known, the value of n can be found from the table above. The value of E must be assumed for the material in the beam. The depth of the section d can readily be measured, also δ and L.

The above method is extremely convenient for finding approximately the stress in any given beam. The error cannot well exceed 10 per cent., and usually will not amount to more than 5 per cent.

CHAPTER XIV.

COMBINED BENDING AND DIRECT STRESSES.

In the figure, let a weight W be supported by two bars, 1 and 2, whose sectional areas are respectively A_1 and A_2, and the corresponding loads on the bars R_1 and R_2; then, in order that the stress may be the same in each, W must be so placed that R_1 and R_2 are proportional to the sectional areas of the bars, or $\frac{R_1}{R_2} = \frac{A_1}{A_2}$. But $R_1 u = R_2 z$, or $A_1 u = A_2 z$; hence W passes through the centre of gravity of the two bars when the stress is equal on all parts of the section. This relation holds, however many bars may be taken, even if taken so close together as to form a solid section; hence, *in order to obtain a direct stress of uniform intensity all over a section, the external force must be so applied that it passes through the centre of gravity of the section.*

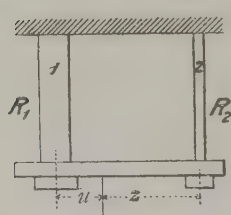


Fig. 530.

If W be not placed at the centre of gravity of the section, but at a distance x from it, we shall have—

$$R_2(u + z) = W(u + x)$$

and when W is at the centre of gravity—

$$R_2(u + z) = Wu$$

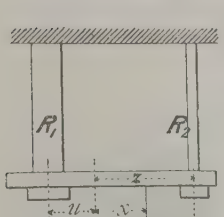


Fig 531.

Thus when W is not placed at the centre of gravity of the section, the section is subjected to a bending moment Wx in addition to the direct force W. Thus—

If an external force W *acts on a section at a distance x from its centre of gravity, it will be subjected to a direct force* W *acting*

uniformly all over the section and a bending moment Wx. The intensity of stress on any part of the section will be the sum of the direct stress and the stress due to bending, tension and compression being regarded as stresses of opposite sign.

In the figure let the bar be subjected to both a direct stress (+), say tension, and bending stresses. The direct stress acting uniformly all over the section may be represented by the diagram *abcd*, where *ab* or *cd* is the intensity of the tensile stress (+); then if the intensity of tensile stress due to bending be represented by *be* (+), and the compressive stress (−) by *fc*, we shall have—

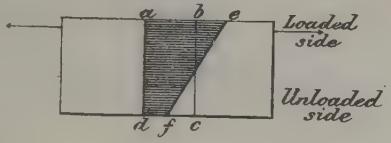


FIG. 532.

The total tensile stress on the outer skin $= ab + be = ae$
" " " inner " $= dc - fc = df$

If the bending moment had been still greater, as shown in

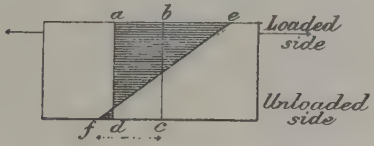


FIG. 533.

Fig. 533, the stress *df* would be −, *i.e.* one side of the bar would have been in compression.

Stresses on Bars loaded out of the Centre.—

Let W = the load on the bar producing either direct tensile or compressive stresses;
A = the sectional area of the bar;
Z = the modulus of the section in bending;
x = the eccentricity of the load, *i.e.* the distance of the point of application of the load from the centre of gravity of the section;
f'_t = the direct tensile stress acting evenly over the section;
f'_c = the direct compressive stress acting evenly over the section;

f_t = the tensile stress due to bending;
f_c = the compressive stress due to bending;
M = the bending moment on the section.

Then $\frac{W}{A} = f'_c$ or f'_t or f' (direct stress)

$$\frac{M}{Z} = \frac{Wx}{Z} = f_t \text{ or } f_c \text{ or } f \text{ (bending stress)}$$

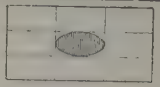
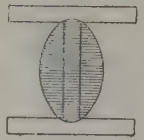
Then the maximum stress on the skin of the section on the loaded side $\Big\} = f' + f = \frac{W}{A} + \frac{Wx}{Z}$

$$= W\left(\frac{1}{A} + \frac{x}{Z}\right)$$

Then the maximum stress on the skin of the section on the unloaded side $\Big\} = f' - f = W\left(\frac{1}{A} - \frac{x}{Z}\right)$

In order that the stress on the unloaded side may not be of opposite sign to the direct stress, the quantity $\frac{1}{A}$ must be greater than $\frac{x}{Z}$. When they are equal, the stress will be zero on the unloaded side, and of twice the intensity of the direct stress on the loaded side; then we have $\frac{1}{A} = \frac{x}{Z}$, or $\frac{Z}{A} = x$. Hence, in order that the stress may not change sign or that there may be no reversal of stress in a section, the line of action of the external force must not be situated at a greater distance than $\frac{Z}{A}$ from the neutral axis.

For convenience of reference, we give various values of $\frac{Z}{A}$ in the following table:—

Section.	Modulus of section Z.	Area of section A.	Eccentricity of load $x = \frac{Z}{A}$	For no change of sign in the stress, the force must act within the	or the shaded area
Rectangular	$\frac{BH^2}{6}$	BH	$\frac{H}{6}$	Middle third	 FIG. 534.
Circular	$\frac{\pi D^3}{32}$	$\frac{\pi D^2}{4}$	$\frac{D}{8}$	Middle fourth	FIG. 535.
Hollow rectangle or I section	$\frac{BH^3 - bh^3}{6H}$	$BH - bh$	$\frac{BH^3 - bh^3}{6H(BH - bh)}$		 FIG. 536.
Do. sideways I	$\frac{bH^3 + Bh^3}{6H}$ (Using the symbols of Fig. 354)	$Bh + bH$	$\frac{bH^3 + Bh^3}{6H(Bh + bH)}$		
Hollow circular	$\frac{\pi(D^4 - D_i^4)}{32D}$	$\frac{\pi(D^2 - D_i^2)}{4}$	$\frac{D^2 + D_i^2}{8D}$		FIG. 537.

General Case of Eccentric Loading.—In the above instances we have only dealt with sections symmetrical about the neutral axis, and we showed that the skin stress was much greater on the one side than the other. In order to equalize the skin stress, we frequently use unsymmetrical sections.

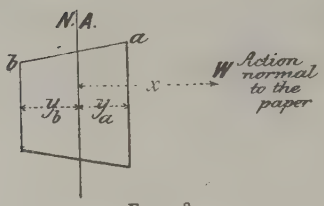


FIG. 538.

Let the skin stress at a due to bending and direct stress $= f'_a$; likewise that at $b = f'_b$.

$$\text{The direct stress all over the section} = f = \frac{W}{A}$$

For bending we have—

$$M = fZ = \frac{f_a I}{y_a} \text{ or } \frac{f_b I}{y_b}$$

according to the side we are considering;

$$\text{or } Wx = \frac{f_a I}{y_a} \text{ or } \frac{f_b I}{y_b}$$

$$\text{hence } f_a = \frac{Wxy_a}{I}$$

$$\text{and } f'_a = f + f = \frac{W}{A} + \frac{Wx \,.\, y_a}{I}$$

$$\text{also } f'_b = f - f = \frac{W}{A} - \frac{Wx \,.\, y_b}{I}$$

When $y_a = y_b$, the expression becomes the same as we had above.

Cranked Tie-bar.—Occasionally tie bars and rods have

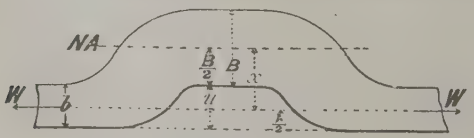


FIG. 539.

to be cranked in order to give clearance or for other reasons, but they are very rarely properly designed, and therefore are a source of constant trouble.

The normal width of the tie-bar is b; the width in the cranked part must be greater as it is subjected to bending as well as to tension. We will calculate the width B to satisfy the condition that the maximum intensity of stress in the wide part shall not be greater than that due to direct tension in the narrow part.

Let the thickness of the bar be t.

Then, using the same notation as before—

$$x = \frac{B}{2} + u - \frac{b}{2}$$

the direct stress on the wide part of the bar $= \frac{W}{Bt} = f'$

the bending stress on the wide part of the bar $= \frac{Wx}{Z} = \frac{6W\left(\frac{B}{2} + u - \frac{b}{2}\right)}{B^2t}$

the maximum skin stress due to both $= \frac{W}{Bt} + \frac{6W\left(\frac{B}{2} + u - \frac{b}{2}\right)}{B^2t}$

But as the stress on the wide part of the bar has to be made equal to the stress on the narrow part, we have—

$$\frac{W}{bt} = \frac{W}{Bt} + \frac{6W(B + 2u - b)}{2B^2t}$$

Then dividing both sides of the equation by $\frac{W}{t}$, and solving, we get—

$$B = \sqrt{6bu + b^2} + 2b$$

Both b and u are known for any given case, hence the width B is readily arrived at. If a rectangular section be retained, the stress on the inner side will be much greater than on the outer. The actual values are easily calculated by the methods given above, hence there will be a considerable waste of material. For economy of material, the section should be tapered off at the back to form a trapezium section. Such a section may be assumed, and the stresses calculated by the method given in the last paragraph; if still unequal, the correct section can be arrived at by one or two trials. An expression can be got out to give the form of the section at once, but it is very cumbersome and more trouble in the end to use than the trial and error method.

Bending of Curved Bars.—Let the curved bar in its original condition be represented by the full lines, and after bending by the broken lines.

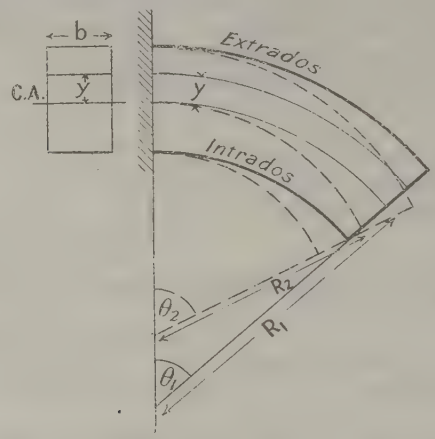


FIG. 540.

Let the distance of any layer from the central axis which passes through the centroid of the section be $+y$ when measured towards the extrados, and $-y$ when measured towards the intrados.

Let the area of the cross section be $A = \Sigma b\delta y = \Sigma\delta a$

The original circumferential length of a layer distant y from the central axis $= (R_1 + y)\theta_1 = l_1$

The final length . . . . . $= (R_2 + y)\theta_2 = l_2$

The strain on the layer $= (R_2 + y)\theta_2 - (R_1 + y)\theta_1 = x$

$$x = R_2\theta_2 - R_1\theta_1 + y(\theta_2 - \theta_1) \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

$$x = 0 \text{ when } y = -\frac{R_2\theta_2 - R_1\theta_1}{\theta_2 - \theta_1} = h \quad . \quad . \quad \text{(ii.)}$$

The only layer on which the strain is zero is that at the neutral axis, hence h is the distance of the neutral axis from the central axis of the section. R_2 and θ_2 are unknown at this stage in the reasoning, hence the expression must be put in another form before h can be calculated.

Substituting the value of h in (i.),

$$x = -h(\theta_2 - \theta_1) + y(\theta_2 - \theta_1)$$

$$x = (y - h)(\theta_2 - \theta_1) = (y - h)\Delta\theta \quad . \quad . \quad \text{(iii.)}$$

For elastic materials we have—

$$\frac{x}{l_1} = \frac{f}{E} = e \quad \text{and} \quad f = \frac{Ex}{l_1} = Ee. \quad . \quad \text{(iv.)}$$

$$e = \frac{(y - h)\Delta\theta}{l_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v.)}$$

Substituting the value of x from (iii.)

$$f = \frac{E(y - h)\Delta\theta}{l_1} = Ee \quad . \quad . \quad . \quad . \quad \text{(vi.)}$$

Since the total tension on the one side of the neutral axis is equal to the total compression on the other, the net force on the whole section $\Sigma(f\delta a) = 0$

$$\text{or } f_1a_1 + f_2a_2 + \text{etc.} = f_1'a_1' + f_2'a_2' + \text{etc.}$$

$$\text{or } E\Delta\theta\left\{\left(\frac{y_1 - h}{l_1}\right)a_1 + \text{etc.}\right\} = E\Delta\theta\left\{\left(\frac{y_1' + h}{l_1'}\right)a_1' + \text{etc.}\right\}$$

$$\text{and } \Sigma\left(\frac{y - h}{l_1}\delta a\right) = 0 = \Sigma\left(\frac{y\delta a}{l_1}\right) - h\Sigma\left(\frac{\delta a}{l_1}\right)$$

$$h = \frac{\Sigma\left(\frac{y}{l_1}\delta a\right)}{\Sigma\left(\frac{\delta a}{l_1}\right)} = \frac{\Sigma\left(\frac{y}{R_1 + y}\delta a\right)}{\Sigma\left(\frac{\delta a}{R_1 + y}\right)} = \frac{\Sigma\left\{\left(1 - \frac{R_1}{R_1 + y}\right)\delta a\right\}}{\Sigma\left(\frac{\delta a}{R_1 + y}\right)}$$

$$h = \frac{A - \frac{AR_1}{R_1 + y}}{\frac{A}{R_1 + y}} = \frac{R_1(A - A')}{A'} \text{ (see p. 546) } . \quad . \quad . \quad \text{(vii.)}$$

We also have $M = fZ = f_1y_1a_1 + f_2y_2a_2 + \text{etc.}$

$$\text{or } M = \Sigma(f \,.\, y \,.\, \delta a) = E\,\Delta\theta\,\Sigma\left\{\frac{y(y - h)}{l_1}\delta a\right\}$$

$$\Delta\theta = \frac{M}{E\Sigma\left\{\frac{y(y - h)}{l_1}\delta a\right\}}$$

Substituting this value in (vi.)

$$f = \frac{E(y - h)M}{El_1\Sigma\left\{\frac{y(y - h)}{l_1}\delta a\right\}}$$

Substituting the value of f_1

$$f = \frac{y - h}{R_1 + y} \frac{M}{\Sigma\left\{\frac{y(y-h)}{R_1 + y}\delta a\right\}}$$

and $$\Sigma\left\{\frac{y(y-h)}{R_1+y}\delta a\right\} = \Sigma\left\{\left(y - h - R_1\frac{y-h}{R_1+y}\right)\delta a\right\}$$

But $$\Sigma\delta a = 0, \text{ and } R_1\frac{y-h}{R_1+y} = 0$$

Hence $$f = \frac{M}{-hA} \times \frac{y-h}{R_1+y}$$

The stress at the intrados—

$$f_i = \frac{M(y_i - h)}{hA(R_1 - y_i)}$$

The stress at the extrados—

$$f_e = \frac{M(y_e + h)}{hA(R_1 + y_e)}$$

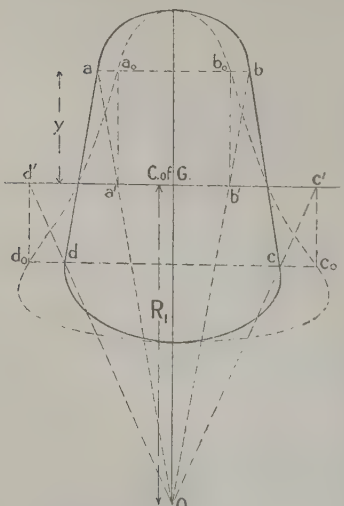


FIG. 541.

where y_i and y_e are the distances of the intrados and the extrados respectively from the central axis.

The value of

$$A' \text{ or } R_1\Sigma\left(\frac{\delta a}{R_1 + y}\right)$$

can be found graphically thus: The section of the curved bar is shown in full lines; the centre of curvature is at O. The points a and b are joined to O; they cut the central axis in a' and b'. At these points erect perpendiculars to cut the line ab, as shown in a_0 and b_0, which are points on the new area A', because

$$\frac{a_0b_0}{ab} = \frac{R_1}{R_1 + y}$$

Similarly $$\frac{e_0 d_0}{cd} = \frac{R_1}{R_1 - y}$$

The two areas, A and A′, must be accurately measured by a planimeter. The author wishes to acknowledge his indebtedness to Morley's "Strength of Materials," from which the above paragraph is largely drawn.

Hooks.—In the commonly accepted, but erroneous, theory of hooks, a hook is regarded as a special case of a cranked tie bar, and the stresses are calculated by the expressions given in the paragraph on the "general case of eccentric loading," but such a treatment gives too low a value for the tensile stress on the intrados, and too high a value for the compressive stress on the extrados. In spite, however, of its inaccuracy, it will probably continue in use on account of its simplicity; and provided the permissible tensile stress be taken somewhat lower to allow for the error, the method gives quite good results in practice.

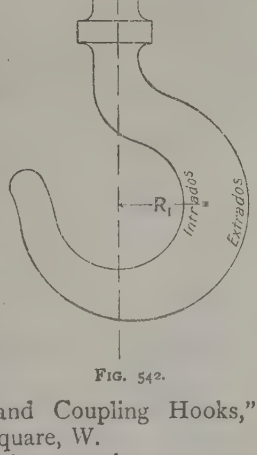


FIG. 542.

The most elaborate and complete treatment of hooks is that by Pearson & Andrews, "On a Theory of the Stresses in Crane and Coupling Hooks," published by Dulau & Co., 37, Soho Square, W.

The application of their theory is, however, by no means simple when applied to such hook sections as are commonly used for cranes. In the graphical treatment the sections must be drawn to a very large scale, since very small errors in drawing produce large errors in the final result. For a comparison of their theory with tests on large crane hooks, see a paper by the author, *Proceedings I.C.E.*, clxvii.

For a comparison of the ordinary theory with tests on drop forged steel hooks, see a paper by the author, *Engineering*, October 18, 1901.

The hook section shown in Fig. 541 gave the following results—

$A = 14{\cdot}53$ sq. ins. $A_1 = 15{\cdot}74$ sq. ins. $R_1 = 5$ ins.
$h = 0{\cdot}384$ in. $M = 85$ in.-tons. $y_i = 2{\cdot}46$ ins.
$y_e = 2{\cdot}81$ ins. Then $f_i = 12{\cdot}5$ tons sq. in.

The Andrews-Pearson theory gave 13·9 tons sq. in., the common theory 8·8 tons sq. in., and by experiment 13·2 tons sq. in.

By this theory $f_e = 6{\cdot}2$ tons sq. in., the Andrews-Pearson theory gave 4·6, and the common theory 7·6 tons sq. in.

Inclined Beam.—Many cases of inclined beams occur in practice, such as in roofs, etc.; they are in reality members subject to combined bending and direct stresses. In Fig. 543, resolve W into two components, W_1 acting normal to the beam, and P acting parallel with the beam; then the bending moment at the section $x = W_1 l_1$.

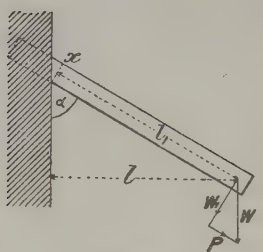


Fig. 543.

$$\text{But } W_1 = W \sin \alpha$$

$$\text{and } l_1 = \frac{l}{\sin \alpha}$$

$$\text{hence } M_x = W \sin \alpha \times \frac{l}{\sin \alpha}$$

$$M_x = Wl = fZ$$

$$f = \frac{Wl}{Z}$$

$$\text{The tension acting all over the section} = \frac{P}{A} = \frac{W \cos \alpha}{A}$$

$$\text{hence } f \text{ max.} = \frac{W \cos \alpha}{A} + \frac{Wl}{Z} = W\left(\frac{\cos \alpha}{A} + \frac{l}{Z}\right)$$

$$\text{and } f \text{ min.} = \frac{W \cos \alpha}{A} - \frac{Wl}{Z} = W\left(\frac{\cos \alpha}{A} - \frac{l}{Z}\right)$$

N.B.—The Z is for the section x taken normal to the beam, *not* a vertical section.

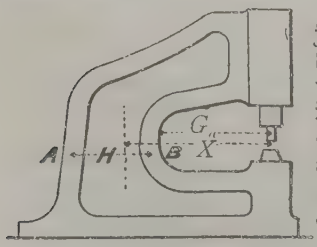


Fig. 544.

Machine Frames subjected to Bending and Direct Stresses.—Many machine frames which have a gap, such as punching and shearing machines, riveters, etc., are subject to both bending and direct stresses. Take, for example, a punching-machine with a box-shaped section through AB. Let the load on the punch = W, and the distance of the punch from the centre of gravity of the section = X. X is at

present unknown, unless a section has been assumed, but if not a fairly close approximation can be obtained thus: We must first of all fix roughly upon the ratio of the compressive to the tensile stress due to bending; the actual ratio will be somewhat less, on account of the uniform tension all over the section, which will diminish the compression and increase the tension. Let the ratio be, say, 3 to 1; then, neglecting the strength of the web, our section will be somewhat as follows:—

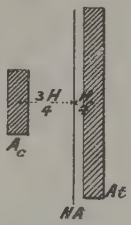


FIG. 545.

Make $A_c = \frac{1}{3}A_t$

then $X = G_a + \frac{H}{4}$ approx.

$$Z = \frac{I}{y} = \frac{A_c\left(\frac{3}{4}H\right)^2 + 3A_c\left(\frac{H}{4}\right)^2}{\frac{H}{4}} \text{ (approx.)}$$

$Z = 3A_cH$ (for tension)

But $WX = fZ$ (f being the tensile stress)

$$W\left(G_a + \frac{H}{4}\right) = 3A_cHf$$

$$A_c = \frac{W\left(G_a + \frac{H}{4}\right)}{3Hf} \text{ or } \frac{W\left(G_a + \frac{H}{n+1}\right)}{nHf}$$

where n is the ratio of the compressive to the tensile stress,

and $A_t = nA_c$

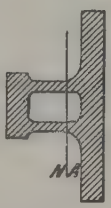


FIG. 546.

Having thus approximately obtained the sectional areas of the flanges, complete the section as shown in Fig. 546; and as a check on the work, calculate the stresses by finding the centre of gravity, also the Z or the I of the complete section by the method given on page 544, or better by the "curved bar" method.

CHAPTER XV.

STRUTS.

General Statement.—The manner in which short compression pieces fail is shown in Chapter X.; but when their length is great in proportion to their diameter, they bend laterally, unless they are initially absolutely straight, exactly centrally loaded, and of perfectly uniform material—three conditions which are never fulfilled in practice. The nature of the stresses occurring in a strut is, therefore, that of a bar subjected to both bending and compressive stresses. In Chapter XIV. it was shown that if the load deviated but very slightly from the centre of gravity of the section, it very greatly increased the stress in the material; thus, in the case of a circular section, if the load only deviated by an amount equal to one-eighth diameter from the centre, the stress was doubled; hence a very slight initial bend in a compression member very seriously affects its strength.

Effects of Imperfect Loading.—Even if a strut be initially straight before loading, it does not follow that it will

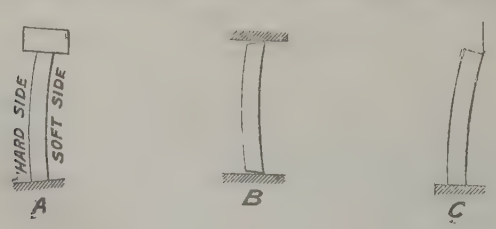


FIG. 547.

remain so when loaded; either or both of the following causes may set up bending:—

(1) The one side of the strut may be harder and stiffer than the other; and consequently the soft side will yield most, and the strut will bend as shown in A, Fig. 547.

(2) The load may not be perfectly centrally applied, either through the ends not being true as shown in B, or through the load acting on one side, as in C.

Possible Discrepancies between Theory and Practice.—We have shown that a very slight amount of bending makes a serious difference in the strength of struts; hence such accidental circumstances as we have just mentioned may not only make a serious discrepancy between theory and experiment, but also between experiment and experiment. Then, again, the theoretical determination of the strength of struts does not rest on a very satisfactory basis, as in all the theories advanced somewhat questionable assumptions have to be made; but, in spite of it, the calculated buckling loads agree fairly well with experiments.

Bending of Long Struts.—The bending moment at the middle of the bent strut shown in Fig. 548 is evidently $W\delta$.

Then $W\delta = fZ$, using the same notation as in the preceding chapters.

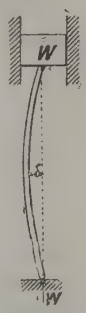


FIG 548.

If we increase the deflection we shall correspondingly increase the bending moment, and consequently the stress.

From above we have—

$$W = \frac{f}{\delta}Z \text{ or } \frac{f_1}{\delta_1}Z, \text{ and so on}$$

But as f varies with δ, $\frac{f}{\delta}$ = a constant, say K;

$$\text{then } W = KZ$$

But Z for any given strut does not vary when the strut bends; hence there is only one value of W that will satisfy the equation.

When the strut is thus loaded, let an external bending moment M, indicated by the arrow (Fig. 549), be applied to it until the deflection is δ_1, and its stress f_1;

$$\text{Then } W\delta_1 + M = f_1Z$$
$$\text{But } W\delta_1 = f_1Z$$
$$\text{therefore } M = 0$$

that is to say, that no external bending moment M is required to keep the strut in its bent position, or the strut, when thus loaded, is in a state of neutral equilibrium, and will remain

when left alone in any position in which it may be placed; this condition, of course, only holds so long as the strut is elastic, *i.e.* before the elastic limit is reached. This state of neutral equilibrium may be proved experimentally, if a long thin piece of elastic material be loaded as shown.

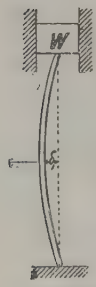


FIG. 549.

Now, place a load W_1 *less* than W on the strut, say $W = W_1 + w$, and let it again be bent by an external bending moment M till its deflection is δ_1 and the stress f_1; then we have, as before—

$$W_1\delta_1 + M = f_1Z = W\delta_1 = W_1\delta_1 + w\delta_1$$
$$\text{hence } M = w\delta_1$$

Thus, in order to keep the strut in its bent position with a deflection δ_1, we must subject it to a + bending moment M, *i.e.* one which tends to bend the strut in the *same* direction as $W_1\delta_1$; hence, if we remove the bending moment M, the deflection will become zero, *i.e.* the strut will straighten itself.

Now, let a load W_2 *greater* than W be placed on the strut, say $W = W_2 - w$, and let it again be bent until its deflection $= \delta_1$, and the stress f_1 by an external bending moment M; then we have as before—

$$W_2\delta_1 + M = f_1Z = W_2\delta_1 - w\delta_1$$
$$\text{hence } M = -w\delta_1$$

Thus, in order to keep the strut in its bent position with a deflection δ_1, we must subject it to a − bending moment M, *i.e.* one which tends to bend the strut in the *opposite* direction to $W_2\delta_1$; hence, if we remove the bending moment M, the deflection will go on increasing, and ere long the elastic limit will be reached when the strain will increase suddenly and much more rapidly than the stress, consequently the deflection will suddenly increase and the strut will buckle.

Thus, the strut may be in one of three conditions—

Condition.	When slightly bent by an external bending moment M, on being released, the strut will—	When supporting a load—
i.	Remain bent	W.
ii.	Straighten itself	*less* than W.
iii.	Bend still further and ultimately buckle	*greater* than W.

Condition ii. is, of course, the only one in which a strut can exist for practical purposes; how much the working load must be less than W is determined by a suitable factor of safety.

Buckling Load of Long Thin Struts. Euler's Formula.—The results arrived at in the paragraph above refer only to very long thin struts.

As a first approximation, mainly for the sake of getting the form of expression for the buckling load of a slender strut, assume that the strut bends to an arc of a circle.

Let l = the effective length of the strut (see Fig. 550);
E = Young's modulus of elasticity;
I = the least moment of inertia of a section of the strut (assumed to be of constant cross-section).

FIG. 550.

Then for a strut loaded thus—

$$\delta = \frac{Ml^2}{8EI} \text{ (see p. 506)} = \frac{W\delta l^2}{8EI}$$

$$\text{or } W = \frac{8EI}{l^2} \text{ (first approximation)}$$

As the strut is very long and the deflection small, the length l remains practically constant, and the other quantities 8, E, I are also constant for any given strut; thus, W is equal to a constant, which we have previously shown must be the case.

Once the strut has begun to bend it cannot remain a circular arc, because the bending moment no longer remains constant at every section, but it will vary directly as the distance of any given section from the line of application of the load. Under these conditions assume as a second approximation that it bends to a parabolic arc, then the deflection—

$$\delta = \frac{2}{3} \times \frac{l}{2} M \times \frac{5}{8} \times \frac{l}{2} \div EI = \frac{Ml^2}{9{\cdot}6EI} = \frac{W\delta l^2}{9{\cdot}6EI}$$

$$\text{and } W = \frac{9{\cdot}6EI}{l^2}$$

The value obtained by Euler was—

$$W = \frac{\pi^2 EI}{l^2} = \frac{9{\cdot}87EI}{l^2} = \frac{10EI}{l^2} \text{ (nearly)}$$

This expression is obtained thus—
The bending moment M at any point distant x from the middle of the strut is

$$M = -W\delta = EI\frac{d^2\delta}{dx^2}$$ (see page 510)

Multiply each side by $\frac{d\delta}{dx}$

$$\frac{d\delta}{dx}\cdot\frac{d^2\delta}{dx^2} = -\frac{W\delta}{EI}\frac{d\delta}{dx}$$

Integrating $$\left(\frac{d\delta}{dx}\right)^2 = -\frac{W}{EI}\left(\delta^2 + C\right)$$

When $\frac{d\delta}{dx} = 0$, $\delta = \Delta$, hence $C = -\Delta^2$

and $$\left(\frac{d\delta}{dx}\right)^2 = \frac{W}{EI}(\Delta^2 - \delta^2)$$

$$dx = \sqrt{\frac{EI}{W}}\frac{d\delta}{\sqrt{\Delta^2 - \delta^2}}$$

Integrating again—

$$x = \sqrt{\frac{EI}{W}}\sin^{-1}\frac{\delta}{\Delta} + K$$

When $x = 0$, $\delta = 0$, therefore $K = 0$

Hence $$\delta = \Delta \sin\left(x\sqrt{\frac{W}{EI}}\right)$$

When $x = \frac{l}{2}$ $\delta = \Delta$

$$\text{and } \sin\left(\frac{l}{2}\sqrt{\frac{W}{EI}}\right) = 1$$

The only angles whose sines are $= 1$ are $\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc. We require the *least* value of W, hence—

$$\frac{l}{2}\sqrt{\frac{W}{EI}} = \frac{\pi}{2}$$

$$\text{and } W = \frac{\pi^2 EI}{l^2}$$

It must not be forgotten that this expression is only an approximation, since the direct stress on the strut is neglected. When the strut is very long and slender the direct compressive stress is very small and therefore negligible, but in short struts the direct stress is not negligible consequently for such cases the above expression gives results very far from the truth.

Effect of End holding on the Buckling Load.—In the case we have just considered the strut was supposed to be free or pivoted at the ends, but if the ends are not free the strut behaves in a different manner, as shown in the accompanying diagram.

DIAGRAM SHOWING STRUTS OF EQUAL STRENGTH.

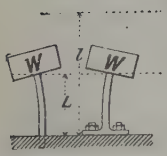
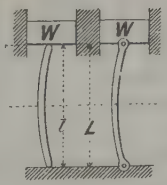
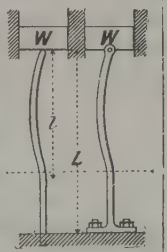
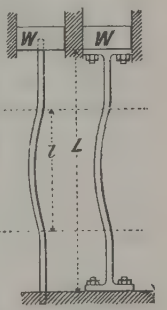
One end free, the other fixed.	Both ends pivoted or rounded.	One end rounded or pivoted, the other end built in or fixed.	Both ends fixed or built in.
$l = 2L$	$l = L$	$l = \frac{L}{\sqrt{2}}$	$l = \frac{L}{2}$
			
$W = \frac{2{\cdot}5EI}{L^2}$	$W = \frac{10EI}{L^2}$	$W = \frac{20EI}{L^2}$	$W = \frac{40EI}{L^2}$
$P = \frac{2{\cdot}5E\rho^2}{L^2}$	$P = \frac{10E\rho^2}{L^2}$	$P = \frac{20E\rho^2}{L^2}$	$P = \frac{40E\rho^2}{L^2}$

FIG. 551.

Each strut is supposed to be of the same section, and loaded with the same weight W.

Let P = the buckling stress of the strut, *i.e.* $\frac{W}{A}$, where

W = the buckling load of the strut;
A = the sectional area of the strut.

We also have $\frac{I}{A} = \rho^2$ (see p. 78), where ρ is the radius of gyration of the section.

Substituting these values in the above equation, we have--

$$P = \frac{\pi^2 E \rho^2}{l^2}$$

The "effective" or "virtual" length l, shown in the diagram, is found by the methods given in Chapter XIII. for finding the virtual length of built-in beams.

The square-ended struts in the diagrams are shown bolted down to emphasize the importance of rigidly fixing the ends; if the ends merely rested on flat flanges without any means of fixing, they much more nearly approximate round-ended struts.

It will be observed that Euler's formula takes no account of the compressive stress on the material; it simply aims at giving the load which will produce neutral equilibrium as regards bending in a long bar, and even this it only does imperfectly, for when a bar is subjected to both direct and bending stresses, the neutral axis no longer passes through the centre of gravity of the section. We have shown above that when the line of application of the load is shifted but one-eighth of the diameter from the centre of a round bar, the neutral axis shifts to the outermost edge of the bar. In the case of a strut subject to bending, the neutral axis shifts away from the line of application of the load; thus the bending moment increases more rapidly than Euler's hypothesis assumes it to do, consequently his formula gives too high results; but in very long columns in which the compressive stress is small compared with the stress due to bending, the error may not be serious. But if the formula be applied to short struts, the result will be absurd. Take, for example, an iron strut of circular section, say 4 inches diameter and 40 inches long; we get $P = \frac{10 \times 29000000 \times 1}{1600} = 181{,}000$ lbs. per square inch, which is far higher than the crushing strength of a short specimen of the material, and obviously absurd.

If Euler's formula be employed, it must be used exclusively

for long struts, whose length l is not less than 30 diameters for wrought iron and steel, or 12 for cast iron and wood.

Notwithstanding the unsatisfactory basis on which it rests, many high authorities prefer it to Gordon's, which we will shortly consider. For a full discussion of the whole question of struts, the reader is referred to Todhunter and Pearson's "History of the Theory of Elasticity."

Gordon's Strut Formula rationalized.—Gordon's strut formula, as usually given, contains empirical constants obtained from experiments by Hodgkinson and others; but by making certain assumptions constants can be obtained rationally which agree remarkably well with those found by experiment.

Gordon's formula certainly has this advantage, that it agrees far better with experiments on the ultimate resistance of columns than does the formula propounded by Euler; and, moreover, it is applicable to columns of any length, short or long, which, we have seen above, is not the case with Euler's formula. The elastic conditions assumed by Euler cease to hold when the elastic limit is passed, hence a long strut always fails at or possibly before that point is reached; but in the case of a short strut, in which the bending stress is small compared with the compressive stress, it does not at all follow that the strut will fail when the elastic limit in compression is reached—indeed, experiments show conclusively that such is not the case. A formula for struts of any length must therefore cover both cases, and be equally applicable to short struts that fail by crushing and to long struts that fail by bending. In constructing this formula we assume that the strut fails either by buckling or by crushing,[1] when the sum of the direct compressive stress and the skin stress, due to bending, are equal to the crushing strength of the material; in using the term "crushing strength" for ductile materials, we mean the stress at which the material becomes plastic. This assumption, we know, is not strictly true, but it cannot be far from the truth, or the calculated values of the constant (α), shortly to be considered, would not agree so well with the experimental values.

[1] Mons. Considère and others have found that for long columns the resistance does not vary directly as the crushing resistance of the material, but for short columns, which fail by crushing and not by bending, the resistance does of course entirely depend upon it, and therefore must appear in any formula professing to cover struts of all lengths.

Let S = the crushing (or plastic) strength of a short specimen of the material;
C = the direct compressive stress on the section of the strut;
then, adopting our former notation, we have—

$$C = \frac{W}{A} \text{ and } f = \frac{W\delta}{Z}$$
$$\text{then } S = C + f$$
$$S = \frac{W}{A} + \frac{W\delta}{Z} \text{ (the } \textit{least} \text{ Z of the section)}$$

We have shown above that, on Euler's hypothesis, the maximum deflection of a strut is—

$$\delta = \frac{Ml^2}{10EI} = \frac{fZl^2}{10EZy}$$

where y is the distance of the most strained skin from the centre of gravity of the section, or from the assumed position of the neutral axis. We shall assume that the same expression holds in the present case. In symmetrical sections $y = \frac{d}{2}$, where d is the *least* diameter of the strut section.

By substitution, we have—

$$\delta = \frac{fl^2}{5Ed}$$
$$\text{also } S = \frac{W}{A} + \frac{Wfl^2}{5EdZ} \quad \text{(i.)}$$
$$= \frac{W}{A}\left(1 + \frac{Afl^2}{5EdZ}\right)$$
$$= \frac{W}{A}\left(1 + \frac{Afd}{5EZ} \times \frac{l^2}{d^2}\right)$$

If W be the buckling load, we may replace $\frac{W}{A}$ by **P**.

Let $\frac{Afd}{5EZ} = \alpha$.

$$\text{Then } S = P\left(1 + \alpha\frac{l^2}{d^2}\right)$$
$$\text{or } P = \frac{S}{1 + \alpha\frac{l^2}{d^2}} = \frac{S}{1 + \alpha r^2}$$

where $r = \frac{l}{d}$, which is a modification of "Gordon's Strut Formula."

P may be termed the buckling stress of the strut.

The d in the above formula is the *least* dimension of the section, thus—

FIG. 552.

It now remains to be seen how the values of the constant a agree with those found by experiment; it, of course, depends upon the values we choose for f and E. The latter presents no difficulty, as it is well known for all materials; but the former is not so obvious at first. In equation (i.), the first term provides for the crushing resistance of the material irrespective of any stress set up by bending; and the second term provides for the bending resistance of the strut. We have already shown that the strut buckles when the elastic limit is reached, hence we may reasonably take f as the elastic limit of the material.

It will be seen that the formula is only true for the two extreme cases, viz. for a very short strut, when W = AS, and for a very long strut, in which S = f; then—

$$W = \frac{5Ed^2Z}{l^2} \text{ or } \frac{10EI}{l^2}$$

which is Euler's formula. It is impossible to get a rational formula for intermediate cases, because any expression for δ only holds up to the elastic limit, and even then only when the neutral axis passes through the centre of gravity of the section, *i.e.* when there is pure bending and no longitudinal stress. However, the fact that the rational value for a agrees so well with the experimental value is strong evidence that the formula is trustworthy.

Values of S, f, and E are given in the table below; they

must be taken as fair average values, to be used in the absence of more precise data.

Material.	Pounds per square inch.		
	S	f	E
Soft wrought iron	40,000	28,000	25,000,000
Hard ,, ...	48,000	32,000	29,000,000
Mild steel	67,000	45,000	30,000,000
Hard ,, ...	110,000	75,000	32,000,000
Cast iron	80,000 (no marked limit)	80,000	13,000,000
,, (hard close-grained metal)	130,000	130,000	22,000,000
Pitchpine and oak	8,000	8,000	900,000

Material.	Form of section.	$a = \frac{Afd}{5EZ}$	a by experiment.
Wrought iron ...		$\frac{1}{740}$ to $\frac{1}{750}$	$\frac{1}{750}$ to $\frac{1}{700}$
		$\frac{1}{580}$ to $\frac{1}{570}$	$\frac{1}{580}$
		$\frac{1}{930}$ to $\frac{1}{940}$	$\frac{1}{1370}$ to $\frac{1}{900}$
		$\frac{1}{450}$ to $\frac{1}{460}$	$\frac{1}{460}$ to $\frac{1}{225}$
Mild steel ...		$\frac{1}{340}$	$\frac{1}{503}$ to $\frac{1}{820}$
		$\frac{1}{400}$	$\frac{1}{350}$
		$\frac{1}{670}$	$\frac{1}{830}$
		$\frac{1}{320}$	$\frac{1}{503}$ to $\frac{1}{350}$ (author)
Hard steel ...		$\frac{1}{330}$	$\frac{1}{330}$ to $\frac{1}{403}$
		$\frac{1}{260}$	$\frac{1}{230}$ to $\frac{1}{225}$
		$\frac{1}{440}$	$\frac{1}{380}$ [1]
		$\frac{1}{210}$	—

[1] The discrepancies in these cases may be due to the section being thicker or thinner than the one assumed in calculating the value of a. In the case of hollow sections, angles, tees, etc., the value of a should be worked out.

Material.	Form of section.	$a = \frac{Af d}{5EZ}$	a by experiment.
Cast iron ...		$\frac{1}{130}$ to $\frac{1}{140}$	$\frac{1}{172}$
		$\frac{1}{100}$ to $\frac{1}{110}$	$\frac{1}{100}$
		$\frac{1}{130}$ to $\frac{1}{170}$	$\frac{1}{130}$ (Rankine) $\frac{1}{200}$
		$\frac{1}{70}$ to $\frac{1}{80}$	$\frac{1}{34}$ [1]
Pitchpine and oak		$\frac{1}{90}$	$\frac{1}{82}$
		$\frac{1}{70}$	$\frac{1}{82}$ (author) $\frac{1}{80}$

N.B.—The values of a given in the last column are four times as great as those usually given, due to the l used in our formula being taken equal to L for rounded ends, whereas some other writers take it for square or fixed ends.

The values of the constant a have been worked out for the various materials, and are given in tabulated form above; also values found by experiment as given in Rankine's "Applied Mechanics," and by Bovey, "Theory of Structures and Strength of Materials" (Wiley, New York).

Rankine's Strut Formula.—In the above tables it will be noticed that the value of a as found by calculation quite closely agrees with that found by experiment for the solid sections, but the agreement is not so good in the case of hollow or rolled sections, largely due to the fact that a varies with each form of section and with the thickness of the metal. If the value of a be calculated for each section there is no objection to the use of the Gordon formula, but if one value of a be taken to cover all the cases shown in the tables above it is possible that considerable errors may creep in. For such cases it is better to use Rankine's modification of Gordon's formula.

Instead of writing in the Gordon formula—

$$S = \frac{W}{A}\left(1 + \frac{Af l^2}{5EdZ}\right)$$

we may write—

$$S = \frac{W}{A}\left(1 + \frac{Afyl^2}{10EyAK^2}\right) = \frac{W}{A}\left(1 + \frac{f}{10E}\frac{l^2}{K^2}\right)$$

$$W = \frac{AS}{1 + bR^2}$$

where $b = \frac{f}{10E}$ and $R = \frac{l}{k}$, *i.e.* the equivalent length divided by the *least* radius of gyration of the section about a line passing through the centroid of the section. Values of k will be found in Chapters III. and XI.

The values of b are as follows :—

Material.	Value of b.	
	$\frac{f}{10E}$	By experiment.
Wrought iron	$\frac{1}{9000}$	$\frac{1}{9000}$
Mild steel	$\frac{1}{6700}$	$\frac{1}{7500}$
Hard steel	$\frac{1}{4300}$	$\frac{1}{3000}$
Cast iron	$\frac{1}{1630}$ to $\frac{1}{1690}$	$\frac{1}{1800}$

The discrepancies are due to the assumed value of f not being suitable for the material experimented upon. The terms "mild" and "hard" steel are very vague. If the properties of the material in the tested struts were known, the discrepancies would probably be smaller. It must, however, be borne in mind that the strength of struts cannot be calculated with the same degree of accuracy as beams, shafts, etc.

In the case of *long* cast-iron struts the failure is usually due to tension on the convex side, and not to compression on the concave side. The expression then becomes—

$$P = \frac{T}{ar^2 - 1}$$

where T = the tensile strength of the material, or rather the tension modulus of rupture, *i.e.* the tensile stress as found from a bending experiment. The values of f and T then vary from 30,000 to 45,000 lbs. per square inch, and E (at the breaking point) varies from 11,000,000 to 16,000,000 lbs. per square inch. The value of a then becomes $\frac{1}{300}$ for a rectangular section. On calculating some values for P, it will be seen that for *long* struts where the fracture might occur through the excessive

stress on the tension skin, the value given by this formula agrees fairly well with the values calculated from the original formula; hence we see that such struts are about as likely to fail by tension on the convex side as by compression on the concave side.

The following tables have been worked out by the formula given above to the nearest 100 lbs. per square inch. For those who are constantly designing struts, it will be found convenient to plot them to a large scale, in the same manner as shown in Fig. 553. In order better to compare the results obtained by Euler's and by Gordon's formula, curves representing both are

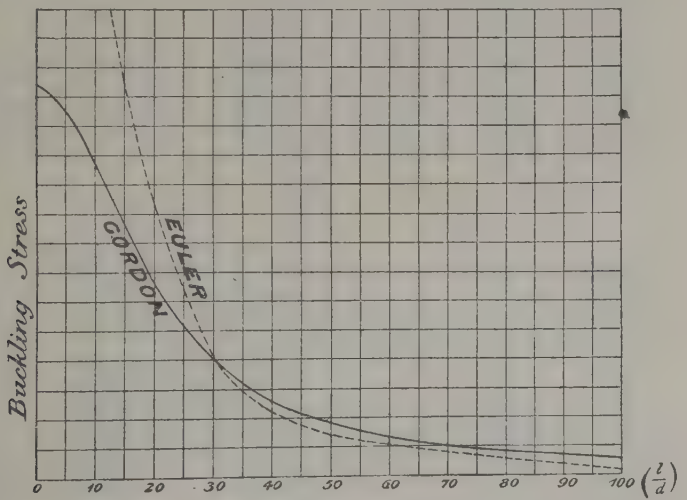


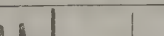
FIG. 553.

NOTE.—The two curves in many cases practically coincide after 40 diameters. In the figure the Gordon curve has been shifted bodily up, to better show the relation.

given, from which it will be seen that they agree fairly well for very long struts, but that Euler's is quite out of it for short struts.

The table on the opposite page gives the ultimate or the *buckling* loads; they must be divided by a suitable factor of safety to get the safe working load.

BUCKLING LOAD OF STRUTS IN POUNDS PER SQUARE INCH OF SECTION.

$\frac{r}{d}$ or $\frac{l}{d}$								
	Wrought iron.				Mild steel.			
5	42,600	42,000	43,000	41,700	64,100	63,100	64,700	62,000
10	38,800	37,200	39,900	36,000	56,600	56,300	58,300	51,000
20	28,800	25,600	30,900	23,200	38,500	33,500	41,900	29,800
30	20,000	16,800	22,200	14,700	25,000	20,600	28,600	17,500
40	14,000	11,400	16,300	9,600	17,000	13,400	19,700	11,100
50	10,400	8,000	12,400	6,700	11,900	9,200	14,100	7,800
60	7,600	5,900	9,100	4,900	8,700	6,700	10,500	5,400
70	5,800	4,500	7,100	3,700	6,700	5,000	8,100	4,100
80	4,600	3,500	5,600	2,900	5,200	3,900	6,300	3,200
90	3,700	2,800	4,500	2,100	4,200	3,100	5,100	2,500
100	3,100	2,300	3,800	1,900	3,400	2,500	4,200	2,100
	Hard steel.				Cast iron (soft).			
5	102,000	100,300	104,000	98,600	67,100	64,000	69,200	58,900
10	85,500	79,400	89,600	74,500	45,200	40,000	49,200	32,900
20	51,500	43,300	57,500	37,900	19,600	16,000	22,900	11,900
30	30,800	24,600	36,100	20,400	10,100	8,000	12,100	5,800
40	19,700	15,400	23,700	12,700	6,000	4,700	7,300	3,400
50	13,500	10,300	16,400	8,500	3,900	3,100	4,800	2,200
60	9,750	7,400	11,900	6,100	2,800	2,200	3,400	1,500
70	7,300	5,500	9,100	4,500	2,100	1,600	2,500	1,100
80	5,700	4,300	7,100	3,500	1,600	1,200	2,000	870
90	4,600	3,400	5,700	2,800	1,300	980	1,600	690
100	3,700	2,800	4,600	2,200	1,000	790	1,300	560

$\frac{r}{d}$ or $\frac{l}{d}$	Cast iron (hard close-grained).				Pitchpine and oak.	
5	109,000	104,000	112,000	95,700	6,300	5,900
10	73,500	65,000	80,000	53,500	3,800	3,300
20	31,800	26,000	37,200	19,300	1,500	1,200
30	16,400	13,000	19,700	9,400	730	580
40	9,700	7,600	11,900	5,500	430	340
50	6,300	5,000	7,800	3,600	280	220
60	4,600	3,600	5,500	2,400	200	150
70	3,400	2,600	4,100	1,800	140	110
80	2,600	1,900	3,300	1,400	110	90
90	2,100	1,600	2,600	1,100	90	70
100	1,600	1,300	2,100	900	70	60

FACTOR OF SAFETY FOR STRUTS.

	Dead loads.	Live loads.
Wrought iron and steel	4	8
Cast iron	6	12
Timber	5	10

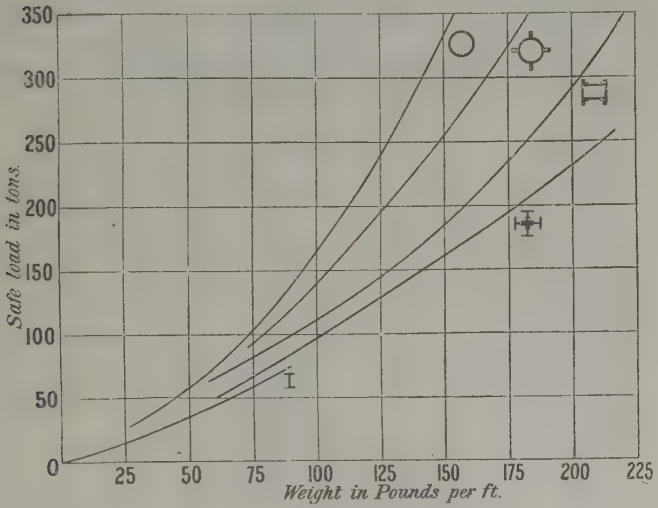


FIG. 554.

In choosing a section for a column, economy in material is not the only and often not the most important matter to be considered; every case must be dealt with on its merits. Even as regards the cost the lightest column is not always the cheapest. In Figs. 554 and 555 we show by means of curves how the weight and cost of different sections vary with the load to be supported. Judging from the weight only, the hollow circle would appear to be the cheapest section, but the cost per ton of drawn tubes is far greater than that of rolled sections; hence on taking this into account, we find the hollow circle the most expensive form of section.

The values given in the figures must not be taken as being rigidly accurate; they vary largely with the state of the market. Designers, however, will find it extremely useful to plot such curves for themselves, not only for struts, but for floorings, cross-girders, roof-coverings, roof-trusses, and many other details which a designer constantly has to deal with.

Straight-line Strut Formula.—The more experience

one gets in the testing of full-sized struts and columns, the more one realizes how futile it is to attempt to calculate the buckling load with any great degree of accuracy. If the struts are of homogeneous material and have been turned or machined all over, and are, moreover, very carefully tested with special holders, which ensure dead accuracy in loading, and every possible care be taken, the results may agree within 5 per cent. of the calculated value; but in the case of commercial struts, which are not always *perfectly* straight, and are

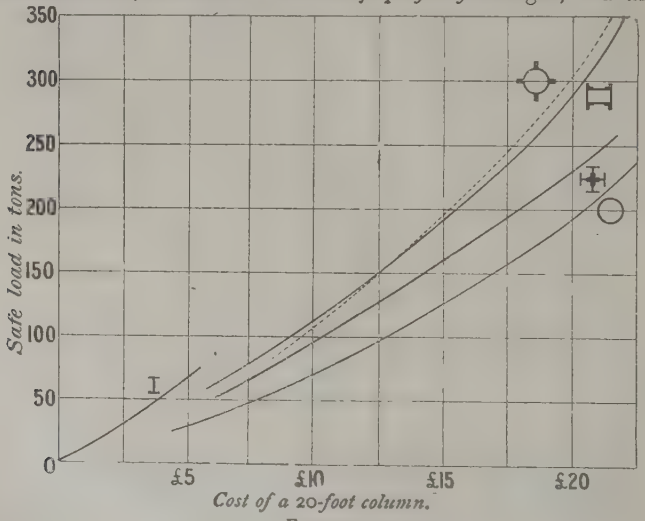
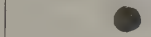
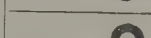
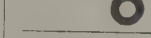
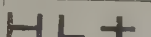
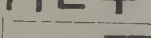
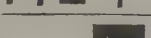
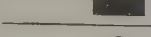
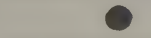
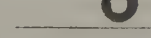
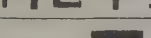
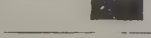
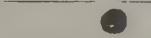
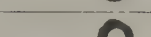


FIG. 555.

not always perfectly centrally loaded, the results are frequently 10 or 15 per cent. out with calculation, even when reasonable care has been taken; hence an approximate expression, such as one of the straight-line formulas, is good enough for many practical purposes, provided the length does not exceed that specified. An expression of this kind is—

$$P = M - N\frac{l}{d}$$

where P = the buckling load in pounds per square inch;
M = a constant depending upon the material;
N = a constant depending upon the form of the strut section;
l = the "equivalent length" of the strut;
d = the *least* diameter of the strut.

Material.	Form of section.	M.	N.	$\frac{l}{d}$ not to exceed
Wrought iron ...		47,000	825	40
		47,000	900	40
		47,000	775	40
		47,000	1070	30
Mild steel ...		71,000	1570	30
		71,000	1700	30
		73,000	1430	30
		71,000	1870	30
Hard steel ...		114,000	3200	30
		114,000	3130	30
		114,000	2700	30
		114,000	3500	30
Soft cast iron ...		90,000	4100	15
		90,000	4700	15
		90,000	3900	15
		90,000	5000	15
Hard cast iron ...		140,000	6600	15
		140,000	7000	15
		140,000	6100	15
		140,000	8000	15
Pitchpine and oak		8000	470	10
		8000	500	10

Columns loaded on Side Brackets.—The barbarous practice of loading columns on side brackets is unfortunately far too common. As usually carried out, the practice reduces the strength of the column to one-tenth[1] of its strength when centrally loaded.

FIG. 556.

In Fig. 557 the height of the shaded figure on the bracket of the column shows the relative loads that may be safely placed at the various distances from the axis of the column. It will be perceived how very rapidly the value of the safe load falls as the eccentricity is increased. If a designer will take the trouble to go carefully into the matter, he will find that it is positively cheaper to use two separate centrally loaded columns instead of putting a side bracket on the much larger column that is required for equal strength.

Let f_c = the maximum compressive stress on the material due to both direct and bending stresses;

f_t = the maximum tensile stress on the material due to both direct and bending stresses;

f = the skin stress due to bending;

C = the compressive stress acting all over the section due to the weight W;

A = the sectional area of the column.

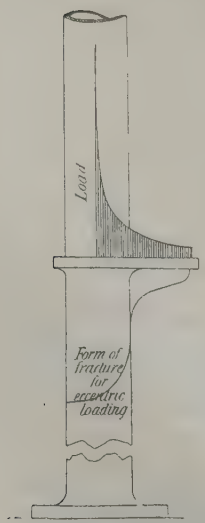


FIG. 557.

$$\text{Then } f_c = f + C = \frac{WX}{Z} + \frac{W}{A}$$

$$\text{and } f_t = \frac{WX}{Z} - \frac{W}{A}$$

If the column also carries a central load W_1, the above become—

$$f_c = \frac{WX}{Z} + \frac{W + W_1}{A}$$

$$f_t = \frac{WX}{Z} - \frac{W + W_1}{A}$$

Columns loaded thus almost invariably fail in tension, therefore the strength must be calculated on the f_t basis. We have neglected the deflection due to loading

[1] The *ten* is not used with any special significance here; may be one-tenth or even one-twentieth.

(Fig. 558), which makes matters still worse; the tensile stress then becomes—

$$f_t = \frac{W(X + \delta)}{Z} - \frac{W + W_1}{A}$$

The deflection of a column loaded in this way may be obtained in the following manner :—

The bending moment = WX

area of bending-moment diagram = WXL

$$\delta = \frac{WXL^2}{2EI} \text{ (approximately)}$$

After the column has bent, the bending moment of course is greater than WX, and approximates to $W(X + \delta)$, but δ is

Fig. 558.

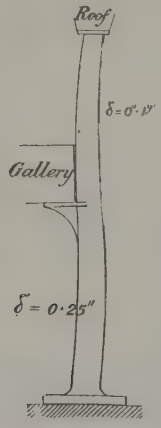


Fig. 559.

usually small compared with X, therefore no serious error arises from taking this approximation.

A column in a public building was loaded as shown in Fig. 559; the deflections given were taken when the gallery was empty. The deflections were so serious that when the gallery was full, an experienced eye immediately detected them on entering the building.

The building in question has been condemned, the galleries have been removed, and larger columns without brackets have

been substituted. The column, as shown, was tested to destruction by the author, with the following results—

External diameter...	4·95 inches
Internal diameter	3·70 ,,
Sectional area	8·49 sq. inches
Modulus of the section	8·18
Distance of load from centre of column ...	6 inches
Height of bracket above base	8′ 6″
Deflection measured, above base	4′ 3″

W in tons...	2	4	6	8	10	12	14	16	18
δ in inches...	0·035	0·069	0·100	0·148	0·195	0·245	0·300	0·355	0·420

The column broke at 18·17 tons; the modulus of rupture was 13·3 tons per sq. inch.

Judging from the deflection when the weight of the gallery rested on the bracket, it will be seen that the column was in a perilously dangerous state.

Another test of a column by the author will serve to emphasize the folly of loading columns on side brackets.

ESTIMATED BUCKLING LOAD IF CENTRALLY LOADED, ABOUT **1000 tons.**

Length 10 feet, end flat, not fixed.

Sectional area of metal at fracture ...	34·3 sq. inches
Modulus of section at fracture	75·0
Distance of point of application of load from centre of column, neglecting slight amount of deflection when loaded ...	17 inches
Breaking load applied at edge of bracket	**65·5 tons**
Bending moment on section when fracture occurred	1114 tons-inches
Compressive stress all over section when fracture occurred	1·91 tons per sq. inch
Skin stress on the material due to bending, assuming the bending formula to hold up to the breaking point	14·85 ,, ,,
Total tensile stress on material due to combined bending and compression[1] ...	12·94 tons per sq. inch
Total compressive stress on material due to combined bending and compression[1]	16·76 ,, ,,
Tensile strength of material as ascertained from subsequent tests	8·45 ,, ,,
Compressive strength of material as ascertained from subsequent tests	30·4 ,, ,,

Thus we see that the column failed by tension in the material on the off side, *i.e.* the side remote from the load.

[1] The discrepancy between this and the tensile strength is due to the bending formula not holding good at the breaking point, as previously explained.

CHAPTER XVI.

TORSION. GENERAL THEORY.

LET Fig. 560 represent two pieces of shafting provided with

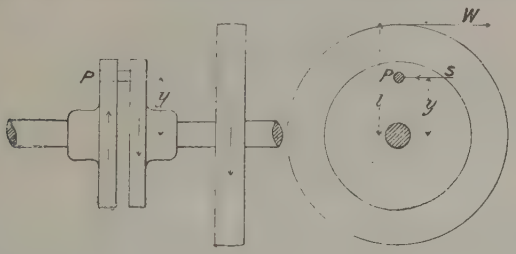


FIG. 560.

disc couplings as shown, the one being driven from the other through the pin P, which is evidently in shear.

Let S = the shearing resistance of the pin.

Then we have $Wl = Sy$

Let the area of the pin = a, and the shear stress on the pin be f_s. Then we may write the above equation—

$$Wl = f_s a y$$

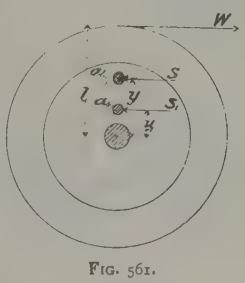


FIG. 561.

Now consider the case in which there are two pins, then—

$$Wl = Sy + S_1 y_1 = f_s a y + f_s a_1 y_1$$

The dotted holes in the figure are supposed to represent the pin-holes in the other disc coupling. Before W was applied the pin-holes were exactly opposite one another, but after the application of W the yielding or the shear of the pins caused a

slight movement of the one disc relatively to the other, but shown very much exaggerated in the figure. It will be seen that the yielding or the strain varies directly as the distance from the axis of revolution (the centre of the shaft). When the material is elastic, the stress varies directly as the strain; hence—

$$\frac{f_{s1}}{f_s} = \frac{y_1}{y}, \text{ or } f_{s1} = \frac{f_s y_1}{y}$$

Substituting this value in the equation above, we have—

$$Wl = f_s a y + \frac{f_s a_1 y_1^2}{y}$$

$$= \frac{f_s}{y}(a y^2 + a_1 y_1^2)$$

Then, if $a = a_1$, and say $y_1 = \frac{y}{2}$

$$Wl = f_s a y + \frac{f_s a y}{4} = f_s a y(1 + \tfrac{1}{4})$$

Thus the inner pin, as in the beam (see p. 432), has only increased the strength by $\frac{1}{4}$. Now consider a similar arrangement with a great number of pins, such a number as to form a hollow or a solid section, the areas of each little pin or element being a, a_1, a_2, etc., distant y, y_1, y_2, etc., respectively from the axis of revolution. Then, as before, we have—

$$Wl = \frac{f_s}{y}(a y^2 + a_1 y_1^2 + a_2 y_2^2 +, \text{ etc.})$$

But the quantity in brackets, viz. each little area multiplied by the square of its distance from the axis of revolution, is the polar moment of inertia of the section (see p. 77), which we will term I_p. Then—

$$Wl = \frac{f_s I_p}{y} = f_s Z_p$$

The Wl is termed the twisting moment, M_t. f_s is the skin shear stress on the material furthest from the centre, and is therefore the maximum stress on the material, often termed the skin stress.

y is the distance of the skin from the axis of revolution.

$\frac{I_p}{y}$ = the modulus of the section = Z_p. To prevent confusion, we shall use the suffix p to indicate that it is the polar modulus of the section, and not the modulus for bending.

Thus we have $M_t = f_s Z_p$

or the twisting moment = the skin stress × the polar modulus of the section

Shafts subject to Torsion.—To return to the shaft couplings. When power is transmitted from one disc to the other, the pin evidently will be in shear, and will be distorted

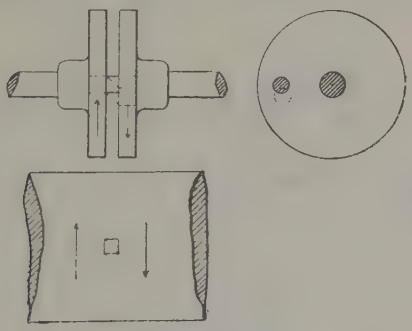


FIG. 562.

as shown (exaggerated). Likewise, if a small square be marked on the surface of a shaft, when the shaft is twisted it will also become a rhombus, as shown dotted on the shaft below.

In Chapter X. we showed that when an element was distorted by shear, as shown in Fig. 563 (*a*), it was

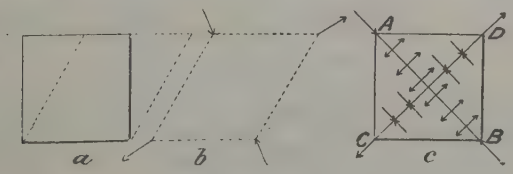


FIG. 563.

equivalent to the element being pulled out at two opposite corners and pushed in at the others, as shown in Fig. 563, (*b*) and (*c*), hence all along the diagonal section AB there

is a tension tending to pull the two triangles ADB, ACB apart; similarly there is a compression along the diagonal CD. These diagonals make an angle of 45° with their sides. Thus, if two lines be marked on a shaft at an angle of 45° with the axis, there will be a tension normal to the one diagonal, and a compression normal to the other. That this is the case can be shown very clearly by getting a piece of thin tube and sawing a diagonal slot along it at an angle of 45°. When the outer end is twisted in the direction of the arrow A, there will be compression normal to the slot, shown by a full line, and the slot will close; but if it be twisted in the direction of the arrow B, there will be tension normal to the slot, and will cause it to open.

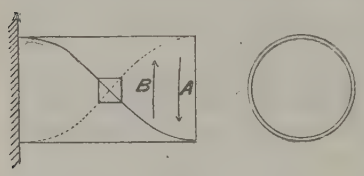


FIG. 564.

Graphical Method of finding the Polar Modulus for a Circular Section.—The method of graphically finding the polar modulus of the section is precisely similar in principle to that given for bending (see Chap. XI.), hence we shall not do more than briefly indicate the construction of the modulus figure. It is of very limited application, as it is *only true for circular sections.*

As in the beam modulus figure, we want to construct a figure to show the distribution of stress in the section.

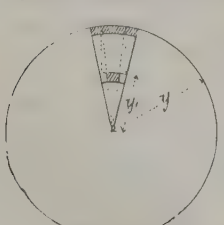


FIG. 565.

Consider a small piece of a circular section as shown, with two blocks equivalent to the pins we used in the disc couplings above. The stress on the inner block $= f_{s1}$, and on the outer block $= f_s$; then $\frac{f_{s1}}{f_s} = \frac{y_1}{y}$. Then by projecting the width of the inner block on to the outer circle, and joining down to the centre of the circle, it is evident, from similar triangles, that we reduce the width and area of the inner block in the ratio $\frac{y_1}{y}$, or in the ratio of $\frac{f_{s1}}{f_s}$. The reduced area of the inner block, shown shaded, we will now term a_1', where $\frac{a_1'}{a_1} = \frac{f_{s1}}{f_s} = \frac{y_1}{y}$, or $a_1' f_s = a_1 f_{s1}$.

Then the *magnitude* of the resultant force acting on the two blocks $= af_s + a_1 f_{s1}$

$= af_s + a_1' f_s = f_s(a + a_1')$

$= f_s$ (shaded area or area of modulus figure)

And the *position* of the resultant is distant y_0 from the centre, where—

$$y_0 = \frac{ay + a_1' y_1}{a + a_1'}$$

i.e. at the centre of gravity of the blocks.

$$\text{Then } f_s Z_p = f_s(a + a_1') \times \frac{ay + a_1' y_1}{a + a_1'}$$

$$= f_s(ay + a_1' y_1) = \frac{f_s}{y}(ay^2 + a_1 y_1^2)$$

which is the same result as we had before for W*l*, thus proving the correctness of the graphical method.

In the figure above we have only taken a small portion of a circle; we will now use the same method to find the Z_p for a

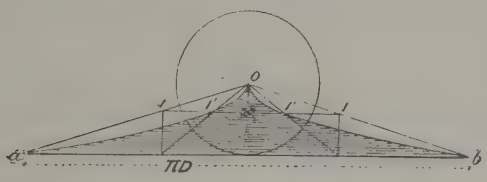


Fig. 566.

circle. For convenience in working, we will set it off on a straight base thus: Draw a tangent *ab* to the circle, making the length $= \pi D$; join the ends to the centre O; draw a series of lines parallel to the tangent; then their lengths intercepted between *ao* and *bo* are equal to the circumference of circles of radii O_1, O_2, etc. Thus the triangle O*ab* represents the circle rolled out to a straight base. Project each of these lines on to the tangent, and join up to the centre; then the width of the line 1′1′, etc., represents the stress in the metal at that layer in precisely the same manner as in the beam modulus figures. Then—

$$\left.\begin{array}{l}\text{The polar modulus of}\\ \text{the section } Z_p\end{array}\right\} = \left\{\begin{array}{l}\text{area of modulus figure} \times \text{distance}\\ \text{of c. of g. of modulus figure from}\\ \text{centre of circle}\end{array}\right.$$

$$\text{or } Z_p = Ay_e$$

The construction for a hollow circle is precisely the same as for the solid circle. It is given for the sake of graphically illustrating the very small amount that a shaft is weakened by making it hollow.

This construction can be applied to any form of section,

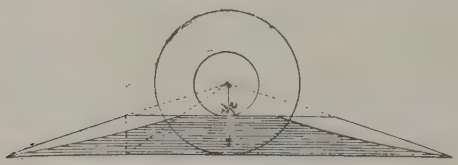


Fig. 567.

but the strengths of shafts other than circular do not vary as their polar moments of inertia or moduli of their sections; serious errors will be involved if they are assumed to be so. The calculation of the stresses in irregular figures in torsion involves fairly high mathematical work. The results of such calculations by St. Venant and Lord Kelvin will be given in tabulated form later on in this chapter.

Strength of Circular Shafts in Torsion.—We have shown above that the strength of a cylindrical shaft varies as $\frac{I_p}{y} = Z_p$. In Chapter III., we showed that $I_p = \frac{\pi D^4}{32}$, where D is the diameter, and y in this case $= \frac{D}{2}$; hence—

$$Z_p = \frac{\frac{\pi D^4}{32}}{\frac{D}{2}} = \frac{\pi D^3}{16} = \frac{D^3}{5{\cdot}1} = 0{\cdot}196D^3$$

which, it will be noticed, is just twice the value of the Z for bending. In order to recollect which is which, it should be remembered that the material in a circular shaft is in the very best form to resist torsion, but in a very bad form to resist bending; hence the torsion modulus will be greater than the bending modulus.

For a hollow shaft—

$$I_p = \frac{\pi(D^4 - D_i^4)}{32}, \text{ where } D_i = \text{the internal diameter}$$

$$\text{hence } Z_p = \frac{\pi(D^4 - D_i^4)}{16D} = 0{\cdot}196\left(\frac{D^4 - D_i^4}{D}\right)$$

If $\frac{1}{n}$ of the metal be removed from the centre of the shaft, we have—

$$\text{The internal area} = \frac{\text{the external area}}{n}$$

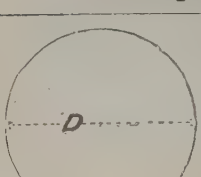
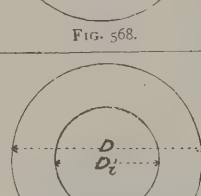
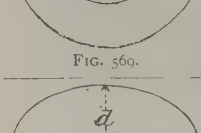
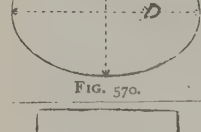
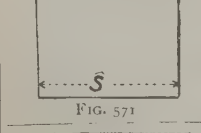
$$\frac{\pi D_i^2}{4} = \frac{\pi D^2}{4n} \qquad \text{or } D_i^4 = \frac{D^4}{n^2}$$

$$Z_p = 0{\cdot}196D^3\left(1 - \frac{1}{n^2}\right)$$

The strength of a shaft with a sunk keyway has never been arrived at by a mathematical process. Experiments show that if the key be made to the usual proportions, viz. the width of the key $= \frac{1}{4}$ diameter of shaft, and the depth of the keyway $= \frac{1}{2}$ width of the key, the shaft is thereby weakened about 19 per cent. See *Engineering*, March 3rd, 1911, page 287.

Another empirical rule which closely agrees with experiments is: The strength of a shaft having a sunk keyway is equal to that of a shaft whose diameter is less than that of the actual shaft by one-half the depth of the keyway; thus, the strength of a 2-inch shaft having a sunk keyway 0·25 inch deep is equal to a shaft $\left(2 - \frac{0{\cdot}25}{2}\right) = 1{\cdot}875$ inches diameter. This rule gives practically the same result as that quoted above.

Strength of Shafts of Various Sections.

 Fig. 568.	$Z_p = \frac{\pi D^3}{16}$, or $Z_p = \frac{D^3}{5{\cdot}1}$, or $Z_p = 0{\cdot}196D^3$	$\theta = \frac{584M_t l}{GD^4}$
 Fig. 569.	$Z_p = \frac{D^4 - D_i^4}{5{\cdot}1D}$, or $Z_p = 0{\cdot}196\left(\frac{D^4 - D_i^4}{D}\right)$ If $D_i = \frac{D}{m}$ $Z = 0{\cdot}196D^3\left(1 - \frac{1}{m^4}\right)$	$\theta = \frac{584M_t l}{G(D^4 - D_i^4)}$
 Fig. 570.	$Z_p = \frac{\pi Dd^2}{16}$, or $Z_p = \frac{Dd^2}{5{\cdot}1}$, or $Z_p = 0{\cdot}196Dd^2$	$\theta = \frac{292M_t l(d^2 + D^2)}{GD^3d^3}$
 Fig. 571	$Z_p = 0{\cdot}208S^3$	$\theta = \frac{410M_t l}{S^4G}$
 Fig. 572.	$Z_p = \frac{Bb^2}{3 + 1{\cdot}8m}$ where $m = \frac{b}{B}$	$\theta = \frac{205M_t l(b^2 + B^2)}{b^3B^3G}$
Any section not containing re-entrant angles (due to St. Venant).	$Z_p = \frac{A^4}{4c\,I_p y}$ (approx.) where A = area of section; I_p = polar moment of inertia of section; y = distance of furthest edge from centre of section.	$\theta = \frac{2290I_pM_t l}{A^4G}$

Twist of Shafts.—In Chapter X., we showed that when an element was sheared, the amount of slide x bore the following relation :—

$$\frac{x}{l} = \frac{f_s}{G} \quad . \quad . \quad . \quad \text{(i.)}$$

where f_s is the shear stress on the material;

G is the coefficient of rigidity.

In the case of a shaft, the x is measured on the curved surface. It will be more convenient if we express it in terms of the angle of twist.

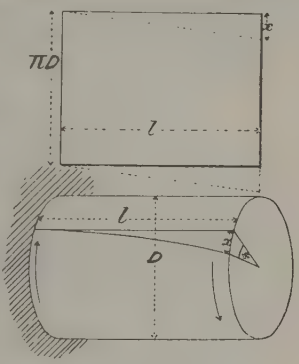


FIG. 573.

If θ_r be expressed in radians, then—

$$x = \frac{\theta_r D}{2} \quad \text{and} \quad \theta_r = \frac{2f_s l}{GD}$$

If θ be expressed in degrees—

$$x = \frac{\pi D \theta}{360}$$

Substituting the value of x in equation (i.), we have—

$$\frac{\pi D \theta}{360 l} = \frac{f_s}{G}, \text{ or } \theta = \frac{360 f_s l}{\pi GD}$$

$$\text{But } M_t = f_s Z_p = f_s \frac{\pi D^3}{16}, \text{ and } f_s = \frac{16 M_t}{\pi D^3}$$

$$\text{hence } \theta = \frac{360 \times 16 \times M_t l}{\pi^2 G D^4} = \frac{584 M_t l}{GD^4}$$

for solid circular shafts. Substituting the value of Z_p for a hollow shaft in the above, we get—

$$\theta = \frac{584 M_t l}{G(D^4 - D_1^4)} \text{ for hollow circular shafts.}$$

N.B.—The stiffness of a hollow shaft is the difference of the stiffness of two solid shafts whose diameters are respectively the outer and inner diameters of the hollow shaft.

When it is desired to keep the twist or spring of shafts within narrow limits, the stress has to be correspondingly

reduced. Long shafts are frequently made very much stronger than they need be in order to reduce the spring. A common limit to the amount of spring is 1° in 20 diameters; the stress corresponding to this is arrived at thus—

$$\text{We have above } \theta = \frac{360 f_s l}{\pi G D}$$

$$\text{But when } \theta = 1^\circ,\ l = 20D$$

$$\text{then } f_s = \frac{\pi G D}{360 \times 20 D} = \frac{G}{2292}$$

For steel, G = 13,000,000; f_s = 5670 lbs. per sq. inch
Wrought iron, G = 11,000,000; f_s = 4800 „ „
Cast iron, G = 6,000,000; f_s = 2620 „ „

In the case of short shafts, in which the spring is of no importance, the following stresses may be allowed:—

Steel, f_s = 10,000 lbs. per sq. inch
Wrought iron, f_s = 8000 „ „
Cast iron, f_s = 3000 „ „

Horse-power transmitted by Shafts.—Let a mean force of P lbs. act at a distance r inches from the centre of a shaft; then—

The twisting moment on the shaft in lbs.-inches } = P (lbs.) × r (inches)

The work done per revolution in foot-lbs. } = $\frac{\text{P (lbs.)} \times r \text{ (inches)} \times 2\pi}{12}$

The work done per minute in foot-lbs. } = $\frac{\text{P (lbs.)} \times r \text{ (inches)} \times 2\pi \text{N (revs.)}}{12}$

where N = number of revolutions per minute.

$$\text{The horse-power transmitted} = \frac{2\pi P r N}{12 \times 33000} = \text{H.P.}$$

then—

$$Pr = \frac{12 \times 33000 \times \text{H.P.}}{2\pi N} = \frac{f_s D^3}{5{\cdot}1}$$

$$D^3 = \frac{12 \times 33000 \times \text{H.P.} \times 5{\cdot}1}{2\pi N f_s} = \frac{321400 \text{ H.P.}}{N f_s}$$

$$= \frac{64{\cdot}3 \text{ H.P.}}{N}$$

taking f_s at 5000 lbs. per square inch.

$$D = 4\sqrt[3]{\frac{H.P.}{N}} \text{ (nearly) for 5000 lbs. per sq. inch}$$

$$= 3{\cdot}5\sqrt[3]{\frac{H.P.}{N}} \text{ for 7500 lbs. per sq. inch}$$

$$= 3\sqrt[3]{\frac{H.P.}{N}} \text{ for 12,000 lbs. per sq. inch}$$

In the case of crank shafts the maximum effort is often much greater than the mean, hence in arriving at the diameter of the shaft the maximum twisting moment should be taken rather than the mean, and where there is bending as well as twisting, it must be allowed for as shown in the next paragraph.

Combined Torsion and Bending.—In Fig. 574 a shaft is shown subjected to torsion only. We have previously seen

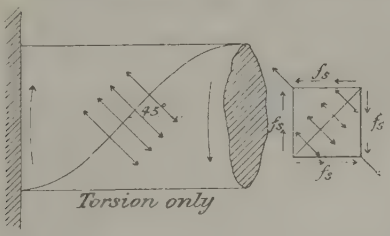


FIG 574.

(Chapter X.) that in such a case there is a tension acting

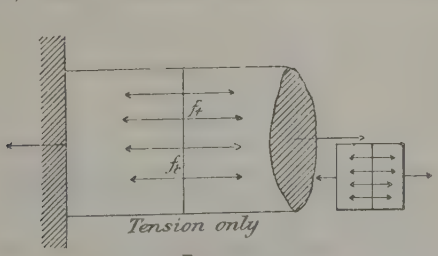


FIG. 575.

normal to a diagonal drawn at an angle of 45° with the axis of the shaft, as shown by the arrows in the figure. In Fig. 575 a shaft is shown subjected to tension only. In this case the tension acts normally to a face at 90° with the axis. In Fig. 576 a shaft is shown subjected to both torsion and

tension; the face on which the normal tensile stress is a maximum will therefore lie between the two faces mentioned above, and the intensity of the stress on this face will be

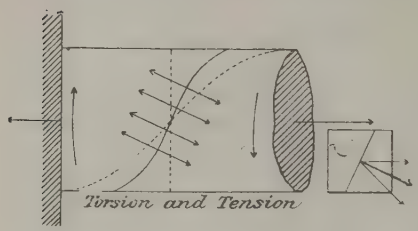


FIG. 576.

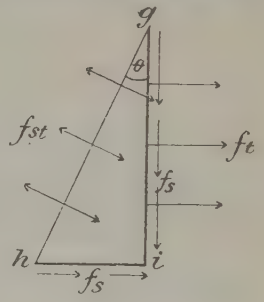


FIG. 577.

greater than that on either of the other faces, when subjected to torsion or tension only.

We have shown in Chapter X. that the stress f_{st} normal to the face g/h due to combined tension and shear is —

$$f_{st} = \frac{f_t}{2} + \sqrt{\frac{f_t^2}{4} + f_s^2}$$

If the tension be produced by bending, we have—

$$f_t = \frac{M}{Z}$$

If the shear be produced by twisting—

$$f_s = \frac{M_t}{Z_p} = \frac{M_t}{2Z}$$

Substituting these values in the above equation—

$$f_{st} = \frac{M}{2Z} + \sqrt{\frac{M^2}{4Z^2} + \frac{M_t^2}{4Z^2}} = \frac{1}{2Z}(M + \sqrt{M^2 + M_t^2})$$

$$f_{st}Z = M_e = \frac{M + \sqrt{M^2 + M_t^2}}{2}$$

also $f_{st} \times 2Z = f_{st}Z_p = M_{st} = M + \sqrt{M^2 + M_t^2}$

The M_e is termed the equivalent bending moment, the M_{et} the equivalent twisting moment, that would produce the same intensity of stress in the material as the combined bending and twisting.

The construction shown in Fig. 578 is a convenient graphical method of finding M_{et}.

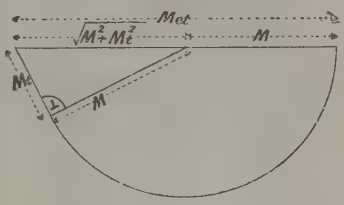


Fig. 578.

In Chapter X. we also showed that—

$$f_s \overline{gi} = f_{st} \overline{gh} \sin \theta \quad \text{and}$$

$$\frac{f_s \overline{gi}}{f_{st} \overline{gh}} = \frac{f_s}{f_{st}} \cos \theta = \sin \theta$$

$$\frac{f_s}{f_{st}} = \frac{\sin \theta}{\cos \theta} = \tan \theta, \text{ or}$$

$$\frac{M_t}{M_{et}} = \tan \theta$$

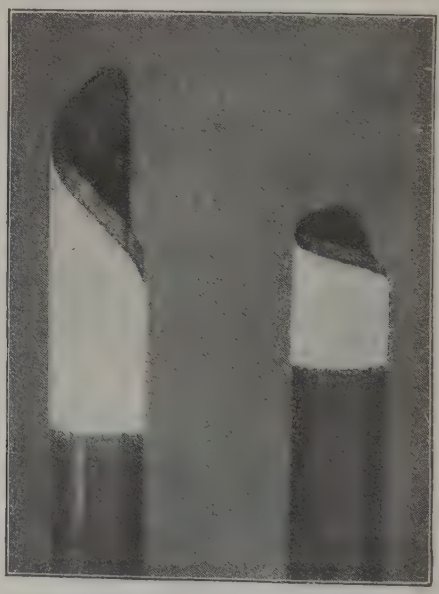
From this expression we can find the angle of greatest normal tensile stress θ, and therefore the angle at which fracture will probably occur, in the case of materials which are weaker in tension than in shear, such as cast iron and other brittle materials.

In Fig. 579 we show the fractures of two cast-iron torsion test-pieces, the one broken by pure torsion, the other by combined torsion and bending. Around each a spiral piece of paper, cut to the theoretical angle, has been wrapped in order to show how the angle of fracture agreed with the theoretical angle θ; the agreement is remarkably close.

The following results of tests made in the author's laboratory show the results that are obtained when cast-iron bars are tested in combined torsion and bending as compared with pure torsion and pure bending tests. The reason why the shear stress calculated from the combined tests is greater than when obtained from pure torsion or shear, is due to the fact that neither of the formulæ ought to be used for stresses up to rupture; however, the results are interesting as a comparison. The angles of fracture agree well with the calculated values.

Twisting moment M_t Pounds-	Bending moment M inches.	Equivalent twisting moment M_{et}	Modulus of rupture f_{st} tons per sq. inch	Angle of fracture. Actual.	Angle of fracture. Calculated.
Zero	2300	4600	25·5	0°	0°
777	1925	4000	26·7	12°	11°
1170	2240	4750	27·1	14°	14°
1228	2255	4820	23·1	17°	15°
1308	2128	4628	24·0	19°	16°
2606	1375	4320	20·8	33°	31°
2644	766	3520	16·2	38°	37°
3084	Zero	3084	16·0	43°	45°
Pure shear ...	...	...	13·0	0°	0°
.. tension ...	...	...	11·5	0°	0°

Mean of a large number of tests.



Pure torsion. Combined torsion and bending.

FIG. 579.

In the case of materials which are distinctly weaker in shear than in tension it is more important to determine the maximum shear stress than the maximum tensile stress, because a shaft subjected to combined bending and torsion will fail in shear rather than in tension.

In Chapter X. it is shown that the maximum shear stress—

$$f_{s\ max.} = \sqrt{\frac{f_t^2}{4} + f_s^2}$$

Hence the equivalent twisting moment which would produce the same intensity of shear stress as the combined bending and twisting is—

$$M_e = \sqrt{M^2 + M_t^2}$$

and the angle at which fracture occurs is at 45° to the face *gh*. Fig. 577.

In the case of ductile materials, in their normal state, the angle of fracture, as found by experiment, undoubtedly does approximately agree with this theory, but in the case of crank shafts broken by repeated stress the fracture more often is in accordance with the maximum tension theory.

The maximum tension theory is generally known as the "Rankine Theory," and the maximum shear theory as the "Guest Theory," named after the respective originators of the two hypotheses.

Example.—A crank shaft is subjected to a maximum bending moment of 300 inch-tons, and a maximum twisting moment of 450 inch-tons. The safe intensity of tensile stress for the material is 5 tons per sq. inch, and for shear 3 tons per sq. inch. Find the diameter of the shaft by the Rankine and the Guest methods.

The equivalent bending moment (Rankine)—

$$= \frac{300 + \sqrt{300^2 + 450^2}}{2} = \frac{300}{2}(1 + \sqrt{1 + 1{\cdot}5^2})$$

$$= 420 \text{ inch-tons.}$$

$$\frac{D^3}{10{\cdot}2} = \frac{420}{5} \qquad D = 9{\cdot}5 \text{ inches.}$$

The equivalent twisting moment (Guest)—

$$= \sqrt{300^2 + 450^2} = 540 \text{ inch-tons.}$$

$$\frac{D^3}{5{\cdot}1} = \frac{540}{3} \qquad D = 9{\cdot}7 \text{ inches.}$$

Helical Springs.—The wire in a helical spring is, to all intents and purposes, subject to pure torsion, hence we can readily determine the amount such a spring will stretch or compress under a given load, and the load it will safely carry.

We may regard a helical spring as a long thin shaft coiled into a helix, hence we may represent our helical spring thus—

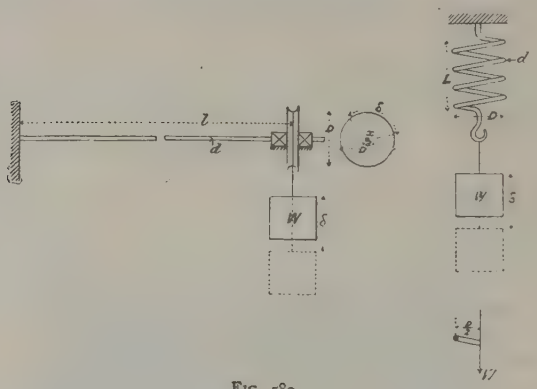


Fig. 580.

In the figure to the left we have the wire of the helical spring straightened out into a shaft, and provided with a grooved pulley of diameter D, *i.e.* the mean diameter of the coils in the spring; hence the twisting moment upon it is $\frac{WD}{2}$. That the twisting moment on the wire when coiled into a helix is also $\frac{WD}{2}$ will be clear from the bottom right-hand figure. The length of wire in the spring (not including the ends and hook) is equal to l. Let n = the number of coils; then $l = \pi Dn$ nearly, or more accurately $l = \sqrt{(\pi Dn)^2 + L^2}$, Fig. 581, a refinement which is quite unnecessary for springs as ordinarily made.

When the load W is applied, the end of the shaft twists, so that a point on the surface moves through a distance x, and a point on the rim of the pulley moves through a distance δ,

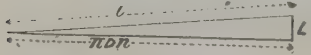


Fig. 581.

where $\frac{\delta}{D} = \frac{x}{d}$, and $\delta = \frac{Dx}{d}$.

But we have $\frac{x}{l} = \frac{f_s}{G}$ $\qquad x = \frac{lf_s}{G}$

hence $\delta = \frac{Dlf_s}{Gd}$ . . . . . (i)

$$\text{Also } \frac{WD}{2} = f_s Z_p = \frac{f_s \pi d^3}{16}$$

$$f_s = \frac{16WD}{2\pi d^3} = \frac{2{\cdot}55WD}{d^3} \quad \cdot \cdot \text{ (ii.)}.$$

$$\text{then } \delta = \frac{2{\cdot}55D^2lW}{Gd^4} \text{ (from i. and ii.)}$$

Substituting the value of $l = n\pi D$—

$$\delta = \frac{2{\cdot}55D^2Wn\pi D}{Gd^4} = \frac{8D^3Wn}{Gd^4}$$

$$\text{G for steel} = 12{,}000{,}000 \quad \delta = \frac{D^3Wn}{1{,}500{,}000d^4}$$

$$\text{G for hard brass} = 5{,}000{,}000 \quad \delta = \frac{D^3Wn}{625{,}000d^4}$$

Safe Load.—From equation (ii.), $W = \frac{f_s d^3}{2{\cdot}55D}$

Experiments by Mr. Wilson Hartnell show that for steel wire the following stresses are permissible :—

Diameter of wire.	Safe stress (f_s).
$\frac{1}{4}$ inch . . .	70,000 lbs. per square inch
$\frac{3}{8}$,, . . .	60,000 lbs. ,, ,,
$\frac{1}{2}$,, . . .	50,000 lbs. ,, ,,

When a spring is required to stretch M times its initial length L, let the initial pitch of the coils be p, then $L = np$

$$\text{From (i.) } \delta = \frac{Dlf_s}{Gd}$$

$$\delta = L(M - 1) = np(M - 1) = \frac{D^2\pi nf_s}{Gd}$$

$$\frac{pd(M - 1)}{D^2} = \frac{\pi f_s}{G} = \frac{3{\cdot}14 \times 70{,}000}{11{,}000{,}000} = \frac{1}{50}$$

$$\text{and } 50(M - 1) = \frac{D^2}{pd}$$

For a close coiled spring, $p = d$, then—

$$\sqrt{50(M - 1)} = \frac{D}{d}$$

Work stored in Springs.

$$\left.\begin{array}{l}\text{The work done in stretching}\\ \text{or compressing a spring}\end{array}\right\} = \frac{W\delta}{2}$$

$$= \frac{f_s d^3 \times D l f_s}{2 \times 2{\cdot}55D \times Gd} \text{ (from i. and iii.)}$$

$$= \frac{f_s^2 d^2 l}{5{\cdot}1G}$$

$$\text{(substituting the value of } l) = \frac{f_s^2 d^2 n D}{1{\cdot}62G} \text{ (inch-lbs.)}$$

$$= \frac{f_s^2 d^2 n D}{19{,}500{,}000}$$

$$\text{putting } G = 12{,}000{,}000$$

Weight of Spring.—Taking the weight of 1 cub. inch of steel = 0·28 lb., then—

The weight of the spring $w = 0{\cdot}785 d^2 l \times 0{\cdot}28 = 0{\cdot}22 d^2 l$

Substituting the value of l, we have—

$$w = 0{\cdot}69 n d^2 D$$

Height a Steel Spring will lift itself (h).

$$h = \frac{\text{work stored in spring}}{\text{weight of spring}}$$

$$h = \frac{f_s^2 d^2 n D}{1{\cdot}62G \times 0{\cdot}69 n d^2 D} = \frac{f_s^2}{1{\cdot}12G} \text{ inches}$$

$$= \frac{f_s^2}{13{\cdot}4G} = \frac{f_s^2}{161{,}000{,}000} \text{ feet}$$

The value of h is given in the following table corresponding to various values of f:—

f_s (lbs. per square inch) ...	30,000	60,000	90,000	120,000	150,000
h (feet)	5·56	22·4	50·3	89·5	139·8

h also gives the number of foot-pounds of energy stored per pound of spring.

All the quantities given above are for springs made of wire of circular section; for wire of square section of side S, and taking the same value for G as before, we get—

$$\left.\begin{aligned}
\delta &= \frac{D^3Wn}{2{,}135{,}000S^4} \text{ (steel springs)} \\
\delta &= \frac{D^3Wn}{890{,}000S^4} \text{ (brass springs)} \\
W &= \frac{2{,}135{,}000S^4\delta}{D^3n} \text{ steel} \\
W &= \frac{890{,}000S^4\delta}{D^3n} \text{ brass}
\end{aligned}\right\} \text{square section}$$

Safe Load for Springs of Square Section.

$$W = \frac{29{,}120S^3}{D} \text{ stress } 70{,}000 \text{ lbs. per square inch}$$

$$= \frac{24{,}960S^3}{D} \quad \text{,, } 60{,}000 \text{ lbs.} \quad \text{,, } \text{,,}$$

$$= \frac{20{,}800S^3}{D} \quad \text{,, } 50{,}000 \text{ lbs.} \quad \text{,, } \text{,,}$$

Taking a mean value, we have—

$$W = \frac{25{,}000S^3}{D} \text{ (square section)}$$

$$\text{work stored} = \frac{f_s^2S^2nD}{24{,}680{,}000} \text{ (inch-lbs.) steel}$$

$$\text{weight} = 0{\cdot}88nS^2D \text{ (steel)}$$

$$\left.\begin{aligned}\text{Height a square-section spring} \\ \text{will lift itself (steel)}\end{aligned}\right\} h = \frac{f_s^2}{260{,}600{,}000} \text{ (feet)}$$

f_s (lbs. per square inch) ...	30,000	60,000	90,000	120,000	150,000
h (feet)	3·45	13·8	31·1	55·3	86·3

It will be observed that in no respect is a square-section spring so economical in material as a spring of circular section.

Helical Spring in Torsion.—When a helical spring is twisted the wire is subjected to a bending moment due to the change of curvature of the spring, which is proportional to the twisting moment.

Let ρ_1 and ρ_2 = the mean radii of the spring in inches before and after twisting respectively;
n_1 and n_2 = the number of free coils before and after twisting respectively;
θ_1 and θ_2 = the angles subtended by the wire in inches before and after twisting respectively;
$= 360n_1$ and $360n_2$ respectively;
$\theta = \theta_1 - \theta_2$, or the angle twisted through by the free end of the spring in degrees;
M_t = the twisting moment in pounds-inches;
I = the moment of inertia of the wire section about the neutral axis in inch units;
d = the diameter or side of the wire in inches;
L = the length of free wire in the spring in inches;
$= 2\pi\rho_1 n_1 = 2\pi\rho_2 n_2$.

$$M_t = EI\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = EI\left(\frac{2\pi n_1}{L} - \frac{2\pi n_2}{L}\right)$$

$$E = \frac{M_t L}{2\pi I(n_1 - n_2)} = \frac{360 M_t L}{2\pi I\theta}$$

$$E = \frac{7340\rho_1 n_1 M_t}{d^4\theta} \text{ for wire of circular section}$$

$$E = \frac{4320\rho_1 n_1 M_t}{d^4\theta} \text{ for wire of square section}$$

If θ_r be the angle of twist expressed in radians, we have—

$$E = \frac{LM_t}{\theta_r I}$$

Open-coiled Helical Spring.—In the treatment given above for helical springs, we took the case in which the coils were close, and assumed that the wire was subjected to torsion only; but if the coils be open, and the angle of the helix be considerable, this is no longer an admissible assumption. Instead of there being a simple twisting moment WR acting on the wire, we have a twisting moment $M_t = WR\cos\alpha$, which twists the wire about an axis *ab*. Think of *ab* as a little shaft attached to the spring wire at *a*, and *ei* as the side view of a circular disc attached to it, then, by twisting this disc, the wire will be subjected to a torsional stress. In addition to this, let *ab* represent the side view of half an annular disc, suitably attached to the wire at *a*, and which rotates about an axis *cd*. Then, by

twisting this disc, the spring wire can be bent; thereby its radius of curvature will be altered in much the same manner as that described in the article on the "Helical Spring in Torsion," the bending moment $M = WR \sin \alpha$. The force which produces the twisting moment acts in the plane of the disc *ei*, and

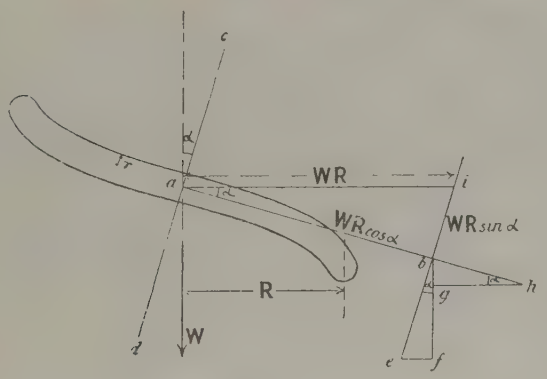


FIG. 582.

that which produces the bending moment in the plane *ab*, *i.e.* normal to the respective sides of the triangle of moments *abi*.

In our expression for the twist of a shaft on p. 579, we gave the angle of twist θ_c in degrees; but if we take it in circular measure, we get $x = r\theta_c$, where r is the radius of the wire, and—

$$\frac{r\theta_c}{l} = \frac{f_s}{G}$$

$$\frac{\theta_c}{l} = \frac{f_s}{rG} = \frac{M_t}{rGZ_p} = \frac{M_t}{GI_p}$$

$$\text{and } \theta_c = \frac{lWR \cos \alpha}{GI_p}$$

Likewise due to bending we have (see p. 590)—

$$\theta_c' = \frac{lM}{EI} = \frac{lWR \sin \alpha}{EI}$$

We must now find how these straining actions affect the axial deflection of the spring.

The twisting moment about *ab* produces a strain $be = R\theta_c$, which may be resolved into two components, viz. one, $ef = R\theta_c \sin \alpha$, which alters the radius of curvature of the coils, and which we are not at present concerned with; and the other, $bf = R\theta_c \cos \alpha$, which alters the axial extension of the spring by an amount $\frac{lWR^2 \cos^2 \alpha}{GI_p}$.

The bending moment about *cd* in the plane of the imaginary disc *ab* produces a strain $bh = R\theta_c'$, of which hg alters the radius of curvature in a horizontal plane, normal to the axis of the spring; and $bg = R\theta_c' \sin \alpha$ alters the axial extension of the spring in the same direction as that due to twisting, by an amount $\frac{lWR^2 \sin^2 \alpha}{EI}$;

$$\left.\begin{array}{l}\text{whence the total axial}\\ \text{extension } \delta\end{array}\right\} = lWR^2\left(\frac{\cos^2 \alpha}{GI_p} + \frac{\sin^2 \alpha}{EI}\right)$$

On substitution and reduction, we get—

$$\delta = \frac{8nD^3W}{Gd^4 \cos \alpha}\left(\cos^2 \alpha + \frac{\sin^2 \alpha}{1{\cdot}25}\right)$$

and for the case of springs in which the bending action is neglected, we get—

$$\delta = \frac{8nD^3W}{Gd^4 \cos \alpha}$$

Angle.	Ratio of $\frac{\text{deflection allowing for bending and torsion}}{\text{deflection allowing for torsion only}}$
5°	0·998
10°	0·992
15°	0·986
20°	0·978
30°	0·950
45°	0·900

Thus for helix angles up to 15° there is no serious error due to the bending of the coils, and when one remembers how many other uncertain factors there are in connection with helical springs, such as finding the *exact* diameter of the wire and coils, the number of free coils, the variation in the value of G, it will be apparent that such a refinement as allowing for the bending of the wire is rarely, if ever, necessary.

CHAPTER XVII.

STRUCTURES.

Wind Pressures.—Nearly all structures at times are exposed to wind pressure. In many instances, the pressure of the wind is the greatest force a structure ever has to withstand.

Let v = velocity of the wind in feet per second;
V = velocity of the wind in miles per hour;
M = mass of air delivered per square foot per second;
W = weight of air delivered per square foot per second (pounds);
w = weight of 1 cubic foot of air (say 0·0807 lb.);
P = pressure of wind per square foot of surface exposed (pounds).

Then, when a stream of air of finite cross-section impinges normally on a flat surface, whose area is much greater than that of the stream, the change of momentum per second per square foot of air stream is—

$$Mv = P$$
$$\text{or } \frac{Wv}{g} = P$$
$$\text{But } W = wv$$
$$\text{hence } \frac{wv^2}{g} = P$$

or expressed in miles per hour by substituting $v = 1{\cdot}466V$, and putting in the value of w, we have—

$$P = \frac{0{\cdot}0807 \times 1{\cdot}466^2 \times V^2}{32{\cdot}2} = 0{\cdot}0054V^2$$

If, however, the section of the air stream is much greater than the area of the flat surface on which it impinges, the change of direction of the air stream is not complete, and consequently the change of momentum is considerably less than (approximately one-half) the value just obtained. Smeaton, from experiments by Rouse, obtained the coefficient 0·005, but

later experimenters have shown that such a value is probably too high. Martin gives 0·004, Kernot 0·0033, Dines 0·0029, and the most recent experiments by Stanton (see *Proceedings I.C.E.*, vol. clvi.) give 0·0027 for the maximum pressure in the middle of the surface on the windward side. In all cases of wind pressure the resultant pressure on the surface is composed of the positive pressure on the windward side, and a suction or negative pressure on the leeward side. Stanton found in the case of circular plates that the ratio of the maximum pressure on the windward side to the negative pressure on the leeward side was 2·1 to 1 in the case of circular plates, and a mean of 1·5 to 1 for various rectangular plates. Hence, for the resultant pressure on plates, Stanton's experiments give as an average $P = 0{\cdot}0036V^2$.

Recent experiments by Eiffel, Hagen, and others show that the total pressure on a surface depends partly on the area of the surface exposed to the wind, and partly on the periphery of the surface. The total pressure P is fairly well represented by an expression of the form—

$$P = (a + bp)SV^2$$

where S = normal surface exposed to the wind in square feet;
p = periphery of the surface in feet.
a and b are constants.

When a wind blowing horizontally impinges on a flat, inclined surface, the pressure in horizontal, vertical, and normal directions may be arrived at thus—

$$\begin{aligned} \text{the normal pressure } P_n &= P \,.\, \sin\theta \\ \text{the horizontal pressure } P_h &= P_n \sin\theta \\ &= P \sin^2\theta \\ \text{the vertical pressure } P_v &= P_n \cos\theta \\ &= P \,.\, \cos\theta \,.\, \sin\theta \end{aligned}$$

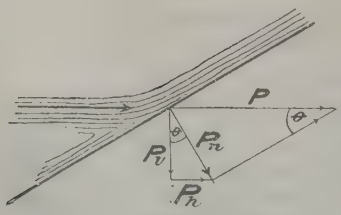


FIG. 583.

In the above we have neglected the friction of the air moving over the inclined surface, which will largely account for the discrepancy between the calculated pressure and that found by experiment. The following table will enable a comparison to be made. The experimental values have been reduced to a *horizontal* pressure of 40 lbs. per

square foot of vertical section of air stream acting on a flat vertical surface.

Angle of roof.	Normal pressure.		Vertical pressure.		Horizontal pressure.	
	Experiment.[1]	$P \sin \theta$.	Experiment.	$P \cos \theta \sin \theta$.	Experiment.	$P \sin^2 \theta$.
10°	9·7	7·0	9·6	6·9	1·7	1·2
20°	18·1	13·7	17·0	12·9	6·2	4·7
30°	26·4	20·0	22·8	17·3	13·2	10·0
40°	33·3	25·7	25·5	19·7	21·4	16·5
50°	38·1	30·6	24·5	19·7	29·2	23·5
60°	40·0	34·6	20·0	17·3	34·0	30·0
70°	41·0	37·6	14·0	12·9	38·5	35·4

When the wind blows upon a surface other than plane, the pressure on the projected area depends upon the form of the surface. The following table gives some idea of the relative wind-resistance of various surfaces, as found by various experiments :—

Flat plate	1
Parachute (concave surface), depth $= \frac{\text{diameter}}{3}$ ...	1·2–2
Sphere	0·36–0·41
Elongated projectile	0·5
Cylinder	0·54–0·57
Wedge (base to wind)	0·8–0·97
,, (edge to wind), vertex angle 90°	0·6–0·7
Cone (base to wind)	0·95
,, (apex to wind), vertex angle 90°	0·69–0·72
,, ,, ,, 60°	0·54
Lattice girders	about 0·8

The pressure and velocity of the wind increase very much as the height above the ground increases (Stevenson's experiments).

Feet above ground	5	9	15	25	52
Velocity in miles per hour ...	4	6	6	6·5	7·5
	7	17	18	21	23
	13	23	25	30	32
	19	28	31	35	40
	26	32	34	37	43

[1] Deduced from Hutton's experiments by Unwin (see "Iron Roofs and Bridges").

These figures appear to show that the pressure varies roughly as the square root of the height above the ground.

The wind pressure as measured by small gauges is always higher than that found from gauges offering a large surface to the wind, probably because the highest pressures are only confined to very small areas, and are much greater than the mean taken on a larger surface.

FORTH BRIDGE EXPERIMENTS.

Date.	Small revolving gauge.	Small fixed gauge.	Large fixed gauge, 15′ × 20′.		
			Mean.	Centre.	Corner.
Mar. 31, 1886	26	31	19	28½	22
Jan. 25, 1890	27	24	18	23½	22

In designing structures, it is usual to allow for a pressure of 40 lbs. per square foot. In very exposed positions this may not be excessive, but for inland structures, unless exceptionally exposed, 40 lbs. is unquestionably far too high an estimate.

In sheltered positions in towns and cities the wind pressure rarely exceeds 10 lbs. per square foot.

For further information on this question the reader should refer to special works on the subject, such as Walmisley's "Iron Roofs" and Charnock's "Graphic Statics" (Broadbent and Co., Huddersfield), Chatley's "The Force of the Wind" (Griffin), Husband and Harby's "Structural Engineering" (Longmans).

Weight of Roof Coverings.—For preliminary estimates, the weights of various coverings may be taken as—

Covering.	Weight per sq. foot in pounds.
Slates	8–9
Tiles (flat)	12–20
Corrugated iron	1½–3½
Asphalted felt	2–4
Lead	5–8
Copper	1–1½
Snow	5

Weight of Roof Structures.—For preliminary estimates, the following formulas will give a fair idea of the probable weight of the ironwork in a roof.

Let W = weight of ironwork per square foot of *covered area* (*i.e.* floor area) in pounds;
D = distance apart of principals in feet;
S = span of roof in feet.
Then for trusses—

$$W = \frac{\sqrt{DS}}{8} + 3$$

and for arched roofs—

$$W = \frac{\sqrt{DS}}{4} + 3$$

Distribution of Load on a Roof.—It will often save trouble and errors if a sketch be made of the load distribution on a roof in this manner.

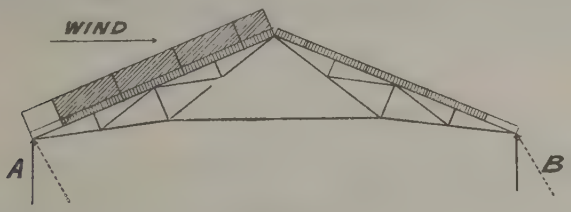


FIG. 584.

The height of the diagram shaded normal to the roof is the weight of the covering and ironwork (assumed uniformly distributed). The height of the diagonally shaded diagram represents the wind pressure on the one side. The lowest section of the diagram on each side is left unshaded, to indicate that if both ends of the structure are rigidly fixed to the supporting walls, that portion of the load may be neglected as far as the structure is concerned. But if A be on rollers, and B be fixed, then the wind-load only on the A side must be taken into account.

When the slope of the roof varies, as in curved roofs, the height of the wind diagram must be altered accordingly. An instance of this will be given shortly.

The whole of the covering and wind-load must be concentrated at the joints, otherwise bending stresses will be set up in the bars.

Method of Sections.—Sometimes it is convenient to check the force acting on a bar by a method known as the

method of sections—usually attributed to Ritter, but really due to Rankine—termed the method of sections, because the structure is supposed to be cut in two, and the forces required to keep it in equilibrium are calculated by taking moments.

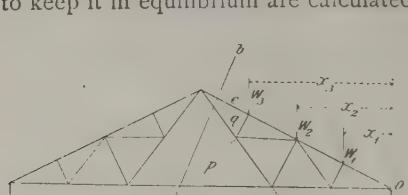


FIG. 585.

Suppose it be required to find the force acting along the bar *pq*. Take a section through the structure *ab*; then three forces, *pa*, *qe*, *pq*, must be applied to the cut bars to keep the structure in equilibrium. Take moments about the point O. The forces *pa* and *qe* pass through O, and therefore have no moment about it; but *pq* has a moment $pq \times y$ about O.

$$\overline{pq} \times y = W_1x_1 + W_2x_2 + W_3x_3$$

$$\overline{pq} = \frac{W_1x_1 + W_2x_2 + W_3x_3}{y}$$

By this method forces may often be arrived at which are difficult by other methods.

Forces in Roof Structures.—We have already shown in Chapter IV. how to construct force or reciprocal diagrams for simple roof structures. Space will only allow of our now dealing with one or two cases in which difficulties may arise.

In the truss shown (Fig. 586), a difficulty arises after the force in the bar *tu* has been found. Some writers, in order to get over the difficulty, assume that the force in the bar *rs* is the same as in *ut*. This may be the case when the structure is evenly loaded, but it certainly is not so when wind is acting on one side of the structure. We have taken the simplest case of loading possible, in order to show clearly the special method of dealing with such a case.

The method of drawing the reciprocal diagram has already been described. We go ahead in the ordinary way till we reach the bar *st* (Fig. 586). In the place of *sr* and *rq* substitute a temporary bar *xy*, shown dotted in the side figure. With this alteration we can now get the force *ey* or *eq*; then *qr*, *rs*, etc., follow quite readily; also the other half of the structure.

There are other methods of solving this problem, but the one given is believed to be the best and simplest. The author

is indebted to Professor Barr, of Glasgow University, for this method.

When the wind acts on a structure, having one side fixed and the other on rollers, the only difficulty is in finding the reactions. The method of doing this by means of a funicular polygon is shown in Fig. 587. The funicular polygon has been fully described in Chapters IV. and XII., hence no further description is necessary. The direction of the reaction at the fixed support

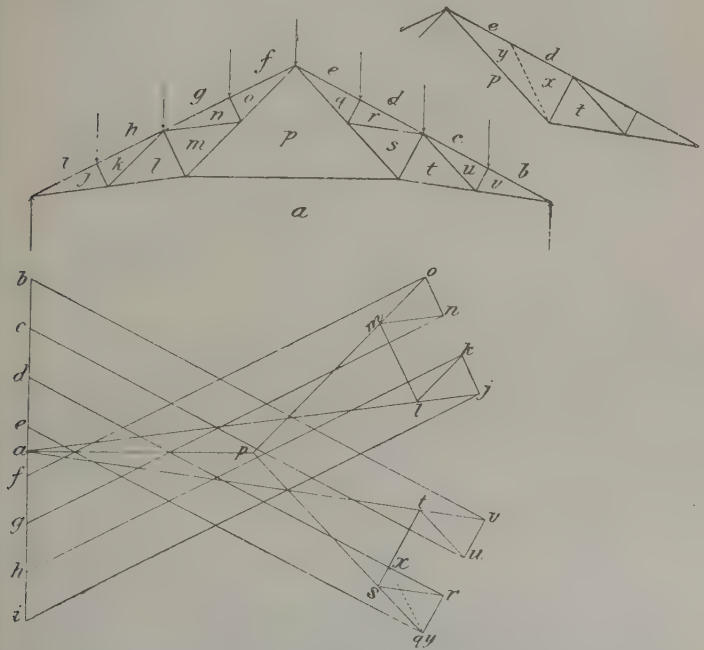


Fig. 586.

is unknown, but as it must pass through the point where the roof is fixed, the funicular polygon should be started from this point.

The direction of the reaction at the roller end is vertical, hence from *j* in the vector polygon a perpendicular is dropped to meet the ray drawn parallel to the closing line of the funicular polygon. This gives us the point *a*; then, joining *ba*, we get the direction of the fixed reaction. The reciprocal

diagram is also constructed; it presents no difficulties beyond that mentioned in the last paragraph.

In the figure, the vertical forces represent the dead weight on the structure, and the inclined forces the wind. The two are combined by the parallelogram of forces.

In designing a structure, a reciprocal diagram must be drawn for the structure, both when the wind is on the roller

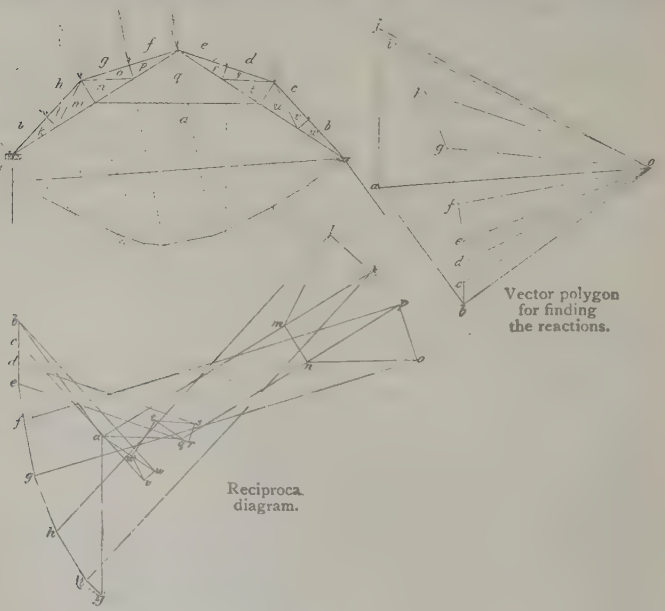


FIG. 587.

and on the fixed side of the structure, and each member of the structure must be designed for the greatest load.

The nature of the forces, whether compressive or tensional, must be obtained by the method described in Chapter IV.

Island Station Roof.—This roof presents one or two interesting problems, especially the stresses in the main post. The determination of the resultant of the wind and dead load at each joint is a simple matter. The resultant of all the forces is given by *pa* on the vector polygon in magnitude and direction (Fig. 588). Its position on the structure must then be determined. This has been done by constructing a funicular polygon

in the usual way, and producing the first and last links to meet in the point Q. Through Q a line is drawn parallel to *pa* in

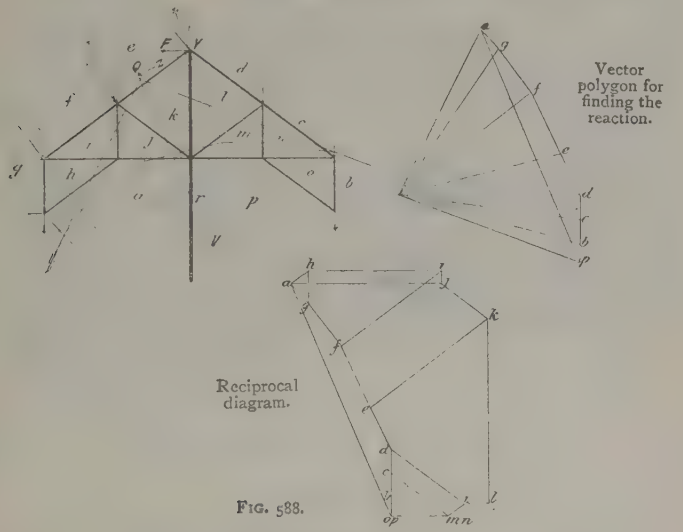


FIG. 588.

the vector polygon. This resultant cuts the post in *r*, and may be resolved into its horizontal and vertical components, the horizontal component producing bending moments of different sign, thus giving the post a double curvature (Fig. 589).

FIG. 589.

The bending moment on the post is obtained by the product $\overline{pa} \times Z$, where Z is the perpendicular distance of Q from the apex of the post.

When using reciprocal diagrams for determining the stresses in structures, we can only deal with direct tensions and compressions. But in the present instance, where there is bending in one of the members, we must introduce an imaginary external force to prevent this bending action. It will be convenient to assume that the structure is pivoted at the virtual joint *r*, and that an external horizontal force F is introduced at the apex Y to keep the structure in equilibrium. The value of F is readily found thus. Taking moments about *r*, we have—

$$F \times rY = \overline{pa} \times z$$

$$\text{or } F = \frac{\overline{pa} \cdot z}{rY}$$

z is the perpendicular distance from the point Y to the resultant.

On drawing the reciprocal diagram, neglecting F, it will be found that it will not close. This force is shown dotted on the reciprocal diagram $l\,i$, and on measurement will be found to be equal to F.

Dead and Live Loads on Bridges.—The dead loads consist of the weight of the main and cross girders, floor, ballast, etc., and, if a railway bridge, the permanent way; and the live loads consist of the train or other traffic passing over the bridge, and the wind pressure.

The determination of the amount of the dead loads and the resulting bending moment is generally quite a simple

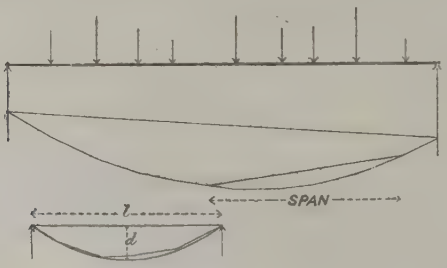


FIG. 590.

matter. In order to simplify matters, it is usual to assume (in small bridges) that the dead load is evenly distributed, and consequently that the bending-moment diagram is parabolic.

In arriving at the bending moment on railway bridges, an equivalent evenly distributed load is often taken to represent the actual but somewhat unevenly distributed load due to a passing train. The maximum bending moment produced by a train which covers a bridge (treated as a standing load) can be arrived at thus (Fig. 590). Take a span greater than twice the actual span, so as to get every possible combination of loads that may come on the structure. Construct a bending-moment diagram in the ordinary way, then find by trial where the greatest bending moment occurs, by fitting in a line whose length is equal to the span. A parabola may then be drawn to enclose

this diagram as shown in the lower figure; then, if d = depth of this parabola to proper scale, we have $\frac{wl^2}{8} = d$, where w is the equivalent evenly distributed load due to the train. (The small diagram is not to scale in this case.)

Let W_r = total rolling load in tons distributed on each pair of rails;
S = span in feet.

Then for English railways $W_r = 1{\cdot}6S + 20$

Maximum Shear due to a Rolling Load. *Concentrated Rolling Load.*—The shear at any section is equal to the algebraic sum of the forces acting to the right or left of any section, hence the shear between the load and either abutment is equal to the reaction at the abutment. We have above—

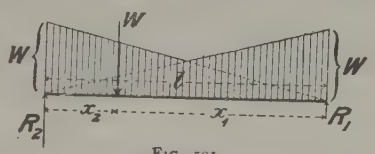


Fig. 591.

$$R_1 l = W x_2$$

$$R_1 = W\frac{x_2}{l}$$

$$\text{likewise } R_2 = W\frac{x_1}{l}$$

When the load reaches the abutment, the shear becomes W, and when in the middle of the span the shear is $\frac{W}{2}$. The shear diagram is shown shaded vertically.

Uniform Rolling Load, whose Length exceeds the Span.—Let

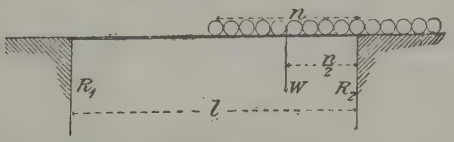


Fig. 592.

w = the uniform live load per foot run, and w_d = the uniform dead load per foot run.

$$\text{Let } M = \frac{w_d}{w}$$

The total load on the structure $= wn = W$

$$\text{then } R_1 l = \frac{Wn}{2} = \frac{wn^2}{2}$$

$$R_1 = \frac{w}{2l} n^2 = Kn^2$$

where K is a constant for any given case. But as the train passes from the right to the left abutment, the shear between the head of the train and the left abutment is equal to the left reaction R_1, but this varies as the square of the covered length of the bridge, hence the curve of shear is a parabola.

When the train reaches the abutment, $l = n$;

$$\text{then } R_1 = \frac{wn}{2} = \frac{W}{2}$$

The curves of shear for both dead and live loads are shown in Fig. 593. When a train passes from right to left over the point f, the shear is reversed in sign, because the one shear is positive, and the other negative. The distance x between the two points f, f is known as the "focal distance" of the bridge.

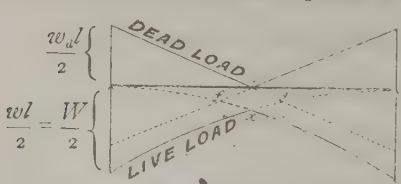


FIG. 593.

The focal distance x can be calculated thus: The point f occurs where the shear due to the live load is equal to that due to the dead load.

$$\text{The shear due to the live load} = \frac{wn^2}{2l}$$

$$\text{,, ,, dead ,,} = \frac{w_d x}{2} = w_d\left(\frac{l}{2} - n\right)$$

$$\frac{wn^2}{2l} = w_d\left(\frac{l}{2} - n\right)$$

Solving, we get—

$$n = l\sqrt{M + M^2} - lM$$

$$\text{and } x = l - 2n$$

$$= l(1 - 2\sqrt{M + M^2} + 2M)$$

Determination of the Forces acting on the Members of a Girder.—In the case of the girder given

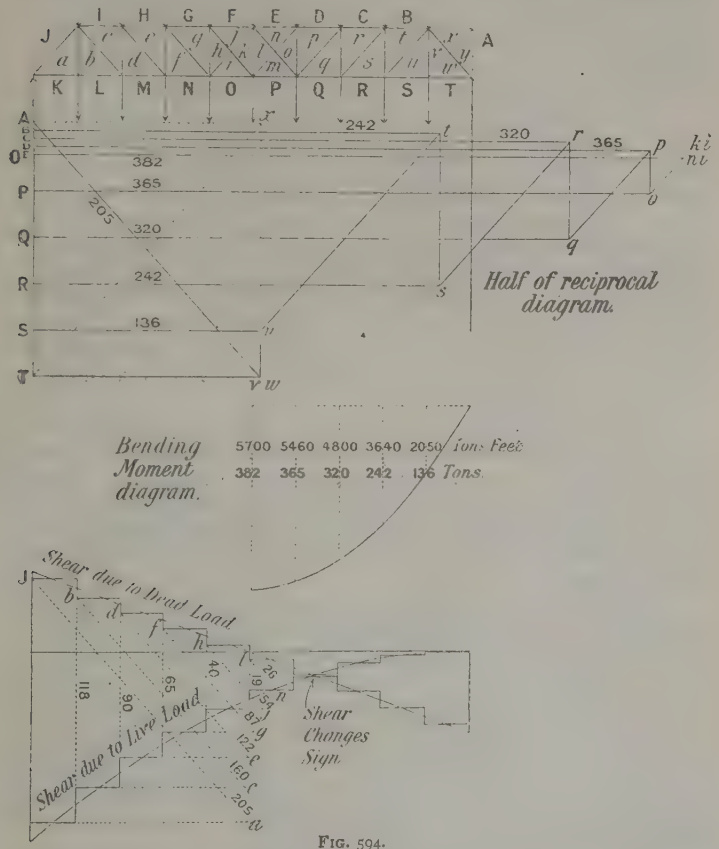


FIG. 594.

as an example, the dead load $w_d = 0{\cdot}75$ ton per foot, or $\frac{0{\cdot}75 \times 13{\cdot}5}{2} = 5$ tons per joint.

The live load $w = 1{\cdot}75$ ton per foot run, or $1{\cdot}75 \times 13{\cdot}5 = 23{\cdot}6$ tons on each bottom joint, thus giving 5 tons on each top joint, and 28·6 tons on each bottom joint.

The forces acting on the booms can be obtained either by

constructing a reciprocal diagram for the structure when fully loaded as shown, or by constructing the bending-moment diagram for the same conditions. The depth of this diagram at any section measured on the moment scale, divided by the depth of the structure, gives the force acting on the boom at that section. The results should agree if the diagrams are carefully constructed.

The forces in the bracing bars, however, cannot be obtained by these methods, unless separate reciprocal diagrams are constructed for several (in this instance six) positions of the train, since the force in each bar varies with each position of the train. The nature of the stress on the bracing bars within the focal length changes as the train passes; hence, instead of designing the focal members to withstand the reversal of stress, it is usual, for economic reasons, to counterbrace these panels with two tie-bars, and to assume that at any given instant only one of the bars is subjected to stress, viz. that bar which at the instant is in tension, since the other tie-bar is not of a suitable section to resist compression.

All questions relating to moving loads on structures are much more readily solved by "Influence Lines" than by the tedious method of constructing reciprocal diagrams for each position of the moving load. By this means the greatest stresses which occur in the various members of the structure can be determined in a fraction of the time required by the older methods.

Deflection of Braced Structures.—The deflection produced by any system of loading can be calculated either algebraically by equating the work done by the external forces to the internal elastic work done on the various members of the structure, or graphically by means of the Williott diagram.

Readers should refer to Warren's "Engineering Construction in Iron, Steel, and Timber," Fidler's "Practical Treatise on Bridge Construction," Husband and Harby's "Structural Engineering," Burr and Falk's "Influence Lines for Bridges and Roofs."

Girder with a Double System of Triangulation.—Most girders with double triangulation are statically indeterminate, and have to be treated by special methods. They can, however, generally be treated by reciprocal diagrams without any material error (Fig. 595). We will take one simple case to illustrate two methods of treatment. In the first each system is treated separately, and where the members overlap, the forces must be added: in the second the whole diagram will

be constructed in one operation. The same result will, of course, be obtained by both methods.

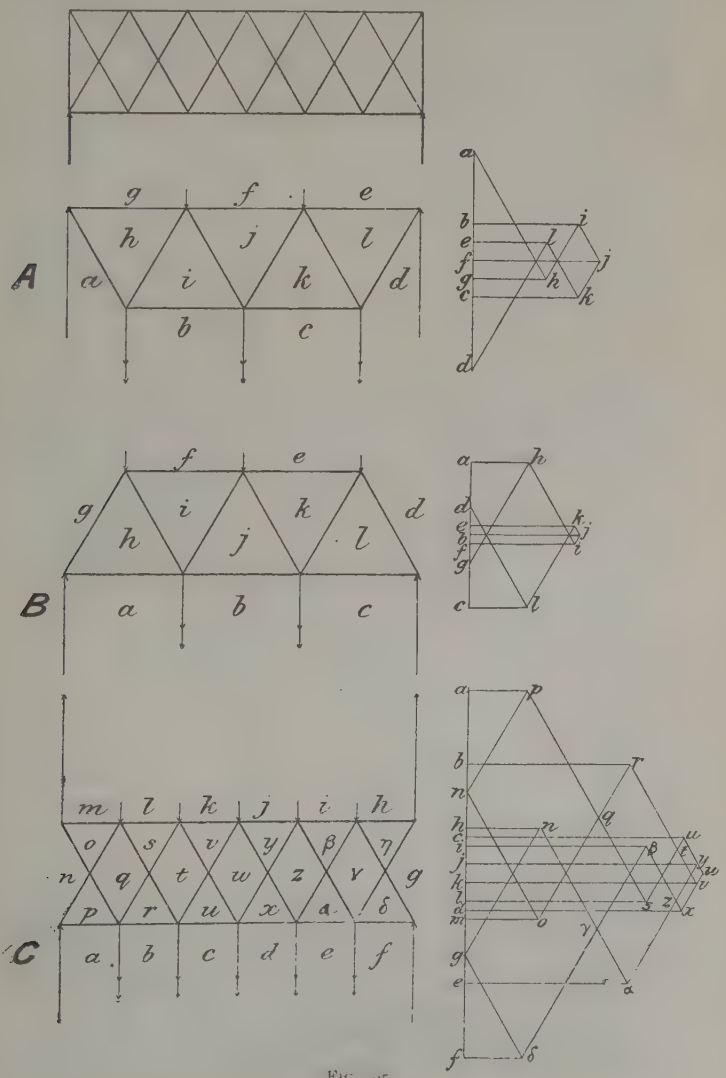


FIG. 395.

In dealing with the second method, the forces *mn*, *hg* acting on the two end verticals are simply the reactions of Fig. A. There is less liability to error if they are treated as two upward forces, as shown in Fig. C, than if they are left in as two vertical bars. It will be seen, from the reciprocal diagram, that the force in *qs* is the same as that in *rt*, which, of course, must be the case, as they are one and the same bar.

Incomplete and Redundant Framed Structures.—If a jointed structure have not sufficient bars to make it retain its original shape under all conditions of loading, it is termed an "incomplete" structure. Such a structure may, however, be used in practice for one special distribution of loading which never varies, but if the distribution should ever be altered, the structure will change its shape. The determination of the forces acting on the various members can be found by the reciprocal diagram.

But if a structure have more than sufficient bars to make it retain its original shape, it is termed a "redundant" structure. Then the stress on the bars depends entirely upon their relative yielding when loaded, and cannot be obtained from a reciprocal diagram. Such structures are termed "statically indeterminate structures." Even the most superficial treatment would occupy far too much space. If the reader wishes to follow up the subject, he cannot do better than consult an excellent little book on the subject, "Statically Indeterminate Structures," by Martin, published at *Engineering* Office.

Pin Joints.—In all the above cases we have assumed that all the bars are jointed with frictionless pin joints, a condition which, of course, is never obtained in an actual structure. In American bridge practice pin joints are nearly always used, but in Europe the more rigid riveted joint finds favour. When a structure deflects under its load, its shape is slightly altered, and consequently bending stresses are set up in the bars when rigidly jointed. Generally speaking, such stresses are neglected by designers.

Plate Girders.—It is always assumed that the flanges of a rectangular plate girder resist the whole of the bending stresses, and that the web resists the whole of the shear stresses. That such an assumption is not far from the truth is evident from the shear diagram given on p. 596.

In the case of a parabolic plate girder, the flanges take some of the shear, the amount of which is easily determined.

The determination of the bending moment by means of a diagram has already been fully explained. The bending

moment at any point divided by the corresponding depth of the girder gives the total stress in the flanges, and this, divided by the intensity of the stress, gives the net area of one flange. In the rectangular girder the total flange stress will be greatest in the middle, and will diminish towards the abutments, consequently the section of the flanges should correspondingly diminish. This is usually accomplished by keeping the width of the flanges the same throughout, and reducing the thickness by reducing the number of plates. The bending-moment diagram lends itself very readily to the stepping of the plates. Thus suppose it were found that four thicknesses of plate were required to resist the bending stresses in the flanges in the middle of the girder; then, if the bending-moment diagram be divided into four strips of equal thickness, each strip will represent one plate. If these strips be projected up on to the flange as shown, it gives the position where the plates may be stepped.[1]

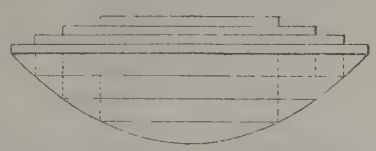


FIG. 597.

The shear in the web may be conveniently obtained from the shear diagram (see Chapter XII.).

Then if S = shear at any point in tons,

f_s = permissible shear stress, usually not exceeding 3 tons per square inch of gross section of web,

d_w = depth of web in inches,

t = thickness of plate in inches (rarely less than $\frac{3}{8}$ inch),

we have—

$$S = f_s d_w t$$

The depth is usually decided upon when scheming the girder; it is frequently made from $\frac{1}{8}$ to $\frac{1}{12}$ span. The thickness of web is then readily obtained. If on calculation the thickness comes out less than $\frac{3}{8}$ inch, and it has been decided not to use a thinner web, the depth in some cases is decreased accordingly within reasonable limits.

[1] It is usual to allow from 6 inches to 12 inches overlap of the plates beyond the points thus obtained.

The web is attached to the flanges or booms by means of angle irons arranged thus:

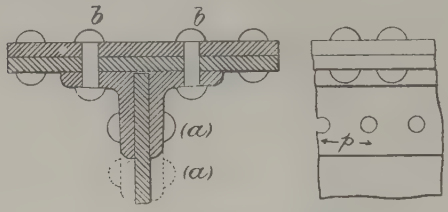


FIG. 598.

The pitch p of the rivets must be such that the bearing and shearing stresses are within the prescribed limits.

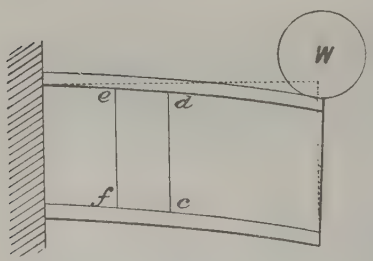


FIG. 599.

On p. 389 we showed that, in the case of any rectangular element subject to shear, the shear stress is equal in two directions at right angles, *i.e.* the shear stress along ef = shear stress along ed, which has to be taken by the rivets a, a, Fig. 598.

The shear per (gross) inch run of web plate along ef is $f_s t = \frac{S}{d_w}$.

The shearing resistance of each rivet is (in double shear) $\frac{2\pi d^2 f_r}{4} = 1{\cdot}57 d^2 f_r = f_s t p$.

Whence, to satisfy shearing conditions—

$$p = \frac{1{\cdot}57 d^2}{t} \cdot \frac{f_r}{f_s}$$

This pitch is, however, often too small to be convenient; then two (zigzag) rows of rivets are used, and p = twice the above value.

The bearing resistance of each rivet is—

$$d t f_b = f_s t p$$

The bearing pressure f_b is usually taken at about 8 tons per square inch. We get, to satisfy bearing-pressure conditions—

$p = 3d$ (for a single row of rivets) (approximately)
$p = 6d$ (for a double row of rivets) „

The joint-bearing area of the two rivets *b*, *b* attaching the angles to the booms is about twice that of a single rivet (*a*) through the web; hence, as far as bearing pressure is concerned, single rows are sufficient at *b*, *b*. A very common practice is to adopt a pitch of 4 inches, putting two rows in the web at *a*, *a*, and single rows at *b*, *b*.

The pitch of the rivets in the vertical joints of the web (with double cover plates) is the same as in the angles.

The shear diminishes from the abutments to the middle of the span, hence the thickness of the web plates may be diminished accordingly. It often happens, however, that it is more convenient on the whole to keep the web plate of the same thickness throughout. The pitch p of the rivets may then

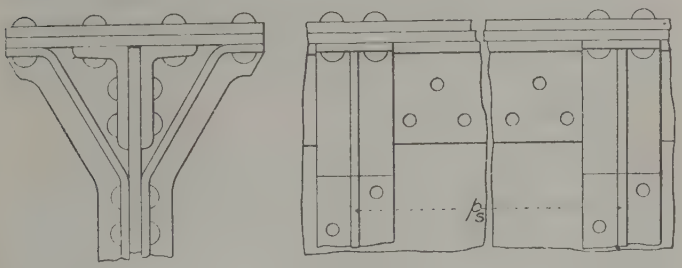


FIG. 600.

be increased towards the middle. It should be remembered, however, that several changes in the pitch may in the end cost more in manufacture than keeping the pitch constant, and using more rivets.

The rivets should always be arranged in such a manner that not more than two occur in any one section, in order to reduce the section of the angles as little as possible.

Practical experience shows that if a deep plate girder be constructed with simply a web and two flanges, the girder will not possess sufficient lateral stiffness when loaded. In order to provide against failure from this cause, vertical tees, or angles, are riveted to the web and flanges, as shown in Fig. 600. That

such stiffeners are absolutely necessary in many cases none will deny, but up to the present no one appears to have arrived at a satisfactory theory as to the dimensions or pitch required. Rankine, considering that the web was liable to buckle diagonally, due to the compression component of the shear, treated a narrow diagonal strip of the web as a strut, and proceeded to calculate the longest length permissible against buckling. Having arrived at this length, it becomes a simple matter to find the pitch of the stiffeners, but, unfortunately for this theory, there are a large number of plate girders that have been in constant use for many years which show no signs of weakness, although they ought to have buckled up under their ordinary working load if the Rankine theory were correct. The Rankine method of treatment is, however, so common that we must give our reasons for considering it to be wrong in principle. From the theory of shear, we know that a pure shear consists of two forces of equal magnitude and opposite in kind, acting at right angles to one another, each making an angle of 45° with the roadway; hence, whenever one diagonal strip of a web is subjected to a compressive stress, the other diagonal is necessarily subjected to a tensile stress of equal intensity. Further, we know that if a long strut of length l be supported laterally (even very flimsily) in the middle, the effective length of that strut is thereby reduced to $\frac{l}{2}$, and in general the effective length of any strut is the length of its longest unstayed segment. But even designers who adhere to the Rankinian theory of plate webs act in accordance with this principle when designing latticed girders, in which they use thin flat bars for compression members, which are quite incapable of acting as long struts. But, as is well known, they do not buckle simply because the diagonal ties to which they are attached prevent lateral deflection, and the closer the lattice bracing the smaller is the liability to buckling; hence there is no tendency to buckle in the case in which the lattice bars become so numerous that they touch one another, or become one continuous plate, since the diagonal tension in the plate web effectually prevents buckling along the other diagonal, *provided that the web is subjected to shear only*. What, then, is the object of using stiffeners? Much light has recently been thrown on this question by Mr. A. E. Guy (see "The Flexure of Beams:" Crosby Lockwood and Son), who has very thoroughly, both experimentally and analytically, treated the question of the twisting of deep narrow sections. It is well

known that for a given amount of material the deeper and narrower we make a beam of rectangular section the stronger will it be if we can only prevent it from twisting sideways. Mr. Guy has investigated this point, and has made the most important discovery that the load at which such a beam will buckle sideways is that load which would buckle the same beam if it were placed vertically, and thereby converted into a strut.

If readers will refer to the published accounts ("Menai and Conway Tubular Bridges," by Sir William Fairbairn) of the original experiments made on the large models of the Menai tubular bridge by Sir William Fairbairn, they will see that failure repeatedly occurred through the twisting of the girders; and in the later experiments two diagonals were put in in order to prevent this side twisting, and finally in the bridge itself the ends of the girders were supported in such a way as to prevent this action, and in addition substantial gusset stays were riveted into the corners for the same purpose. Some tests by the author on a series of small plate girders of 15 feet span, showed in every case that failure occurred through their twisting.

The primary function, then, of web stiffeners is believed to be that of giving torsional rigidity to the girder to prevent side twisting, but the author regrets that he does not see any way of calculating the pitch of plate or tee-stiffeners to secure the necessary stiffness; he trusts, however, that, having pointed out what he believes to be the true function of stiffeners, others may be persuaded to pursue the question further.

This twisting action appears to show itself most clearly when the girder is loaded along its tension flange, *i.e.* when the compression flange is free to buckle. Probably if the load were evenly distributed on flooring attached to the compression flange, there would be no need for any stiffeners, because the flooring itself would prevent side twisting; in fact, in the United States one sees a great many plate girders used without any web stiffeners at all when they are loaded in this manner.

But if the flooring is attached to the bottom of the girder, leaving the top compression flange without much lateral support, stiffeners will certainly be required to keep the top flange straight and parallel with the bottom flange. The top flange in such a case tends to pivot about a vertical axis passing through its centre. For this reason the ends should be more rigidly stiffened than the middle of the girder, which is, of course, the common practice, but it is usually assigned to another cause.

There are, however, other reasons for using web stiffeners. Whenever a concentrated load is applied to either flange it

produces severe local stresses; for example, when testing rolled sections and riveted girders which, owing to their shortness, do not fail by twisting, the web always locally buckles just under the point of application of the load. This local buckling is totally different from the supposed buckling propounded by Rankine. By riveting tee-stiffeners on both sides of the web, the local loading is more evenly distributed over it, and the buckling is thereby prevented. Again, when a concentrated load is locally applied to the lower flange, it tends to tear the flange and angles away from the web. Here again a tee or plate stiffener well riveted to the flange and web very effectually prevents this by distributing the load over the web.

In deciding upon the necessary pitch of stiffeners p_s, there should certainly be one at every cross girder, or other concentrated load, and for the prevention of twisting, well-fitted plate stiffeners near the ends, pitched empirically about 2 feet 6 inches to 3 feet apart; then alternate plate and tee stiffeners, increasing in pitch to not more than 4 feet, appear to accord with the best modern practice.

Weight of Plate Girders.—For preliminary estimates, the weight of a plate girder may be arrived at thus—

Let w = weight of girder in tons per foot run;

W = total load on the girder, not including its own weight, in tons.

$$\text{Then } w = \frac{W}{500} \text{ roughly}$$

Arched Structures.—We have already shown how to determine the forces acting on the various segments of a suspension-bridge chain. If such a chain were made of suitable form and material to resist compression, it would, when inverted, simply become an arch. The exact profile taken up by a suspension-bridge chain depends entirely upon the distribution of the load, but as the chain is in tension, and, moreover, in stable equilibrium, it immediately and automatically adjusts itself to any altered condition of loading; but if such a chain were inverted and brought into compression, it would be in a state of unstable equilibrium, and the smallest disturbance of the load distribution would cause it to collapse immediately. Hence arched structures must be made of such a section that they will resist change of shape in profile; in other words, they must be capable of resisting bending as well as direct stresses.

Masonry Arches.—In a masonry arch the permissible

bending stress is small in order to ensure that there may be no, or only a small amount of, tensile stress on the joints of the voussoirs, or arch stones. Assuming for the present that there may be no tension, then the resultant line of thrust must lie within the middle third of the voussoir (see p. 541). In order to secure this condition, the form of the arch must be such that under its normal system of loading the line of thrust must pass through or near the middle line of the voussoirs. Then, when under the most trying conditions of live loading, the line of thrust must not pass outside the middle third. This condition can be secured either by increasing the depth of the voussoirs, or by increasing the dead load on the arch in order to reduce the ratio of the live to the dead load. Many writers still insist on the condition that there shall be no tension in the joints of a masonry structure, but every one who has had any experience of such structures is perfectly well aware that there are very few masonry structures in which the joints do not tend to open, and yet show no signs of instability or unsafeness. There is a limit, of course, to the amount of permissible tension. If the line of thrust pass *through* the middle third, the maximum intensity of compressive stress on the edge of the voussoir is twice the mean, and if there be no adhesion between the mortar and the stones, the intensity of compressive stress is found thus—

Let T = the thrust on the voussoirs at any given joint per unit width;

d = the depth or thickness of the voussoirs;

x = the distance of the line of thrust from the middle of the joint;

P = the maximum intensity of the compressive stress on the loaded edge of the joint.

The distance of the line of thrust from the loaded edge is $\frac{d}{2} - x$, and since the stress varies directly as the distance from this edge, the diagram of stress distribution will be a triangle with its centre of gravity in the line of thrust, and the area of the triangle represents the total stress; hence—

$$\frac{P \times 3\left(\frac{d}{2} - x\right)}{2} = T$$

$$\text{and } P = \frac{2T}{3\left(\frac{d}{2} - x\right)}$$

$$\text{The mean stress on the section} = \frac{T}{d}$$

hence, when—

$$x = \frac{d}{6}, \quad P = 2 \times \text{mean intensity of stress}$$

$$x = \frac{d}{4}, \quad P = 2\tfrac{2}{3} \times \text{mean intensity}$$

A masonry arch may fail in several ways; the most important are—

1. By the crushing of the voussoirs.
2. By the sliding of one voussoir over the other.
3. By the tilting or rotation of the voussoirs.

The first is avoided by making the voussoirs sufficiently deep, or of sufficient sectional area to keep the compressive stress within that considered safe for the material. This condition is fulfilled if—

$$\frac{2{\cdot}67\,T}{d} = \text{or} < \text{the permissible compressive stress}$$

The depth of the arch stones or voussoirs d in feet at the keystone can be approximately found by the following expressions:—

Let S = the clear span of the arch in feet;
R_a = the radius of the arch in feet.

$$\text{Then } d = \frac{\sqrt{S}}{2{\cdot}2} \text{ for brick to } \frac{\sqrt{S}}{3} \text{ for masonry}$$

or, according to Trautwine—

$$d = \left(\frac{\sqrt{R_a + \frac{S}{2}}}{4} + 0{\cdot}2 \right) K$$

where K = 1 for the best work;
1·12 for second-class work;
1·33 for brickwork.

The second-mentioned method of failure is avoided by so arranging the joints that the line of thrust never cuts the normal to the joint at an angle greater than the friction angle. Let *tt* be the line of thrust, then sliding will occur in the manner

shown by the dotted lines, if the joints be arranged as shown at *bb* –that is, if the angle α exceeds the friction angle ϕ. But if the joint be as shown at *aa*, sliding cannot occur.

The third-mentioned method of failure only occurs when the line of thrust passes right outside the section; the voussoirs then tilt till the line of thrust passes through the pivoting points. An arch can never fail in this way if the line of thrust be kept inside the middle halt.

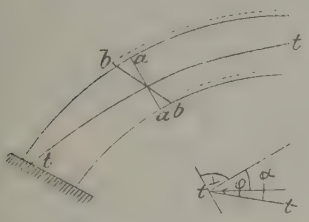


FIG. 601.

FIG. 602.

The rise of the arch R (Fig. 505) will depend upon local conditions, and the lines of thrust for the various conditions of loading are constructed in precisely the same manner as the link-and-vector polygon. A line of thrust is first constructed for the distributed load to give the form of the arch, and if the line of thrust comes too high or too low to suit the desired rise, it is corrected by altering the polar distance. Thus, suppose the rise of the line of thrust were R_0, and it was required to bring it to R_1. If the original polar distance were O_0H, the new polar distance required to bring the rise to R_1 would be $O_1H = O_0H \times \frac{R_0}{R_1}$.

After the median line of the arch has been constructed, other link polygons, such as the bottom right-hand figure, are drawn in for the arch loaded on one side only, for one-third of the length, on the middle only, and any other ways which are likely to throw the line of thrust away from the median line. After these lines have been put in, envelope curves parallel to the median line are drawn in to enclose these lines of thrust at every point; this gives us the middle half of the voussoirs. The outer lines are then drawn in equidistant from the middle half lines, making the total depth of the voussoirs equal to twice the depth of the envelope curves.

An infinite number of lines of thrust may be drawn in for any given distribution of load. Which of these is the right one? is a question by no means easily answered, and whatever

answer may be given, it is to a large extent a matter of opinion. For a full discussion of the question, the reader should refer to I. O. Baker's "Treatise on Masonry" (Wiley and Co., New York); and a paper by H. M. Martin, *I.C.E. Proceedings*, vol. xciii. p. 462.

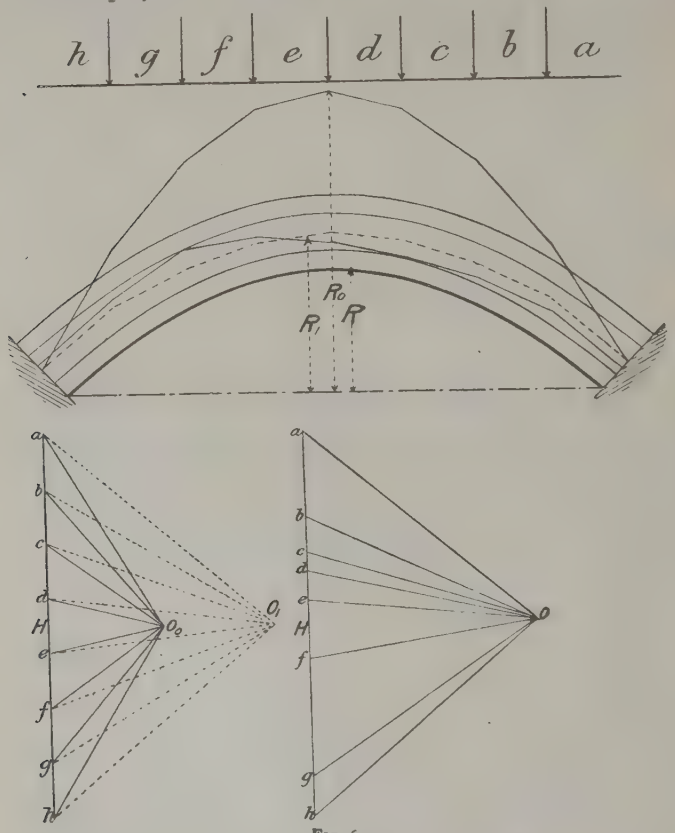


FIG. 603.

From an examination of several successful arches, the author considers that if, by altering the polar distance OH, a line of thrust can be flattened or bent so as to fall within the middle half, it may be concluded that such a line of thrust is admissible. One portion or point of it may touch the inner and one the outer middle half lines. As a matter of fact, an

exact solution of the masonry arch problem in which the voussoirs rest on plane surfaces is indeterminate, and we can only say that a certain assumption is admissible if we find that arches designed on this assumption are successful.

Arched Ribs.—In the case of iron and steel arches, the line of thrust may pass right outside the section, for in a continuous rib capable of resisting tension as well as compression the rib retains its shape by its resistance to bending. The bending moment varies as the distance of the line of thrust from the centre of gravity of the section of the rib. The determination of the position of the line of thrust is therefore important.

Arched ribs are often hinged at three points—at the

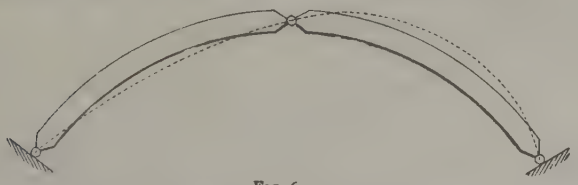


FIG. 604.

springings and at the crown. It is evident in such a case that the line of thrust must pass through the hinges, hence there is no difficulty in finding its exact position. But when the arch is rigidly held at one or both springings, and not hinged at the crown, the position of the line of thrust may be found thus:[1]

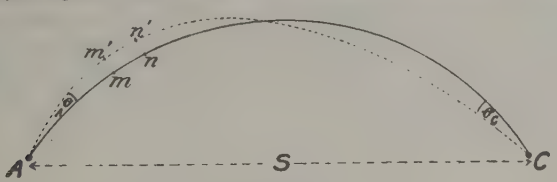


FIG. 605.

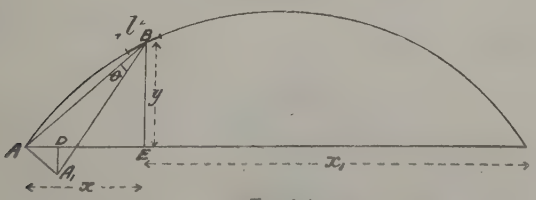


FIG. 606.

[1] See also a paper by Bell, *I.C.E.*, vol. xxxiii. p. 68.

Consider a short element of the rib *mn* of length *l*. When the rib is unequally loaded, it is strained so that *mn* takes up the position *m'n'*. Both the slope and the vertical position of the element are altered by the straining of the rib. First consider the effect of the alteration in slope; for this purpose that portion of the rib from B to A may be considered as being pivoted at B. Join BA; then, when the rib is strained, BA becomes BA_1. Thus the point A has received a horizontal displacement AD, and a vertical displacement A_1D. The two triangles BAE, AA_1D are similar; hence—

$$\frac{AD}{AA_1} = \frac{y}{AB} \quad AD = \frac{AA_1}{AB} \cdot y = \theta \cdot y \quad . \quad . \quad . \quad \text{(i.)}$$

θ being expressed in circular measure. Likewise—

$$\frac{A_1D}{AA_1} = \frac{x}{AB} \quad A_1D = \frac{AA_1}{AB} \cdot x = \theta \cdot x \quad . \quad . \quad \text{(ii.)}$$

A similar relation holds for every other small portion of the rib, but as A does not actually move, it follows that some of the horizontal displacements of A are outwards, and some inwards. Hence the algebraic sum of all the horizontal displacements must be zero, or $\Sigma\theta y = 0$ . . . . . (iii.)

Now consider the vertical movement of the element. If the rib be pivoted at C, and free at A, when *mn* is moved to *m'n'*, the rib moves through an angle θ_c, and the point A receives a vertical displacement $S \cdot \theta_c$. We have previously seen that A has also a vertical displacement due to the bending of the rib; but as the point A does not actually move vertically, some of the vertical displacements must be upwards, and some downwards. Hence the algebraic sum of all the vertical displacements is also zero, or—

$$\Sigma\theta x + S\theta_c = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

By similar reasoning—

$$\Sigma\theta x_1 + S\theta_A = 0$$
$$\text{or } \Sigma(S - x)\theta + S\theta_A = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(v.)}$$

If the arch be rigidly fixed at C—

$$\theta_c = 0$$
$$\text{and } \Sigma\theta x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi.)}$$

If it be fixed at A—

$$\theta_A = 0$$
$$\text{and } \Sigma(S - x)\theta = 0 \text{ from v.}$$

If it be fixed at both ends, we have by addition—

$$\Sigma(S\theta - x\theta + x\theta) = 0$$
$$\Sigma S\theta = 0$$

Then, since S is constant—

$$\Sigma\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii.)}$$

Whence for the three conditions of arches we have—

Arch hinged at both ends, $\Sigma y\theta = 0$ from iii.
„ fixed „ „ $\Sigma y\theta$, and $\Sigma\theta = 0$ from iii. and vii.
„ „ „ one end only, $\Sigma y\theta$, and $\Sigma x\theta = 0$ from i. and vi.

We must now find an expression for θ. Let the full line represent the portion of the unstrained rib, and the dotted line the same when strained.

Let the radius of curvature before straining be ρ_0, and after straining ρ_1.

Then, using the symbols of Chapter XIII., we have—

$$M_0 = \frac{EI}{\rho_0}, \text{ and } M_1 = \frac{EI}{\rho_1}$$

Then the bending moment on the rib due to the change of curvature when strained is—

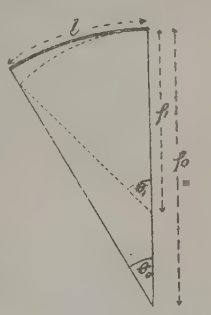


FIG. 607.

$$M = M_1 - M_0 = EI\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right) \quad . \quad . \quad . \quad \text{(viii.)}$$

But as l remains practically constant before and after the strain, we have—

$$\theta_1 = \frac{l}{\rho_1}, \text{ and } \theta_0 = \frac{l}{\rho_0}$$
$$\text{and } \theta_1 - \theta_0 = \theta = l\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)$$
$$\text{or } l = \frac{\theta}{\frac{1}{\rho_1} - \frac{1}{\rho_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ix.)}$$

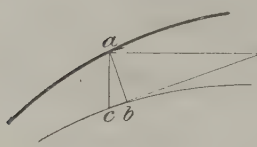
Then we have from viii. and ix.—

$$M\ell = EI\theta$$

$$\text{and } \theta = \frac{M\ell}{EI}$$

If the arched rib be of constant cross-section, $\frac{\ell}{EI}$ is constant; but if it be not so, then the length ℓ must be taken so that $\frac{\ell}{I}$ is constant.

The bending moment M on the rib is $M = F \cdot \overline{ab}$, where the lowest curved line through *cb* is the line of thrust, and the upper dark line the median line of the rib.



Draw *ac* vertical at *c*;
dc tangential at *c*;
ab normal to *dc*;
ad horizontal.

FIG. 608.

Let H be the horizontal thrust on the vector polygon. Then the triangles *adc*, *a'd'c'*, also *adb*, *cda*, are similar; hence—

$$\frac{F}{H} = \frac{cd}{ad} \text{ and } \frac{ab}{ad} = \frac{ac}{cd}$$

$$\text{or } \frac{ab}{ac} = \frac{ad}{cd} = \frac{H}{F}$$

$$\text{or } F \cdot \overline{ab} = H \cdot \overline{ac} = M$$

but H is constant for any given case.

Let $\overline{ac} = z$.

Hence the expression $\Sigma\theta y = 0$

may be written $\Sigma My = 0$, since $\frac{\ell}{EI}$ is constant

$$\text{or } \Sigma z \cdot y = 0$$

Then, substituting in a similar manner in the equations above, we have—

Arch hinged at both ends, $\Sigma z \cdot y = 0$
" fixed " " $\Sigma z \cdot y = 0$, and $\Sigma z = 0$
" " at one end only, $\Sigma z \cdot y = 0$, and $\Sigma xz = 0$

Thus, after the median line of the arch has been drawn, a line of thrust for uneven loading is constructed; and the

median line is divided up into a number of parts of length l, and perpendiculars dropped from each. The horizontal distances between them will, of course, not be equal; then all the values $z \times y$ must be found, some z's being negative, and

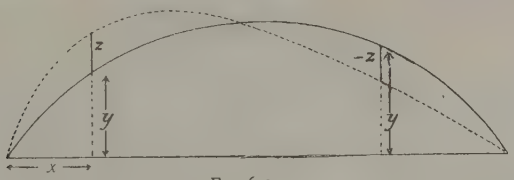


FIG. 609.

some positive, and the sum found. If the sum of the negative values are greater than the positive, the line of thrust must be raised by reducing the polar distance of the vector polygon, and *vice versâ* if the positive are greater than the negative. The line of thrust always passes through the hinged ends. In the case of the arch with fixed ends, the sum of the z's must also

FIG. 610.

be zero; this can be obtained by raising or lowering the line of thrust bodily. When one end only is fixed, the sum of all the quantities $x \cdot z$ must be zero as well as zy; this is obtained by shifting the line of thrust bodily sideways.

Having fixed on the line of thrust, the stresses in the rib are obtained thus:—

The compressive stress all over the rib at any section is—

$$f_c = \frac{T}{A}$$

where T is the thrust obtained from the vector polygon, and A is the sectional area of the rib.

The skin stress due to bending is—

$$f = \frac{M}{Z} = \frac{T \cdot \overline{ab}}{Z} \text{ (see Fig. 608)}$$

$$\text{or } f = \frac{H \cdot z}{Z}$$

and the maximum stress in the material due to both—

$$f_o + f = T\left(\frac{1}{A} + \frac{ab}{Z}\right)$$

$$\text{or} = \frac{T}{A} + \frac{H \cdot z}{Z}$$

Except in the case of very large arches, it is never worth while to spend much time in getting the *exact* position of the worst line of thrust; in many instances its correct position may be detected by eye within very small limits of error.

Effect of Change of Length and Temperature on Arched Ribs.—Long girders are always arranged with expansion rollers at one end to allow for changes in length as the temperature varies. Arched ribs, of course, cannot be so treated —hence, if their length varies due to any cause, the radius of curvature is changed, and bending stresses are thereby set up.

The change of curvature and the stress due to it may be arrived at by the following approximation, assuming the rib to be an arc of a circle:—

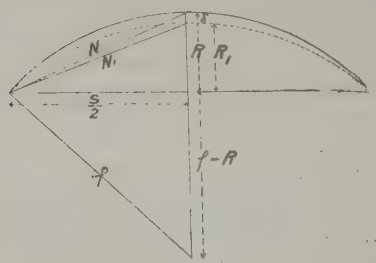


FIG. 611.

$$\text{Let } N_1 = N\left(1 - \frac{1}{n}\right)$$

$$N^2 = R^2 + \frac{S^2}{4}$$

$$N_1^2 = R_1^2 + \frac{S^2}{4}$$

$$N^2 - N_1^2 = R^2 - R_1^2$$

Substituting the value of N_1 and reducing, we get—

$$-\frac{N^2}{n^2} + \frac{2N^2}{n} = (R + R_1)(R - R_1)$$

But $R - R_1 = \delta$

and $R + R_1 = 2R$ (nearly)

The fraction $\frac{1}{n}$ is very small, viz. $\frac{f_c}{E}$, and is rarely more than $\frac{1}{5000}$; hence the quantity involving its square, viz. $\frac{N^2}{9000000}$, is negligible.

Then we get—

$$\delta = \frac{N^2}{nR} \quad \text{(x.)}$$

But $\left(\frac{S}{2}\right)^2 = N^2 - R^2$

also $\left(\frac{S}{2}\right)^2 = \rho^2 - (\rho - R)^2$

$$N^2 - R^2 = \rho^2 - (\rho - R)^2$$

from which we get—

$$N^2 = 2\rho R$$

Substituting in x., we have—

$$\delta = \frac{2\rho}{n}$$

$$\text{and } \rho = \frac{N^2}{2R}$$

$$\text{also } \rho_1 = \frac{N_1^2}{2R_1}$$

The bending moment on the rib due to the change of curvature is—

$$M = EI\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) \text{ from viii.}$$

$$= EI\left(\frac{2R}{N^2} - \frac{2R_1}{N_1^2}\right)$$

and the corresponding skin stress is—

$$f = \frac{M}{Z} = \frac{My}{I} = \frac{Md}{2I}$$

where d is the depth of the rib in inches if R and N are taken in inches.

Then, substituting the value of M, we have—

$$f = \frac{2EId}{2I}\left(\frac{R}{N^2} - \frac{R_1}{N_1^2}\right)$$

N_1^2 may without sensible error (about 1 in 2000) be written N^2; then—

$$f = \frac{Ed}{N^2}(R - R_1)$$

$$f = \frac{Ed\delta}{N^2}$$

The stress at the crown due to the change of curvature on account of the compression of the rib then becomes—

$$f = \frac{2\mathrm{E}d\rho}{n\mathrm{N}^2}$$

$$\text{and } \frac{1}{n} = \frac{f_c}{\mathrm{E}} \text{ (see p. 374)}$$

where f_c is the compressive stress all over the section of the rib at the crown; hence—

$$f = \frac{2d\rho f_c}{\mathrm{N}^2} = \frac{8d\rho}{\mathrm{N}^2}$$

taking f_c as a preliminary estimate at about 4 tons per square inch; then $\frac{1}{n} = \frac{1}{3500}$ (about). In the case of the change of curvature due to change of temperature, it is usual to take the expansion and contraction on either side of the mean temperature of 60° Fahr. as $\frac{1}{4}$ inch per 100 feet for temperate climates such as England, and twice this amount for tropical climates. Hence for England—

$$\frac{1}{n} = \frac{0{\cdot}25}{1200} = \frac{1}{4800}$$

putting E = 12,000 tons per square inch;

$$f = \frac{5d\rho}{\mathrm{N}^2}$$

Thus in England the stress due to temperature change of curvature is about five-eighths as great as that due to the compression change of curvature.

The value of ρ varies from $1{\cdot}25\mathrm{N}$ to $2{\cdot}5\mathrm{N}$, and d from $\frac{\mathrm{N}}{30}$ to $\frac{\mathrm{N}}{20}$.

Hence, due to the compression change of curvature—

$$f = 0{\cdot}6 \text{ to } 0{\cdot}8 \text{ ton per square inch}$$

and due to temperature in England—

$$f = 0{\cdot}4 \text{ to } 0{\cdot}5 \text{ ton per square inch}$$

In the case of ribs fixed rigidly at each end, it can be

shown by a process similar to that given in Chapter XIII., of the beam built in at both ends, that the change of curvature stresses at the abutments of a rigidly held arch is nearly twice as great, and the stress at the crown about 50 per cent. greater than the stress in the hinged arch.

An arched rib must, then, be designed to withstand the direct compression, the bending stresses due to the line of thrust passing outside the section, the bending stresses due to the change of curvature, and finally it must be checked to see that it is safe when regarded as a long strut; to prevent side buckling, all the arched ribs in a structure are braced together.

Effect of Sudden Loads on Structures.—If a bar be subjected to a gradually increasing stress, the strain increases in proportion, provided the elastic limit is not passed. The work done in producing the strain is given by the shaded area *abc* in Fig. 612.

Let x = the elastic strain produced;
l = the unstrained length of the bar;
f = the stress over any cross-section;
E = Young's Modulus of Elasticity;
A = sectional area of bar.

FIG. 612.

$$\text{Then } x = \frac{fl}{E}$$

$$\text{The work done in straining a bar of section A} = \frac{fAx}{2} = \frac{f^2Al}{2E}$$

$$= \frac{Al}{2} \times \frac{f^2}{E}$$

$$= \tfrac{1}{2} \text{ vol. of bar} \times \text{modulus of elastic resilience}$$

The work done in stretching a bar beyond the elastic limit has been treated in Chapter X.

If the stress be produced by a hanging weight (Fig. 613), and the whole load be suddenly applied instead of being gradually increased, then, taking A as 1 square inch to save the constant repetition of the symbol, we have—

$$\text{The work stored in the weight } w_0 \text{ due to falling through a height } x = w_0 x$$

$$= fx$$

But when the weight reaches c (Fig. 613), only a portion of the energy developed during the fall has been expended in stretching the bar, and the remainder is still stored in the

falling weight; the bar, therefore, continues to stretch until the kinetic energy of the weight is absorbed by the bar.

The work done in stretching the bar by an amount x is $\frac{fx}{2}$; hence the kinetic energy of the weight, when it reaches c, is the difference between the total work done by the weight and the work done in stretching the bar, or $fx - \frac{fx}{2} = \frac{fx}{2}$. The bar will therefore continue to stretch until the work taken up by it is equal to the total work done by gravity on the falling weight, or until the area *ahd* is equal to the area *agch*. Then, of course, the area *bed* is equal to the area *acb*.

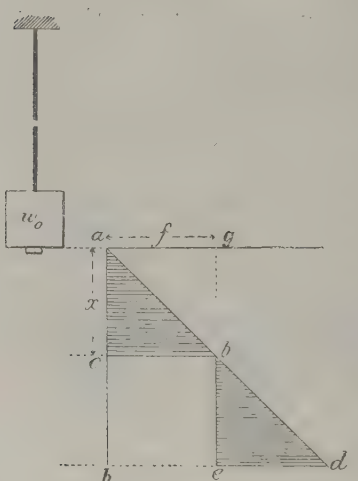


FIG. 613.

When the weight reaches h, the strain, and therefore the stress, is doubled; thus, when a load is suddenly applied to a bar or a structure, the stress produced is twice as great as if the load were gradually applied.

When the weight reaches h, the tension on the bar is greater than the weight; hence the bar contracts, and in doing so raises the weight back to nearly its original position at a; it then drops again, oscillating up and down until it finally comes to rest at c. See page 265.

If the bar in question supported a dead weight W, and a further weight w_1 were suddenly applied, the momentary load on the bar would, by the same process of reasoning, be $W + 2w_0$.

If the suddenly applied load acted upwards, tending to compress the bar, the momentary load would be $W - 2w_0$. Whether or not the bar ever came into compression would entirely depend upon the relative values of W and w_0.

The special case when $w_0 = 2W$ is of interest; the momentary load is then—

$$W - 2(2W) = -3W$$

the negative sign simply indicating that the stress has been

reversed. This is the case of a revolving loaded axle. Consider it as stationary for the moment, and overhanging as shown—

The upper skin of the axle in case (i) is in tension, due to W. In order to relieve it of this stress, *i.e.* to straighten the axle, a force $-W$ must be applied, as in (ii); and, further, to bring the upper skin into compression of the same intensity, as in (i), a force $-2W$ must be applied, as shown in (iii). Now, when an axle revolves, every portion of the skin is alternately brought into tension and compression; hence we may regard a revolving axle as being under the same system of loading as in (iii), or that the momentary load is three times the steady load.

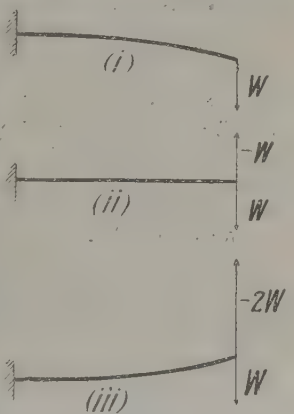


FIG. 614.

The following is a convenient method of arriving at the momentary force produced by a suddenly applied load.

The dead load W is shown by heavy lines with arrows pointing up for tension and down for compression, the suddenly

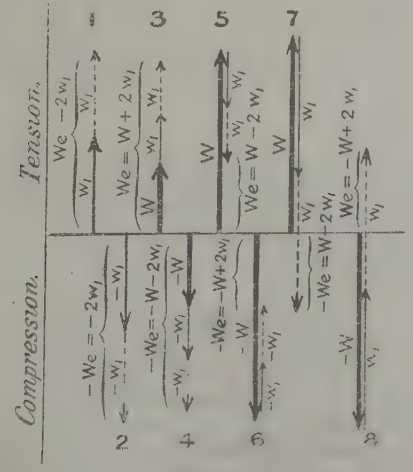


FIG. 615.

applied load W_1 is shown by full light lines, the dynamic increment by broken lines. The equivalent momentary load is denoted W_e. In some instances in which the suddenly applied load is of opposite sign to that of the dead load, the dead load *may* be greater than the equivalent momentary load; for example, cases 5, 6, 7, 8. In case 7 the momentary load is compressive; when designing a member which is subjected to such a system of loading it should be carefully considered from the standpoint of a strut to withstand a load W_e, and a tie to withstand a load W. In case 8 the member must be designed as a strut to support a load $-W$, and be checked as a tie for a load W_e.

In case 3, W may represent the dead load due to the weight of a bridge, and W_1 the weight of a train when standing on the bridge. W_e is the momentary load when a train crosses the bridge.

There are many other methods in use by designers for allowing for the effect of live loads, some of which give higher and some lower results than the above method.

Falling Weight.—When the weight W_o falls through a height h before striking the collar on the bottom of the bar, Fig. 613, the momentary effect produced is considerably greater than $2w_o$.

Let f_s = the static stress due to the load w_o

$$= \frac{w_o}{A}$$

f_e = the equivalent momentary stress produced

$$x = \frac{f_e l}{E}$$

Work done in stretching the bar—or the resilience of the bar per square inch of section $\Big\} = \frac{f_e x}{2} = \frac{f_e^2 l}{2E}$

Work done by the falling weight, per sq. inch of section $\Big\} = \frac{W_o}{A}(h + x) = f_s(h + x)$

$$\frac{f_e^2 l}{2E} = f_s(h + x) = f_s\left(h + \frac{f_e l}{E}\right)$$

$$f_e = f_s\left(1 \pm \sqrt{1 + \frac{2hE}{f_s l}}\right)$$

If a helical buffer spring be placed on the collar, the momentary stress in the bar will be greatly reduced. Let

the spring compress 1 inch for a load of P lbs., and let the maximum compression under the falling weight be δ inches, and h the height the weight falls before it strikes the spring.

The momentary load on the spring $= P\delta$
The momentary load on the bar $= f_c A$
$P\delta = f_c A$

The total work done in stretching the bar and compressing the spring $= \frac{A f_c^2 l}{2E} + \frac{P\delta^2}{2}$
The work done by the falling weight $= W_c(h + \delta + x)$
and $\frac{A f_c^2 l}{2E} + \frac{P\delta^2}{2} = W_c\left(h + \delta + \frac{f_c l}{E}\right)$

from which the value of f_c can be obtained.

Experiments on the Repetition of Stress.—In 1870 Wöhler published the results of some extremely interesting experiments on the effect of repeated stresses on materials, often termed the fatigue of materials; since then many others have done similar work, with the result that a large amount of experimental data is now available, which enables us to construct empirical formulas to approximately represent the general results obtained. In these experiments many materials have been subjected to tensile, compressive, torsional, and bending stresses which were wholly or partially removed, and in some cases reversed as regards the nature of the stress imposed. In all cases the experiments conclusively showed that when the intensity of the stresses imposed approached that of the static breaking strength of the material, the number of repetitions before fracture occurred was small, but as the intensity was reduced the number of repetitions required to produce fracture increased, and finally it was found that if the stress imposed was kept below a certain limit, the bar might be loaded an infinite number of times without producing fracture. This limiting stress appears to depend (i) upon the ultimate strength of the material (*not* upon the elastic limit), and (ii) upon the amount of fluctuation of the stress, often termed the "range of stress" to which the material is subjected.

In a general way the results obtained by the different experimenters are fairly concordant, but small irregularities in the material and in the apparatus used appear to produce marked effects, consequently it is necessary to take the average of large numbers of tests in order to arrive at reliable data.

The following figures, selected mainly from Wöhler's tests, will serve to show the sort of results that may be expected from repetition of stress experiments:—

MATERIAL.

Krupp's Axle Steel.

Tensile strength, varying from 42 to 49 tons per sq. inch.

Tensile stress applied in tons per square inch		Number of repetitions before fracture.	Nominal bending stress in tons per square inch		Number of repetitions before fracture.
from	to		from	to	
0	38·20	18,741	0	26·25	1,762,000
0	33·40	46,286	0	25·07	1,031,200
0	28·65	170,170	0	24·83	1,477,400
0	26·14	123,770	0	23·87	5,234,200
0	23·87	473,766	0	23·87	40,600,000
0	22·92	13,600,000 (unbroken)			(unbroken)

Nominal bending stress in a revolving axle		Number of repetitions before fracture.
from	to	
20·1	−20·1	55,100
17·2	−17·2	127,775
16·3	−16·3	797,525
15·3	−15·3	642,675
,,	,,	1,665,580
,,	,,	3,114,160
14·3	−14·3	4,163,375
,,	,,	45,050,640

MATERIAL.

Krupp's Spring Steel.

Tensile strength, 57·5 tons per sq. inch.

Tensile stress applied in tons per square inch		Number of repetitions before fracture.	Tensile stress applied in tons per square inch		Number of repetitions before fracture.
from	to		from	to	
47·75	7·92	62,000	38·20	4·77	99,700
,,	15·92	149,800	,,	9·55	176,300
,,	23·87	400,050	,,	14·33	619,600
,,	27·83	376,700	,,	,,	2,135,670
,,	31·52	19,673,000 (unbroken)	,,	19·10	35,800,000 (unbroken)
42·95	9·55	81,200	33·41	4·77	286,100
,,	14·33	1,562,000	,,	9·55	701,800
,,	19·10	225,300	,,	11·94	36,600,000 (unbroken)
,,	23·87	1,238,900			
,,	,,	300,900			
,,	28·65	33,600,000 (unbroken)			

MATERIAL.

Phœnix Iron for Axles.

Tensile strength, 21·3 tons per sq. inch.

Nominal bending stress in a revolving axle.		Rounded shoulder.	Square shoulder.	Conical shoulder.
From	To	Number of repetitions before fracture.		
15·3	— 15·3	56,430		
13·4	— 13·4	183,145		
11·5	— 11·5	909,840		
9·6	— 9·6	4,917,992		
8·6	— 8·6	19,186,791	2,063,760	535,302
7·6	— 7·6	132,250,000 (not broken)	14,695,000	1,386,072

The above figures show very clearly the importance of having well rounded shoulders in revolving axles.

MATERIAL (TESTS BY AUTHOR).

Mild steel cut from an over-annealed crank shaft.

Elastic limit, 8·28 tons sq. inch ; tensile strength, 27·98 tons sq. inch.

Nominal bending stress in a revolving axle.		Number of repetitions before fracture.
From	To	
10·0	— 10·0	418,100
9·5	— 9·5	842,228
9·0	— 9·0	729,221
8·5	— 8·5	5,000,000 not broken
8·0	— 8·0	5,000,000 not broken

The figures given in the above table were obtained from a series of tests made on bars cut from a broken crank-shaft ; the material was in a very abnormal condition owing to prolonged annealing, which seriously lowered the elastic limit. Other specimens were tested after being subjected to heat by a steel specialist, which raised the elastic limit nearly 50 per cent., but it did not materially affect the safe limit of stress under repeated loading, thus supporting the view, held by many, that the capacity of a given material to withstand repeated loading depends more upon its ultimate or breaking stress than upon its elastic limit.

Various theories have been advanced to account for the results obtained by repeated loading tests, but all are more or less unsatisfactory; hence in designing structures we are obliged to make use of empirical formulas, which only approximately fit experimental data.

Many of the empirical formulas in use are needlessly complicated, and are not always easy of application; by far the simplest is the "dynamic theory" equation, in which it is assumed that the varying loads applied to test bars by Wöhler and others produce the same effects as suddenly applied loads. In this theory it is assumed that a bar will not break under repeated loadings if the "momentary stress" (see Fig. 614) does not exceed the stress which would produce failure if statically applied. Whether the assumptions are justified or not is quite an open question, and the only excuse for adopting such a theory is that it gives results fairly in accord with experimental values, and moreover it is easily remembered and applied.

The diagram, Fig. 616, is a convenient method of showing

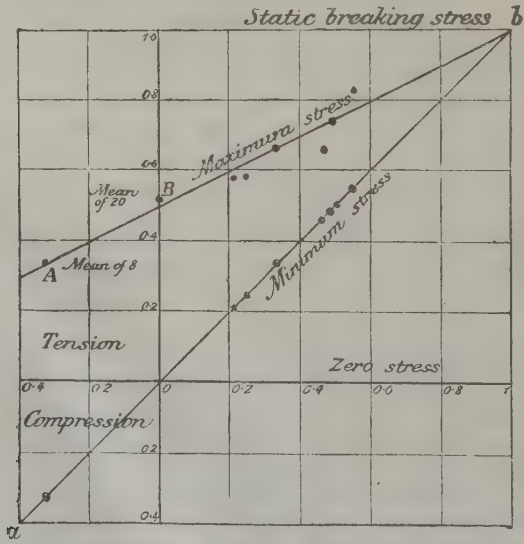


FIG. 616.

to what extent repetition of stress experiments give results in accordance with the "dynamic theory." In this diagram all

stresses are expressed as fractions of the ultimate static stress, which will cause fracture of the material. The minimum stress due to the dead load on the material is plotted on the line *aob*, and the corresponding maximum stress, which may be applied over four million (and therefore presumably an infinite number of) times is shown by the spots along the maximum stress line. If the "dynamic theory" held perfectly for repeated loading, or fatigue tests, the spots would all lie on the line AB*b*, since the stress due to the dead load, *i.e.* the vertical height between the zero stress line and the minimum stress line *plus* twice the live load stress, represented by the vertical distance between the minimum and maximum stress lines, together are equal to the static breaking stress.

The results of tests of revolving axles are shown in group A; the dynamic theory demands that they should be represented by a point situated 0·33 from the zero stress axis.

Likewise, when the stress varies from 0 to a maximum, the results are shown at B; by the dynamic theory, they should be represented by a point situated 0·5 from the zero axis. For all other cases the upper points should lie on the maximum stress line. Whether they do lie reasonably near this line must be judged from the diagram. When one considers the many accidental occurrences that may upset such experiments as these, one can hardly wonder at the points not lying regularly on the mean line.

For an application of this diagram to the most recent work on the effect of repeated loading, readers should refer to the *Proceedings of the Institution of Mechanical Engineers*, November, 1911, p. 910.

Assuming that the dynamic theory is applicable to members of structures when subjected to repeated loads, we proceed thus—let the dead or steady load be termed $W_{min.}$, and the live or fluctuating load $(W_{max.} - W_{min.})$, then the equivalent static load is—

$$W_e = W_{min.} \mp 2(W_{max.} - W_{min.})$$

or using the nomenclature of Fig. 614

$$W_e = W \mp 2(W + w_1 - W)$$
$$W_e = W \mp 2w_1$$

The *plus* sign is used when both the dead and the live loads act together, *i.e.* when both are tension or both compression, and the *minus* when the one is tension and the other compression.

For a fuller discussion of this question, readers are referred to the following sources:—Wöhler's original tests, see *Engineering*, vol. xi., 1871; *British Association Report*, 1887, p. 424; Unwin's "Testing of Materials"; Fidler's "Practical Treatise on Bridge Construction"; Morley's "Theory of Structures"; Stanton and Bairstow, "On the Resistance of Iron and Steel to Reversals of Direct Stress," *Proc. Inst. Civil Engineers*, 1906, vol. clxvi.; Eden, Rose, and Cunningham, "The Endurance of Metals," *Proc. Inst. Mech. Engineers*, November, 1911.

CHAPTER XVIII.

HYDRAULICS.

In Chapter X. we stated that a body which resists a change of form when under the action of a distorting stress is termed a *solid* body, and if the body returns to its original form after the removal of the stress, the body is said to be an *elastic solid* (*e.g.* wrought iron, steel, etc., under small stresses); but if it retains the distorted form it assumed when under stress, it is said to be a *plastic solid* (*e.g.* putty, clay, etc.). If, on the other hand, the body does not resist a change of form when under the action of a distorting stress, it is said to be a *fluid* body; if the change of form takes place immediately it comes under the action of the distorting stress, the body is said to be a *perfect fluid* (*e.g.* alcohol, ether, water, etc., are very nearly so); if, however, the change of form takes place gradually after it has come under the action of the distorting stress, the body is said to be a *viscous fluid* (*e.g.* tar, treacle, etc.). The viscosity is measured by the rate of change of form under a given distorting stress.

In nearly all that follows in this chapter, we shall assume that water is a perfect fluid; in some instances, however, we shall have to carefully consider some points depending upon its viscosity.

Weight of Water.—The weight of water for all practical purposes is taken at 62·5 lbs. per cubic foot, or 0·036 lb. per cubic inch. It varies slightly with the temperature, as shown in the table on the following page, which is for pure distilled water.

The volume corresponding to any temperature can be found very closely by the following empirical formula:—

$$\text{Volume at absolute temperature T, taking the volume at } 39{\cdot}2^{\circ} \text{ Fahr. or } 500^{\circ} \text{ absolute as 1} \Bigg\} = \frac{T^2 + 250{,}000}{1000\,T}$$

Pressure due to a Given Head.—If a cube of water of

1 foot side be imagined to be composed of a series of vertical columns, each of 1 square inch section, and 1 foot high, each will weigh $\frac{62{\cdot}5}{144} = 0{\cdot}434$ lb. Hence a column of water 1 foot high produces a pressure of 0·434 lb. per square inch.

Temp. Fahr.	32°.		39·2°.	50°.	100°.	150°.	200°.	212°.
	Ice.	Water.						
Weight per cubic foot in lbs.	57·2	62·417	62·425	62·409	62·00	61·20	60·14	59·84
Volume of a given weight, taking water at 39·2° Fahr. as 1	1·091	1·0001	1·0000	1·0002	1·007	1·020	1·038	1·043

The height of the column of water above the point in question is termed the *head.*

Let h = the head of water in feet above any surface;

p = the pressure in pounds per square inch on that surface;

w = the weight of a column of water 1 foot high and 1 square inch section;

= 0·434 lb.

Then $p = wh$, or $0{\cdot}434h$

or $h = \frac{p}{w} = 2{\cdot}305p$, or say $2{\cdot}31p$

Thus a head of 2·31 feet of water produces a pressure of 1 lb. per square inch.

Taking the pressure due to the atmosphere as 14·7 lbs. per square inch, we have the head of water corresponding to the pressure of the atmosphere—

$$14{\cdot}7 \times 2{\cdot}31 = 34 \text{ feet (nearly)}$$

This pressure is the same in all directions, and is entirely independent of the shape of the containing vessel. Thus in Fig. 617—

The pressure over any unit area of surface at $a = p_a = 0{\cdot}434h_a$
" " " " $b = p_b = 0{\cdot}434h_b$

and so on.

The horizontal width of the triangular diagram at the side shows the pressure per square inch at any depth below the surface. Thus, if the height of the triangle be made to a scale of 1 inch to the foot, and the width of the base $0{\cdot}434h$, the width of the triangle measured in inches will give the pressure in pounds per square inch at any point, at the same depth below the surface.

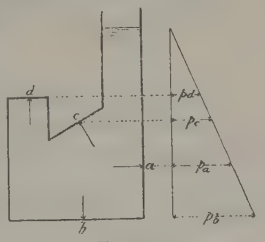


FIG. 617.

Compressibility of Water.—The popular notion that water is incompressible is erroneous; the alteration of volume under such pressures as are usually used is, however, very small. Experiments show that the alteration in volume is proportional to the pressure, hence the relation between the change of volume when under pressure may be expressed in the same form as we used for Young's modulus on p. 374.

Let v = the diminution of volume under any given pressure p in pounds per square inch (corresponding to x on p. 374);
V = the original volume (corresponding to l on p. 374);
K = the modulus of elasticity of volume of water;
p = the pressure in pounds per square inch.

$$\text{Then } \frac{v}{\text{V}} = \frac{p}{\text{K}}, \text{ or } \text{K} = \frac{p\text{V}}{v}$$

K = from 320,000 to 300,000 lbs. per square inch.

Thus water is reduced in bulk or increased in density by 1 per cent. when under a pressure of 3000 lbs. per square inch. This is quite apart from the stretch of the containing vessel.

Total Pressure on an Immersed Surface.—If, for any purpose, we require the total normal pressure acting on an immersed surface, we must find the mean pressure acting on the surface, and multiply it by the area of the surface. We shall show that the mean pressure acting on a surface is the pressure due to the head of water above the centre of gravity of the surface.

Let Fig. 618 represent an immersed surface. Let it be divided up into a large number of horizontal strips of length l_1, l_2, etc., and of width b each at a depth h_1, h_2, etc., respectively from the surface. Then the total pressure on each strip is $p_1 l_1 b$, $p_2 l_2 b$, etc., where p_1, p_2, etc., are the pressures corresponding to h_1, h_2, etc.

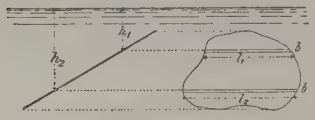


FIG. 618.

But $p = wh$, and $l_1 b = a_1$, $l_2 b = a_2$, etc.
The total pressure on each strip $= wa_1h_1$, wa_2h_2, etc.
Total pressure on whole surface $= P_o = w(a_1h_1 + a_2h_2 + \text{etc.})$

But the sum of all the areas a_1, a_2, etc., make up the whole area of the surface A, and by the principle of the centre of gravity (p. 58) we have—

$$a_1h_1 + a_2h_2 +, \text{ etc.}, = AH_0$$

where H_0 is the depth of the centre of gravity of the immersed surface from the surface of the water, or—

$$P_0 = wAH_0$$

Thus the total pressure in pounds on the immersed surface is the area of the surface in square units × the pressure in pounds per square unit due to the head of water above the centre of gravity of the surface.

Centre of Pressure.—The centre of pressure of a plane immersed surface is the point in the surface through which the resultant of all the pressure on the surface acts.

It can be found thus—

Let H_o = the head of water above the centre of pressure;
H_0 = the head of water above the centre of gravity of the surface;
θ = the angle the immersed surface makes with the surface of the water;
I_o = the second moment, or moment of inertia of the surface about a line lying on the surface of the water and passing through o;
I = the second moment of the surface about a line parallel to the above-mentioned line, and passing through the centre of gravity of the surface;

R_0 = the perpendicular distance between the two axes;

κ^2 = the square of the radius of gyration of the surface about a horizontal axis passing through the c. of g. of the surface;

a_1, a_2, etc. = small areas at depths h_1, h_2, etc., respectively below the surface and at distances x_1, x_2, etc., from O.

A = the area of the surface.

Taking moments about O, we have—

$$p_1a_1x_1 + p_2a_2x_2 +, \text{ etc.} = Px_c$$

$$wh_1a_1x_1 + wh_2a_2x_2 +, \text{ etc.} = wAH_0\frac{H_c}{\sin\theta}$$

$$w\sin\theta(a_1x_1^2 + a_2x_2^2 +, \text{etc.}) = wAH_0\frac{H_c}{\sin\theta}$$

$$\sin^2\theta(a_1x_1^2 + a_2x_2^2 +, \text{etc.}) = AH_0H_c$$

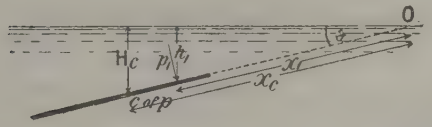


FIG. 619.

On p. 76 we have shown that the quantity in brackets on the left-hand side of the equation is the second moment, or moment of inertia, of the surface about an axis on the surface of the water passing through O. Then we have—

$$\sin^2\theta\, I_0 = \sin^2\theta(I + AR_0^2) = AH_0H_c$$

$$\text{or } \sin^2\theta(A\kappa^2 + AR_0^2) = AH_0H_c$$

Substituting the value of R_0, we get—

$$H_c = \frac{\sin^2\theta\,\kappa^2 + H_0^2}{H_0}$$

The centre of pressure also lies in a vertical plane which passes through the c. of g. of the surface, and which is normal to the surface.

The depth of the centre of pressure from the surface of the water is given for a few cases in the following table:—

Vertical surface.	κ^2.	H_0.	H_c.
Rectangle of depth d with upper edge at surface of water	$\frac{d^2}{12}$	$\frac{d}{2}$	$\frac{2}{3}d$
Circle of diameter d with circumference touching surface of water	$\frac{d^2}{16}$	$\frac{d}{2}$	$\frac{5}{8}d$
Triangle of height d with apex at surface of water and base horizontal	$\frac{d^2}{18}$	$\frac{2d}{3}$	$\frac{3}{4}d$

The methods of finding κ^2 and H_0 have been fully described in Chapter III.

Graphical Method for finding the Centre of Pressure.—In some cases of irregularly shaped surfaces the algebraic method given above is not easy of application, but the following graphic method is extremely simple.

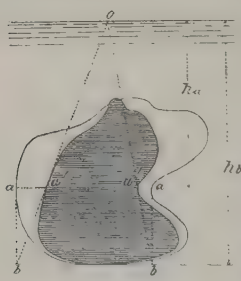


FIG. 620.

In the figure shown draw a series of lines across, not necessarily equidistant; project them on to a base-line drawn parallel to the surface of the water. In the figure shown only one line, *aa*, is projected on to the base-line in *bb*. Join *bb* to a point *o* on the surface of the water and vertically over the centre of gravity of the immersed surface, which cuts off a line $a'a'$; then we have, by similar triangles—

$$\frac{aa}{a'a'} = \frac{bb}{a'a'} = \frac{h_b}{h_a}$$

or the width of the shaded figure $a'a'$ at any depth h_a below the surface is proportional to the total pressure on a very narrow strip *aa* of the surface; hence the shaded figure may be regarded as an equivalent surface on which the pressure is uniform; hence the c. of g. of the shaded figure is the centre of pressure of the original figure.

It will be seen that precisely the same idea is involved here as in the modulus figures of beams given in Chapter XI.

The total normal pressure on the surface is the shaded area A, multiplied by the pressure due to the head at the base-line, or—

$$\text{Total normal pressure} = wA_c h_b$$

Practical Application of the Centre of Pressure.—A good illustration of a practical application of the use of the centre of pressure is shown in Fig. 621, which represents a self-acting movable flood dam. The dam AB, usually of timber, is pivoted to a back stay, CD, the point C being placed at a distance $= \frac{2}{3}$AB from the top; hence, when the level of the water is below A the centre of pressure falls below C, and the dam is stable; if, however, the water flows over A, the centre of pressure rises above C, and the dam tips over. Thus as soon as a flood occurs the dam automatically tips over and prevents the water rising much above its normal. Each section, of course, has to be replaced by those in attendance when the flood has abated.

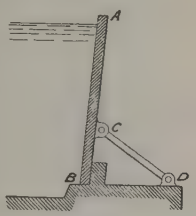


FIG. 621.

Velocity of Flow due to a Given Head.—Let the tank shown in the figure be provided with an orifice in the bottom as shown, through which water flows with a velocity V feet per second. Let the water in the tank be kept level by a supply-pipe as shown, and suppose the tank to be very large compared with the quantity passing the orifice per second, and that the water is sensibly at rest and free from eddies.

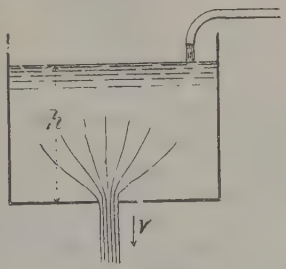


FIG. 622.

Let A = area of orifice in square feet;
A_c = the contracted area of the jet;
Q = quantity of water passing through the orifice in cubic feet per second;
V = velocity of flow in feet per second;
W = weight of water passing in pounds per second;
h = head in feet above the orifice.

Then $Q = A_c V$

Work done per second by W lbs. of water falling through h feet $= Wh$ foot-lbs.

$$\left.\begin{array}{l}\text{Kinetic energy of the water on}\\ \text{leaving the orifice}\end{array}\right\} = \frac{WV^2}{2g}$$

But these two quantities must be equal, or—

$$Wh = \frac{WV^2}{2g}, \text{ and } h = \frac{V^2}{2g}$$
$$\text{and } V = \sqrt{2gh}$$

that is, the velocity of flow is equal to the velocity acquired by a body in falling through a height of h feet.

Contraction and Friction of a Stream passing through an Orifice.—The actual velocity with which water flows through an orifice is less than that due to the head, mainly on account of the friction of the stream on the sides of the orifice; and, moreover, the stream contracts after it leaves the orifice, the reason for which will be seen from the figure. If each side of the orifice be regarded as a ledge over which a stream of water is flowing, it is evident that the path taken by the water will be the resultant of its horizontal and vertical movements, and therefore it does not fall vertically as indicated by the dotted lines, which it would have to do if the area of the stream were equal to the area of the orifice. Both the friction and the contraction can be measured experimentally, but they are usually combined in one coefficient of discharge K_d, which is found experimentally.

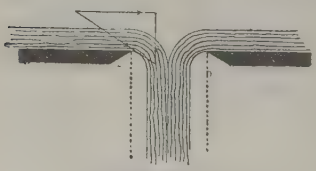


FIG. 623.

Hydraulic Coefficients.—The coefficient of discharge K_d may be split up into the coefficient of velocity K_v, viz.—

$$\frac{\text{actual velocity of flow}}{\sqrt{2gh}}$$

and the coefficient of contraction K_c, viz.—

$$\frac{\text{actual area of the stream}}{\text{area of the orifice}}$$

Then the coefficient of discharge—

$$K_d = K_v K_c$$

The coefficient of resistance—

$$K_r = \frac{\text{actual kinetic energy of the jet of water leaving the orifice}}{\text{kinetic energy of the jet if there were no losses}}$$

The coefficient of velocity for any orifice can be found experimentally by fitting the orifice into the vertical side of a tank and allowing a jet of water to issue from it horizontally. If the jet be allowed to pass through a ring distant h_1 feet below the centre of the orifice and h_2 feet horizontally from it, then any given particle of water falls h_1 feet vertically while travelling h_2 feet horizontally,

$$\text{where } h_1 = \tfrac{1}{2}gt^2$$

$$\text{and } t = \sqrt{\frac{2h_1}{g}}$$

$$\text{also } h_2 = v_2 t$$

where v_2 = the horizontal velocity of the water on leaving the orifice. If there were no resistance in the orifice, it would have a greater velocity of efflux, viz.—

$$v = \sqrt{2gh}$$

where h is the head of water in the tank over the centre of the orifice.

$$\text{Then } v_2 = \frac{h_2\sqrt{g}}{\sqrt{2h_1}}$$

$$\text{and } K_v = \frac{v_2}{v} = \frac{h_2}{2\sqrt{h_1 h}}$$

This coefficient can also be found by means of a Pitot tube, *i.e.* a small sharp-edged tube which is inserted in the jet in such a manner that the water plays axially into its sharp-edged mouth; the other end of the tube, which is usually bent for convenience in handling, is attached to a glass water-gauge. The water rises in this gauge to a height proportional to the velocity with which the water enters the sharp-edged mouthpiece. Let this height be h_3, then $v_2 = \sqrt{2gh_3}$, from which the coefficient of velocity can be obtained. This method is not so good as the last mentioned, because a Pitot tube, however well constructed, has a coefficient of resistance of its own, and therefore this method tends to give too low a value for K_v.

The area of the stream issuing from the orifice can be measured approximately by means of sharp-pointed micrometer screws attached to brackets on the under side of the orifice plate. The screws are adjusted to just touch the issuing stream of water usually taken at a distance of about three diameters from the orifice. On stopping the flow the distance between the screw-points is measured, which is the diameter of the jet; but it is very difficult to thus get satisfactory results. A better way is to get it by working backwards from the coefficient of discharge.

Plain Orifice.—The edges should be chamfered off as shown (Fig. 624); if not, the water dribbles down the sides and makes the coefficient variable. In this case K_v = about 0·97, and K_c = about 0·64, giving $K_d = 0{\cdot}62$. Experiments show that K_d decreases with an increase in the head and the diameter of the orifice, also the sharper the edges the smaller is the coefficient, but it rarely gets below 0·592, and sometimes reaches 0·64. As a mean value $K_d = 0{\cdot}62$.

$$Q = 0{\cdot}62A\sqrt{2gh}$$

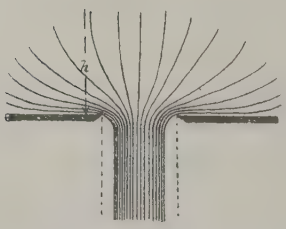


Fig. 624.

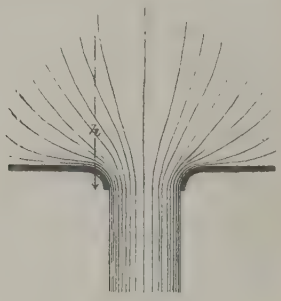


Fig. 625.

Rounded Orifice.—If the orifice be rounded to the same form as a contracted jet, the contraction can be entirely avoided, hence $K_c = 1$; but the friction is rather greater than in the plain orifice, $K_v = 0{\cdot}96$ to 0·98, according to the curvature and the roughness of the surface. The head h and the diameter of the orifice must be measured at the bottom, *i.e.* at the place where the water leaves the orifice; as a mean value we may take—

$$Q = 0{\cdot}97A\sqrt{2gh}$$

Pipe Orifice.—The length of the pipe should be not less than three times the diameter. The jet contracts after leaving the square corner, as in the sharp-edged orifice; it expands again lower down, and fills the tube. It is possible to get a clear jet right through, but a very slight disturbance will make it run as shown. In the case of the clear stream, the value of K is approximately the same as in the plain orifice. When the pipe runs full, there is a sudden change of velocity from the contracted to the full part of the jet, with a consequent loss of energy and velocity of discharge.

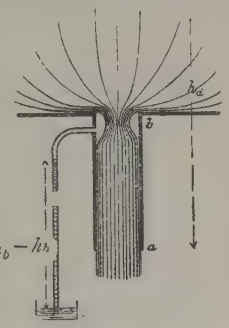


FIG. 626.

Let the velocity at $b = V_b$; and the head $= h_b$
the velocity at $a = V_a$; „ „ $= h_a$

Then the loss of head $= \frac{(V_b - V_a)^2}{2g}$ (see p. 673)

and $h_a = \frac{V_a^2}{2g} + \frac{(V_b - V_a)^2}{2g}$

The velocities at the sections a and b will be inversely as the respective areas. If K_c be the coefficient of contraction at b, we have $V_b = \frac{V_a}{K_c}$; inserting this value in the expression given above, we get—

$$V_a = \frac{1}{\sqrt{1 + \left(\frac{1}{K_c} - 1\right)^2}} \sqrt{2gh_a}$$

The fractional part of this expression is the coefficient of velocity K_v for this particular form of orifice. The coefficient of discharge K_D is from 0·94 to 0·95 of this, on account of friction in the pipe. Then, taking a mean value—

$$Q = 0{\cdot}945 K_v A \sqrt{2gh_a}$$

The pressure at a is atmospheric, but at b it is less (see

p. 666). If the stream ran clear of the sides of the pipe into the atmosphere, the discharge would be—

$$Q = K_d A \sqrt{2gh_b}$$

but in this case it is—

$$Q_1 = K_D A \sqrt{2gh_a}$$

Let $h_a = nh_b$.

$$\text{Then } Q_1 = K_D A \sqrt{2gnh_b}$$

or the discharge is $\frac{K_D \sqrt{n}}{K_d}$ times as great as before; hence we may write—

$$Q_1 = \frac{K_D}{K_d} \times K_d \times A \sqrt{2gnh_b}$$

$$Q_1 = K_d A \sqrt{2gn\left(\frac{K_D}{K_d}\right)^2 h_b}$$

Thus the effective head is $n\left(\frac{K_D}{K_d}\right)^2 h_b$ when running a full stream and only h_b when running clear. The difference is due to a partial vacuum at b amounting to $h_b\left\{n\left(\frac{K_D}{K_d}\right)^2 - 1\right\}$.

If the head h_b be kept constant and the length of the pipe be *increased* it will be found that the quantity passing *diminishes*. The maximum quantity passes when $D = 4fh_b$ where D is the diameter of the pipe in feet, and f is the friction coefficient, see p. 679.

Readers familiar with the Calculus will have no difficulty in obtaining this result, the relation can also be proved by calculating the quantity of water passing for various values of the length *ab*.

When the pipe is horizontal $n = 1$, and the vacuum head is

$$h_b\left\{\left(\frac{K_D}{K_d}\right)^2 - 1\right\} = 0{\cdot}74h_b.$$

The following results were obtained by experiment:—

The diameter of the pipe = 0·945 inches
Length h_{ab} = 2·96 „
K_d = 0·612

Head h_b (inches) ...	16·1	13·1	10·1	8·1	6·1	4·1	3·1
K_D	0·799	0·797	0·797	0·792	0·786	0·780	0·773
Vacuum head by experiment in inches	16·75	14·37	11·95	10·50	9·00	7·45	6·35
Vacuum head by calculation	16·60	14·24	11·97	10·45	8·85	7·37	6·56

Before making this experiment the pipe must be washed out with benzene or other spirit in order to remove all grease, and care must be taken that no water lodges in the flexible pipe which couples the water-gauge to the orifice nozzle.

Re-entrant Orifice or Borda's Mouthpiece.—If a plain orifice in the bottom of a tank be closed by a cover or valve on the upper side, the total pressure on the bottom of the tank will be P, where P is the weight of water in the tank; but if the orifice be opened, the pressure P will be reduced by an amount P_0, equal to (i.) the downward pressure on the valve, viz. whA; and (ii.) by a further amount P_f, due to the flow of water over the surface of the tank all round the orifice. Then we have—

$$P_0 = whA + P_f$$

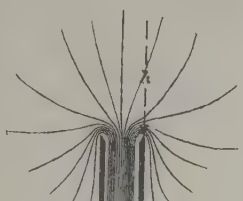


FIG. 627.

In the case of Borda's mouthpiece, the orifice is so far removed from the side of the tank that the velocity of flow over the surface is practically zero; hence no such reduction of pressure occurs, or P_f is zero.

Let the section of the jet be a, and the area of the orifice A.

Then the total pressure due to the column of water over the orifice $= whA$

$$= \frac{wAV^2}{2g}$$

the mass of water flowing per second $= \frac{waV}{g}$

the momentum of the water flowing per second $= \frac{waV^2}{g}$

The water before entering the mouthpiece was sensibly at rest, hence this expression gives us the change of momentum per second.

Change of momentum per second = impulse per second, or pressure

$$\frac{waV^2}{g} = \frac{wAV^2}{2g}$$

hence $a = 0{\cdot}5A$

or $K_c = 0{\cdot}5$

If the pipe be short compared to its diameter, the value of P_f will not be zero, hence the value of K can only have this low value when the pipe is long. The following experiments by the author show the effect of the length of pipe on the coefficient :—

Length of projecting pipe expressed in diameters	0	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	2
K_d	0·61	0·56	0·55	0·54	0·53	0·52

If the mouthpiece be caused to run full, which can be accomplished by stirring the water in the neighbourhood of the mouthpiece for an instant, the coefficient of velocity will be (see "Pipe Orifice")—

$$\frac{1}{\sqrt{1 + \left(\frac{1}{K_c} - 1\right)^2}} = \frac{1}{\sqrt{2}} = 0{\cdot}71$$

Experiments give values from 0·69 to 0·73.

Plain Orifice in a small Approach Channel.—When the area a of the stream passing through the orifice is appreciable as compared with the area of the approach channel A_a, the value of K_c varies with the proportions between the two. With a small approach channel there is an imperfect contraction of the jet, and according to Rankine's empirical formula—

$$\frac{1}{K_c} = \sqrt{2{\cdot}618 - 1{\cdot}618\frac{A^2}{A_a{}^2}}$$

where A is the area of the orifice, and A_a is the area of the approach channel.

The author has, however, obtained a rational value for this coefficient (see *Engineering*, March 11, 1904), but the article is too long for reproduction here. The value obtained is—

$$K_c = \frac{0{\cdot}5}{1 - \frac{K_c A}{A_a}}\left(1 + \frac{n^4 - 2n^2 + 1}{4n^4}\right)$$

where n = the ratio of the radius of the approach channel to the radius of the orifice.

The results obtained by the two formulas are—

n.	K_c Rational.	K_c Rankine.
2	0·679	0·672
3	0·645	0·640
4	0·634	0·631
5	0·631	0·626
6	0·629	0·624
8	0·628	0·622
10	0·627	0·620
100	0·625	0·618
1000	0·625	0·618

Diverging Mouthpiece.—This form of mouthpiece is of great interest, in that the discharge of a pipe can be greatly

increased by adding a nozzle of this form to the outlet end, because the velocity of flow in the throat *a* is greater than the velocity due to the head of water *h* above it. The pressure at *b* is atmospheric;[1] hence the pressure at *a* is less than atmospheric (see p. 666); thus the water is discharging into a partial vacuum. If a water-gauge be attached at *a*, and the vacuum measured, the velocity of flow at *a* will be found to be due to the head of water above it *plus* the vacuum head.

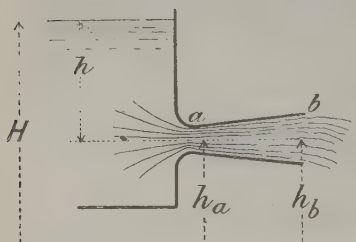


FIG. 628.

We shall shortly show that the energy of any steadily flowing stream of water in a pipe in which the diameter varies gradually is constant at all sections, neglecting friction.

By Bernouilli's theorem we have (see p. 666)—

$$h_a + \frac{p_a}{w} + \frac{V_a^2}{2g} = h_b + \frac{p_b}{w} + \frac{V_b^2}{2g} = H + \frac{p}{w} + \frac{V^2}{2g}$$

where $\frac{p}{w}$ is the atmospheric pressure acting on the free surface of the water. The pressure at the mouth, viz. p_b, is also atmospheric; hence $\frac{p}{w} = \frac{p_b}{w}$.

The velocity V is zero, hence $\frac{V^2}{2g}$ is zero.

Then, assuming no loss by friction, we have—

$$H - h_b = \frac{V_b^2}{2g}$$

$$\text{or } h = \frac{V_b^2}{2g}$$

$$\text{or } V_b = \sqrt{2gh}$$

and the discharge—

$$Q = K_d V_b A_b = K_d A_b \sqrt{2gh}$$

[1] This reasoning will not hold if the mouthpiece discharges into a vacuum.

In the case above, the mouthpiece is horizontal, but if it be placed vertically with b below, the proof given above still holds; the h must then be measured from b, *i.e.* the bottom of the mouthpiece, provided the conditions mentioned below are fulfilled.

Thus we see that the discharge depends upon the area at b, and is independent of the area at a; there is, however, a limit to this, for if the pressure at a be below the boiling point corresponding to the temperature, the stream will not be continuous.

From the above, we have—

$$\frac{p_a}{w} = \frac{V_b^2 - V_a^2}{2g} + \frac{p_b}{w}$$

If $\frac{p_a}{w}$ becomes zero, the stream breaks up, or when—

$$\frac{V_a^2 - V_b^2}{2g} = \frac{p_b}{w} = 34 \text{ feet}$$

$$\text{But } \frac{V_a}{V_b} = \frac{A_b}{A_a} = n, \text{ or } V_a = nV_b$$

$$\text{hence } \frac{n^2V_b^2 - V_b^2}{2g} = \frac{V_b^2}{2g}(n^2 - 1) = 34$$

$$\text{or } h(n^2 - 1) = 34$$

In order that the stream may be continuous, $h(n^2 - 1)$ should be less than 34 feet, and the maximum discharge will occur when the term to the left is slightly less than 34 feet.

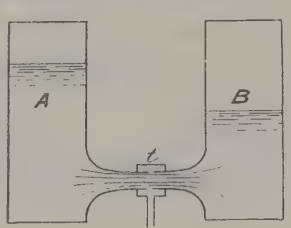


FIG. 629.

The following experiments demonstrate the accuracy of the statement made above, that the discharge is due to the head of water + the vacuum head. The experiments were made by Mr. Brownlee, and are given in the *Proceedings* of the Shipbuilders of Scotland for 1875–6.

The experiments were arranged in such a manner that, in effect, the water flowed from a tank A through a diverging mouthpiece into a tank B, a vacuum gauge being attached at the throat t.

The close agreement between the experimental and the

calculated values as given in the last two columns, is a clear proof of the accuracy of the theory given above.

Head of water in tank A. Feet. H_a.	Head of water in tank B. Feet. H_b.	Vacuum at throat in feet of water. H_v.	Velocity of flow at throat. Feet per second.	
			By experiment.	$\sqrt{2g(H_a + H_v)}$.
69·24	58·85	None	65·97	66·78
69·24	50·78	33·5	80·97	81·34
69·24	None	33·5	81·43	81·34
12·50	8·50	11·3	37·90	38·84
12·50	5·00	33·5	53·98	54·43
12·50	1·50	33·5	54·60	54·43
8·00	None	33·5	51·67	51·70
2·00	None	8·2	24·74	25·63
0·25	None	0·52	6·66	7·04

Jet Pump or Hydraulic Injector.—If the height of the column of water in the vacuum gauge at *t* (Fig. 629) be less than that due to the vacuum produced, the water will be sucked in and carried on with the jet. Several inventors have endeavoured to utilize an arrangement of this kind for saving water in hydraulic machinery when working below their full power. The high-pressure water enters by the pipe A; when passing through the nozzles on its way to the machine cylinder, it sucks in a supply of water from the exhaust sump viâ B, and the greater volume of the combined stream at a lower pressure passes on to the cylinder. All the water thus sucked in is a direct source of gain, but the efficiency of the apparatus as usually constructed is very low, about 30 per cent. The author and Mr. R. H. Thorpe, of New York, made a long series of experiments on jet pumps, and succeeded in designing one which gave an efficiency of 72 per cent.

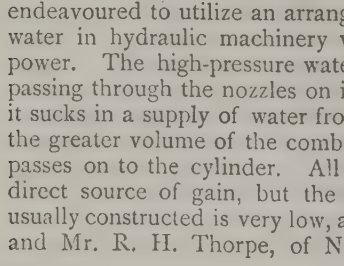


FIG. 630.

An ordinary jet pump is shown in Fig. 630. The main trouble that occurs with such a form of pump is that the water churns round and round the suction spaces of the nozzles instead of going straight through. Each suction space between the nozzles should be in a separate chamber provided with a

back-pressure valve, and the spaces should gradually increase in area as the high-pressure water proceeds—that is to say, the first suction space should be very small, and the next rather larger, and so on.

Rectangular Notch.—An orifice in a vertical plane with an open top is termed a notch, or sometimes a weir. The only two forms of notches commonly used are the rectangular and the triangular.

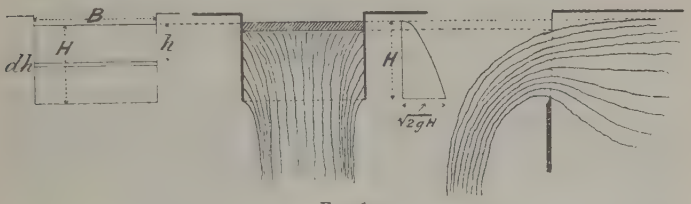


FIG. 631.

From the figure, it will be observed that the head of water immediately over the crest is less than the head measured further back, which is, however, the true head H.

In calculating the quantity of water Q that flows over such a notch, we proceed thus—

The area of any elementary strip as shown $= B \cdot dh$

quantity of water passing strip per second, neglecting contraction $= V \cdot B \cdot dh$

where V = velocity of flow in feet per second,

$$= \sqrt{2gh}, \text{ or } \sqrt{2g}h^{\frac{1}{2}}$$

Hence the quantity of water passing strip per second, neglecting contraction $= \sqrt{2g}Bh^{\frac{1}{2}}dh$

the whole quantity of water Q passing over the notch in cubic feet per second, neglecting contraction $= \sqrt{2g}B\int_{h=0}^{h=H} h^{\frac{1}{2}}dh$

$$Q = \sqrt{2g}B\frac{H^{\frac{3}{2}}}{\frac{3}{2}}$$

$$Q = \sqrt{2g}B\tfrac{2}{3}H\sqrt{H}$$

$$Q = \tfrac{2}{3}BH\sqrt{2gH}$$

introducing a coefficient to allow for contraction $Q = K\tfrac{2}{3}BH\sqrt{2gH}$

where B and H are both measured in feet; where K has values varying from 0·59 to 0·64 depending largely on the

proportions of the section of the stream, *i.e.* the ratio of the depth to the width, and on the relative size of the notch and the section of the stream above it. In the absence of precise data it is usual to take K = 0·62. The following empirical formula by Braschmann gives values of $\frac{2}{3}$K for various heads H allowing for the velocity of approach. Let B_a = the breadth of the approach channel in feet.

$$\tfrac{2}{3}K = \left(0{\cdot}3838 + 0{\cdot}0386\frac{B}{B_a} + \frac{0{\cdot}00174}{H}\right)$$

Triangular Notch.—In order to avoid the uncertainty of the value of K, Professor James Thompson proposed the use of V notches; the form of the section of the stream then always remains constant however the head may vary. Experi-

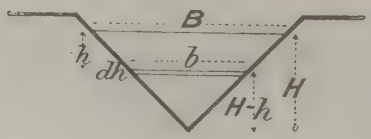


Fig. 632.

ments show that K for such a notch is very nearly constant. Hence, in the absence of precise data, it may be used with much greater confidence than the rectangular notch. The quantity of water that passes is arrived at thus:

Area of elementary strip $= b \,.\, dh$

$$\text{But } \frac{b}{B} = \frac{H - h}{H}, \text{ and } b = \frac{B(H - h)}{H}$$

$$\text{area of elementary strip} = \frac{B(H - h)}{H} \,.\, dh$$

$$\text{velocity of water passing strip} = V = \sqrt{2gh} = \sqrt{2g}\, h^{\frac{1}{2}}$$

$$\left.\begin{array}{l}\text{quantity of water passing}\\ \text{strip per second, neglect-}\\ \text{ing contraction}\end{array}\right\} = \frac{B(H - h)}{H}\sqrt{2g}h^{\frac{1}{2}} \,.\, dh$$

$$\left.\begin{array}{l}\text{whole quantity of water}\\ \text{passing over the notch in}\\ \text{cubic feet per second,}\\ \text{neglecting contraction}\end{array}\right\} = \frac{B}{H}\sqrt{2g}\int_{h=0}^{h=H}(H - h)h^{\frac{1}{2}} \,.\, dh$$

$$Q = \frac{B}{H}\sqrt{2g}\int_{h=0}^{h=H}(Hh^{\frac{1}{2}} - h^{\frac{3}{2}})dh$$

$$Q = \frac{B}{H}\sqrt{2g}\left[\frac{Hh^{\frac{3}{2}}}{\frac{3}{2}} - \frac{h^{\frac{5}{2}}}{\frac{5}{2}}\right]h = H \quad h = 0$$

$$Q = \frac{B}{H}\sqrt{2g}(\tfrac{2}{3}H^{\frac{5}{2}} - \tfrac{2}{5}H^{\frac{5}{2}}) = \frac{B}{H}\sqrt{2g}\tfrac{4}{15}H^{\frac{5}{2}}$$

Introducing a coefficient for the contraction of the steam and putting $B = 2H$ for a right-angled notch.

$$Q = CH^{\frac{5}{2}}$$

where C has the following values. See *Engineering*, April 8th and 15th, 1910.

H (feet) ...	0·05	0·10	0·15	0·20	0·25	0·30	0·40
C	0·289	0·304	0·306	0·306	0·305	0·304	0·303

Rectangular Orifice in a Vertical Plane.—When the vertical height of the orifice is small compared with the depth of water above it, the discharge is commonly taken to be the same as that of an orifice in a horizontal plane, the head being H, *i.e.* the head to the centre of the orifice. When, however, the vertical height of the orifice is not small compared with the

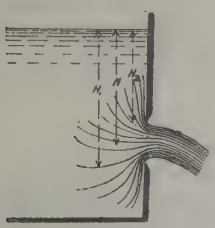


Fig. 633.

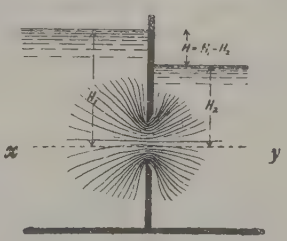


Fig. 634.

depth, the discharge is obtained by precisely the same reasoning as in the two last cases; it is—

$$Q = K\tfrac{2}{3}B\sqrt{2g}(H_1^{\frac{3}{2}} - H_2^{\frac{3}{2}})$$

K, however, is a very uncertain quantity; it varies with the shape of the orifice and its depth below the surface.

Drowned Orifice.—When there is a head of water on

both sides of an orifice, the discharge is not free; the calculation of the flow is, however, a very simple matter. The head producing flow at any section xy (Fig. 634) is $H_1 - H_2 = H$; likewise, if any other section be taken, the head producing flow is also H. Hence the velocity of flow $V = \sqrt{2gH}$, and the quantity discharged—

$$Q = KA\sqrt{2gH}$$

K varies somewhat, but is usually taken 0·62 as a mean value.

Flow under a Constant Head.—It is often found necessary to keep a perfectly constant head in a tank when making careful measurements of the flow of liquids, but it is often very difficult to accomplish by keeping the supply exactly equal to the delivery. It can, however, be easily managed with the device shown in the figure. It consists of a closed tank fitted with an orifice, also a gland and sliding pipe open to the atmosphere. The vessel is filled, or nearly so, with the fluid, and the sliding pipe adjusted to give the required flow. The flow is due to the head H, and the negative pressure p above the surface of the water, for as the water sinks a partial vacuum is formed in the upper part of the vessel, and air bubbles through. Hence the pressure p is always due to the head h, and the effective head producing flow through the orifice is $H - h$, which is independent of the height of water in the vessel, and is constant provided the water does not sink below the bottom of the pipe. The quantity of water delivered is—

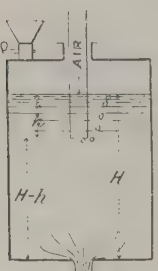


FIG. 635.

$$Q = KA\sqrt{2g(H - h)}$$

where K has the values given above for different orifices.

Velocity of Approach.—If the water approaching a notch or weir have a velocity V_a, the quantity of water passing will be correspondingly greater, but the exact amount will depend upon whether the velocity of the stream is uniform at every part of the cross-section, or whether it varies from point to point as in the section over the crest of a weir or notch.

Let the velocity be uniform, as when approaching an orifice of area a, the area of the approach channel being A.

Let v = velocity due to the head h, *i.e.* the head over the orifice;

V = velocity of water issuing from the orifice.

$$\text{Then } V_a = \frac{Ka}{A}V, \text{ and } V = V_a + v$$

$$V = \frac{Ka}{A}V + v$$

$$V = \frac{v}{1 - \frac{Ka}{A}}$$

Broad-Crested Weir.—The water flows in a parallel stream over the crest of the weir if the sill is of sufficient

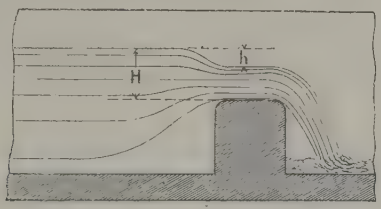


FIG. 636.

breadth to allow the stream lines to take a horizontal direction. Neglecting the velocity of approach, the velocity of the stream passing over the crest is, $V = \sqrt{2gh}$ where h is the depth of the surface below that of the water in the approach channel.

Let B = the breadth (in feet) of the weir at right angles to the direction of flow.

Then the quantity passing over the weir in cubic feet per second is—

$$Q = B(H - h)\sqrt{2gh}$$

The value of h, however, is unknown. If h be large in proportion to H, the section of the stream will be small, and the velocity large; on the other hand, if h be small in proportion to H, the section of the stream will be large and the velocity small, hence there must be some value of h which gives a maximum flow.

Let $h = nH$

$$Q = B\sqrt{2g}(H - nH)\sqrt{nH}$$

$$Q = B\sqrt{2g} \times H^{\frac{3}{2}}(n^{\frac{1}{2}} - n^{\frac{3}{2}})$$

$$\frac{dQ}{dn} = B\sqrt{2g}H^{\frac{3}{2}}(\tfrac{1}{2}n^{-\frac{1}{2}} - \tfrac{3}{2}n^{\frac{1}{2}})$$

This is a maximum when—

$$\tfrac{1}{2}n^{-\frac{1}{2}} = \tfrac{3}{2}n^{\frac{1}{2}} \text{ or when } n = \tfrac{1}{3}$$

Inserting the value of n in the equation for Q, we have—

$$Q = 0{\cdot}385BH\sqrt{2gH}$$

The actual flow in small smooth-topped weirs agrees well with this expression, but in rough masonry weirs the flow is less according to the degree of roughness.

Time required to Lower the Water in a Tank through an Orifice.—The problem of finding the time T required to lower the water in a dock or tank through a sluice-gate, or through an orifice in the bottom, is one that often arises.

(i.) *Tank of uniform cross-section.*

Let the area of the surface of the water be A_a;
" " " stream through the orifice be K_cA;
" greater head of water above the orifice be H_1;
" lesser " " " " " H_2;
" head of water at any given instant be h.

The quantity of water passing through the orifice in the time dt $\Big\} = K_dA\sqrt{2gh}\,.\,dt$

Let the level of the water in the tank be lowered by an amount dh in the interval of time dt.

Then the quantity in the tank is reduced by an amount $A_a dh$, which is equal to that which has passed through the orifice in the interval, or—

$$K_dA\sqrt{2gh}\,.\,dt = A_a dh$$

$$dt = \frac{A_a}{K_dA\sqrt{2g}}h^{-\frac{1}{2}}dh$$

$$T = \frac{A_a}{K_dA\sqrt{2g}}\int_{h=H_2}^{h=H_1} h^{-\frac{1}{2}}dh$$

$$T = \frac{2A_a(\sqrt{H_1} - \sqrt{H_2})}{K_dA\sqrt{2g}}$$

The time required to *empty* the tank is—

$$T = \frac{2A_a\sqrt{H_1}}{K_dA\sqrt{2g}} \text{ (nearly)}$$

It is impossible to get an exact expression for this, because the assumed conditions fail when the head becomes very small; the expression may, however, be used for most practical purposes.

(ii.) *Tapered tank of uniform breadth* B.

In this case the quantity in the tank is reduced by the amount $B \cdot l \cdot dh$ in the given interval of time dt.

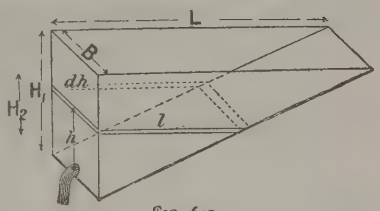


FIG. 637.

$$\text{But } l = \frac{Lh}{H_1}$$

$$\text{hence } B \cdot l \cdot dh = \frac{BL}{H_1} h\, dh$$

$$dt = \frac{BL}{H_1 K_d A \sqrt{2g}} h^{\frac{1}{2}} dh$$

Integrating, we get—

$$T = \frac{2BL(H_1^{\frac{3}{2}} - H_2^{\frac{3}{2}})}{3H_1 K_d A \sqrt{2g}}$$

(iii.) *Hemispherical tank.*

In this case—

$$r^2 = 2Rh - h^2$$

The quantity in the tank is reduced by $\pi(2Rh - h^2)dh$ in the interval dt.

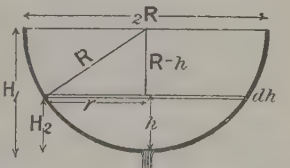


FIG. 638.

$$dt = \frac{\pi}{K_d A \sqrt{2g}} (2Rh - h^2) h^{-\frac{1}{2}} dh$$

$$T = \frac{\pi}{K_d A \sqrt{2g}} \int_{H_2}^{H_1} (2Rh^{\frac{1}{2}} - h^{\frac{3}{2}})\, dh$$

$$T = \frac{\pi\left(\frac{4}{3}RH_1^{\frac{3}{2}} - \frac{2H_1^{\frac{5}{2}}}{5} - \frac{4RH_2^{\frac{3}{2}}}{3} + \frac{2H_2^{\frac{5}{2}}}{5}\right)}{K_d A \sqrt{2g}}$$

(iv.) *Case in which the surface of the water falls at a uniform rate.*

In this case $\frac{dh}{dt}$ is constant;

$$\text{hence } K_d A\sqrt{2gh} = A_a \times \text{a constant}$$

But $K_d A\sqrt{2g}$ is constant in any given case, hence the area of the tank A_a at any height h above the orifice varies as $\sqrt{h}$ or the vertical section of the tank must be parabolic as shown.

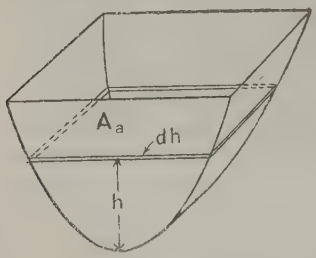


FIG. 638 A.

(v.) *Time required to change the level when water is flowing into a tank at constant rate and leaving by an orifice.*

Let A_a = area of the surface of the water in square feet, when the depth of water above the outlet is h feet.

In a tank of geometrical form we may write

$A_a = Ch^n$ where C and n are constants.

Q_i = the quantity of water in cubic feet per second running into the tank.

A = the area of the orifice in square feet.

T_r = the time required to raise the level of the water from H_1 to H_m feet.

T_l = the time required to lower the level of the water from H_m to H_1 feet.

The quantity of water flowing into the tank in the time dt $\Big\}$ $= Q_i\, dt$

The portion of the water which is retained in the tank in the time dt $\Big\}$ $= A_a\, dh$

The quantity of water flowing out of the tank through the outlet in the time dt $\Big\}$ $= K_d A\sqrt{2gh}\, dt$

When the water is entering the tank at a constant rate and leaving more slowly at a rate dependent upon the head, we have—

$$K_d A\sqrt{2gh}\, dt = Q_i\, dt - A_a\, dh$$

$$dt(Q_i - K_d A\sqrt{2gh}) = A_a\, dh = Ch^n\, dh$$

$$\int dt = C\int_{H_1}^{H_m} \frac{h^n\, dh}{Q_i - K_d A\sqrt{2gh}} = T_r$$

This expression can be integrated, but the final expression is

very long, and moreover in practice the form of the tank or reservoir does not always conform to a geometrical law, hence we use an approximate solution which can be made as accurate as we please by taking a large number of layers between measured contour areas. The above expression then becomes

$$T_r = \sum_{H_1}^{Hm} \frac{A_a\,\delta h}{Q_i - K_d a\sqrt{2gh}}$$

$$T_r = \left\{ \frac{A_1\,\delta h}{2(Q_i - N\sqrt{H_1})} + \frac{A_2\,\delta h}{Q_i - N\sqrt{H_2}} + \frac{A_3\,\delta h}{Q_i - N\sqrt{H_3}} + \cdots + \frac{A_m\,\delta h}{2(Q_i - N\sqrt{H_m})} \right\}$$

The first and last terms are divided by 2 because we start and end at the middle sections of the upper and lower layers, hence the thickness of these layers is only one half as great as that of the intermediate layers.

If we require to find the time necessary to lower the surface, *i.e.* when the water leaves more rapidly than it enters, we have—

$$T_l = \left\{ \frac{A_1\,\delta h}{2(N\sqrt{H_1} - Q_i)} + \frac{A_2\,\delta h}{N\sqrt{H_2} - Q_i} + \frac{A_3\,\delta h}{N\sqrt{H_3} - Q_i} + \cdots + \frac{A_m\,\delta h}{2(N\sqrt{H_m} - Q_i)} \right\}$$

Where A_1 is the area of the surface at a height H_1 above the middle of the culvert; and A_2 is the area at a height $H_2 = H_1 + \delta h$, and A_3 at a height $H_3 = H_2 + \delta h$, and so on, and $N = K_d a\sqrt{2g}$.

Example.—Let $Q_i = 48$ cubic feet per second

$H_1 = 12$ ft., $\delta h = 0{\cdot}5$ ft., $K_d A\sqrt{2g} = 7$

$A_1 = 140$ sq. ft., $A_2 = 151$, $A_3 = 162$, $A_4 = 170$, $A_5 = 190$

Then the time required to raise the level from 12 to 14 feet would be—

$$T_r = \frac{140 \times 0{\cdot}5}{2(48 - 7\sqrt{12})} + \frac{151 \times 0{\cdot}5}{48 - 7\sqrt{12{\cdot}5}} + \frac{162 \times 0{\cdot}5}{48 - 7\sqrt{13}} + \frac{170 \times 0{\cdot}5}{48 - 7\sqrt{13{\cdot}5}} + \frac{190 \times 0{\cdot}5}{2(48 - 7\sqrt{14})} = 14{\cdot}3 \text{ seconds.}$$

(vi.) *Time of discharge through a submerged orifice.*—In Fig. 639 we have—

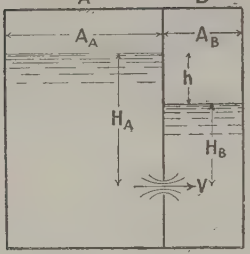


FIG. 639.

$$\delta H_A + \delta H_B = \delta h$$

$$\text{and } \delta H_A + \delta H_A \frac{A_A}{A_B} = \delta h$$

$$\delta H_A = \frac{\delta h}{1 + \frac{A_A}{A_B}} = \frac{A_B\,\delta h}{A_B + A_A}$$

$$A_A \delta H_A = K_d A \sqrt{2gh}\,\delta t$$

$$\delta t = \frac{A_A \times A_B\,\delta h}{(A_A + A_B) K_d A \sqrt{2gh}}$$

$$T = \frac{A_A \times A_B}{(A_A + A_B) K_d A \sqrt{2g}} \int_{H_1}^{H} h^{-\frac{1}{2}}\,\delta h$$

$$T = \frac{2A_A \times A_B}{(A_A + A_B) K_d A \sqrt{2g}} (H^{\frac{1}{2}} - H_1^{\frac{1}{2}})$$

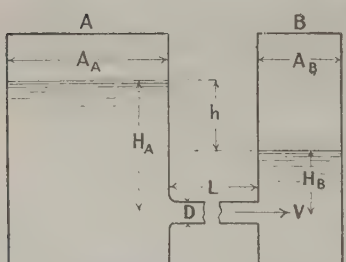


FIG. 640.

where H and H_1 are the initial and final differences of heads.

Time of discharge when the two tanks are connected by a pipe of length L.—The loss of head h_f due to friction in the pipe is $\frac{LV^2}{KD}$ (see page 681), and the head h_e dissipated in eddies at the outlet is $\frac{V^2}{2g}$

$$\text{hence } V = \sqrt{\frac{h + h_e}{\frac{L}{KD} + \frac{1}{2g}}} = \sqrt{\frac{h}{\frac{L}{KD} + \frac{1}{2g}}} = \sqrt{\frac{h}{C}}$$

Hence from similar reasoning to that given in the last paragraph we have—

$$T = \frac{A_A A_B \sqrt{C}}{(A_A + A_B)A} \int_{H_1}^{H} h^{-\frac{1}{2}}\,dh$$

$$T = \frac{2A_A A_B \sqrt{C}}{(A_A + A_B)A} (H^{\frac{1}{2}} - H_1^{\frac{1}{2}})$$

Where A is the area of the pipe which is taken to be bell-mouthed at entry, if it be otherwise the loss at entry (see p. 673) must be added to the friction loss.

Time required to lower the Water in a Tank when it flows over a Weir or Notch.—*Rectangular Notch.*—By the methods already given for orifices we have—

$$A_a dh = \tfrac{2}{3} K_d B H \sqrt{2gH}\, dt$$

$$\int dt = \frac{3A_a}{2K_d B\sqrt{2g}} \int_{H_2}^{H_1} h^{-\frac{1}{2}}\, dh$$

$$T = \frac{3A_a}{2K_d B\sqrt{2g}} \left(\frac{H_1^{-\frac{1}{2}} - H_2^{-\frac{1}{2}}}{-\frac{1}{2}} \right) = \frac{3A_a}{K_d B\sqrt{2g}} \left(\frac{1}{H_2^{\frac{1}{2}}} - \frac{1}{H_1^{\frac{1}{2}}} \right)$$

Right-angled Vee Notch.—In this case we get by similar reasoning—

$$T = \frac{15A_a}{8K_d\sqrt{2g}} \int_{H_2}^{H_1} h^{-2\cdot5}\, dh = \frac{15A_a}{8K_d\sqrt{2g}} \left(\frac{H_1^{-1\cdot5} - H_2^{-1\cdot5}}{-1\cdot5} \right)$$

$$T = \frac{5A_a}{4K_d\sqrt{2g}} \left(\frac{1}{H_2^{1\cdot5}} - \frac{1}{H_1^{1\cdot5}} \right)$$

Flow through Pipes of Variable Section.—For the present we shall only deal with pipes running full, in which the section varies very gradually from point to point. If the variation be abrupt, an entirely different action takes place. This particular case we shall deal with later on. The main point that we have to concern ourselves with at present is to show that the energy of the water at any section of the pipe is constant—neglecting friction.

If W lbs. of water be raised from a given datum to a receiver at a certain height h feet above, the work done in raising the water is Wh foot-lbs., or h foot-lbs. per pound of water. By lowering the water to the datum, Wh foot-lbs. of work will be done. Hence, when the water is in the raised position its energy is termed its energy of position, or—

The energy of position = Wh foot-lbs.

If the water were allowed to fall freely, *i.e.* doing no work in its descent, it would attain a velocity V feet per second, where $V = \sqrt{2gh}$, or $h = \frac{V^2}{2g}$. Then, since no energy is destroyed in the fall, we have $Wh = \frac{WV^2}{2g}$ foot-lbs. of energy stored in the falling water when it reaches the datum, or $\frac{V^2}{2g}$ foot-lbs. per pound of water. This energy, which is due to its velocity, is termed its kinetic energy, or energy of motion; or—

The energy of motion = $\frac{WV^2}{2g}$

If the water in the receiver descends by a pipe to the datum level—for convenience we will take the pipe as one square inch area—the pressure p at the foot of the pipe will be wh lbs. per square inch. This pressure is capable of overcoming a resistance through a distance l feet, and thereby doing pl foot-lbs. of work; then, as no energy is destroyed in passing along the pipe, we have $pl = Wh = \frac{Wp}{w}$ foot-lbs. of work done by the water under pressure, or $\frac{p}{w}$ foot-lbs. per pound of water. This is known as its pressure-energy, or—

$$\text{The pressure-energy} = \frac{Wp}{w}$$

Thus the energy of a given quantity of water may exist exclusively in either of the above forms, or partially in one form and partially in another, or in any combination of the three.

$$\left.\begin{matrix}\text{Total energy per}\\ \text{pound of water}\end{matrix}\right\} = \left\{\begin{matrix}\text{energy of}\\ \text{position}\end{matrix}\right\} + \left\{\begin{matrix}\text{energy of}\\ \text{motion}\end{matrix}\right\} + \left\{\begin{matrix}\text{pressure-}\\ \text{energy}\end{matrix}\right\}$$

$$= h + \frac{V^2}{2g} + \frac{p}{w}$$

This may, perhaps, be more clearly seen by referring to the figure.

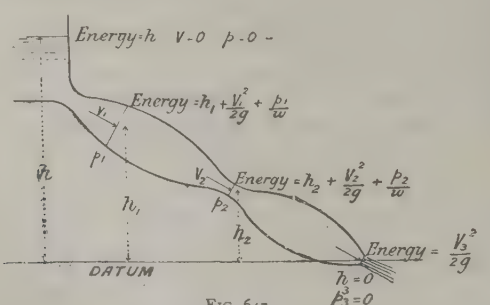


FIG. 641.

Then, as no energy of the water is destroyed on passing through the pipe, the total energy at each section must be the same, or—

$$h_1 + \frac{V_1^2}{2g} + \frac{p_1}{w} = h_2 + \frac{V_2^2}{2g} + \frac{p_2}{w} = \text{constant}$$

The quantity of water passing any given section of the pipe in a given time is the same, or—

$$Q_1 = Q_2$$
$$\text{or } A_1V_1 = A_2V_2.$$
$$\frac{V_1}{V_2} = \frac{A_2}{A_1}$$

or the velocity of the water varies inversely as the sectional area—

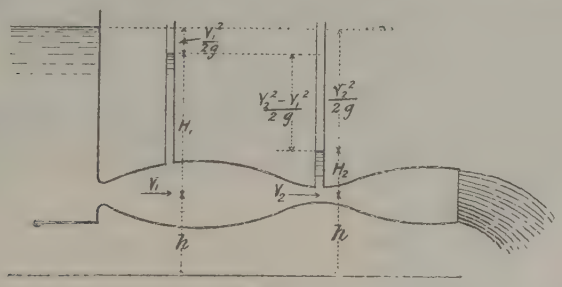


FIG. 642.

Some interesting points in this connection were given by the late Mr. Froude at the British Association in 1875.

Let vertical pipes be inserted in the main pipe as shown; then the height H, to which the water will rise in each, will be proportional to the pressure, or—

$$H_1 = \frac{p_1}{w_1}, \text{ and } H_2 = \frac{p_2}{w}$$

and the total heights of the water-columns above datum—

$$\frac{p_1}{w} + h, \text{ and } \frac{p_2}{w} + h$$

and the differences of the heights—

$$\frac{p_1}{w} + h - \frac{p_2}{w} - h = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$
$$H_1 - H_2 = \frac{V_2^2 - V_1^2}{2g}$$

from the equation given above.

Thus we see that, when water is steadily running through

a full pipe of variable section, the pressure is greatest at the greatest section, and least at the least section.

In addition to many other experiments that can be made to prove that such is the case, one has been devised by Professor Osborne Reynolds that beautifully illustrates this point. Take a piece of glass tube, say $\frac{3}{4}$ inch bore drawn down to a fine waist in the middle of, say, $\frac{1}{20}$ inch diameter; then, when water is forced through it at a high velocity, the pressure is so reduced at the waist that the water boils and hisses loudly. The pressure is atmospheric at the outlet, but very much less at the waist. The hissing in water-injectors and partially opened valves is also due to this cause.

Venturi Water-meter.—An interesting application of this principle is the Venturi water-meter. The water is forced through a very easy waist in a pipe, and the pressure measured at the smallest and largest section; then, if the difference of the heads corresponding to the two pressures be H_0 in feet of water (Fig. 643)—

$$\frac{V_2^2 - V_1^2}{2g} = H_0, \text{ or } V_2^2 - V_1^2 = 2gH_0$$

Let $A_1 = nA_2$; then $V_1 = \frac{V_2}{n}$

$$\text{hence } V_2^2 - \left(\frac{V_2}{n}\right)^2 = 2gH_0 \text{ and } V_2 = \sqrt{\frac{2gH_0}{1 - \frac{1}{n^2}}}$$

$$Q = A_2V_2 = A_2 \sqrt{\frac{2g}{1 - \frac{1}{n^2}}} \sqrt{H_0} = C\sqrt{H_0},$$

$$\text{where } C = A_2 \sqrt{\frac{2g}{1 - \frac{1}{n^2}}}$$

The difference of head is usually measured by a mercury gauge (shown in broken lines in Fig. 643), and the tubes above the surface of the mercury (sp. gravity 13·6) should be kept full of water, the mercury head H_m is most conveniently measured in inches.

$$\text{Then we get, } Q = C\sqrt{\frac{(13{\cdot}6 - 1)H_m}{12}} = C\sqrt{1{\cdot}05H_m}$$

There is a small loss of head in the short cone due to friction which can be allowed for by the use of a coefficient

of velocity K_v which is very nearly constant over a very wide range, its value is from 0·97 to 0·98, or say, 0·975, then—

$$Q = K_v C \sqrt{1{\cdot}05} \sqrt{H_m} = C \sqrt{H_m} \text{ very nearly.}$$

At very low velocities of flow, where errors are usually of no importance, the value of K_v varies in a very erratic fashion, the reason for which is unknown at present; but for such velocities of flow as are likely to be used in practice the meter gives extremely accurate results. When used for waterworks purposes the meter is always fitted with a recorder and integrator, particulars of which can be obtained of Mr. Kent, of High Holborn, London.

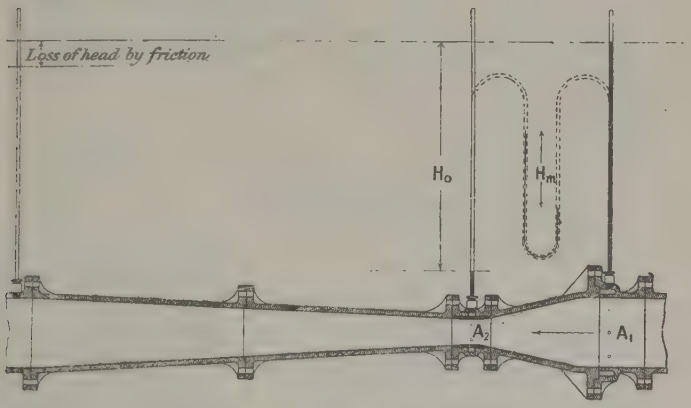


FIG. 643.

The loss of head on the whole meter often amounts to about $\frac{H_0}{7}$. For experimental data on the losses in divergent pipes, readers should refer to a paper by Gibson, "The Resistance to Flow of Water through Pipes or Passages having Divergent Boundaries," *Transactions of the Royal Society of Edinburgh*, vol. xlviii.

Radiating Currents and Free Vortex Motion.—Let the figure represent the section of two circular plates at a small distance apart, and let water flow up the vertical pipe and escape round the circumference of the plates. Take any small portion of the plates as shown; the strips represent portions of rings of water moving towards the outside. Let their areas be a_1, a_2; then, since the flow is constant, we have—

$$v_1 a_1 = v_2 a_2, \text{ or } \frac{v_2}{v_1} = \frac{a_1}{a_2} = \frac{r_1}{r_2} \text{ hence } v_2 = v_1 \frac{r_1}{r_2}$$

or the velocity varies inversely as the radius. The plates being horizontal, the energy of position remains constant; therefore—

$$\frac{v_1^2}{2g} + \frac{p_1}{w} = \frac{v_2^2}{2g} + \frac{p_2}{w} = \text{H}$$

Substituting the value of v_2 found above, we have—

$$\frac{v_1^2}{2g} + \frac{p_1}{w} - \frac{v_1^2 r_1^2}{2g \cdot r_2^2} = \frac{p_2}{w}$$

Then, substituting $\frac{v_1^2}{2g} + \frac{p_1}{w} = \text{H}$ from above, and putting $\left(\frac{v_1^2}{2g}\right)\frac{r_1^2}{r_2^2} = h_1 \frac{r_1^2}{r_2^2} = h_2$, we have—

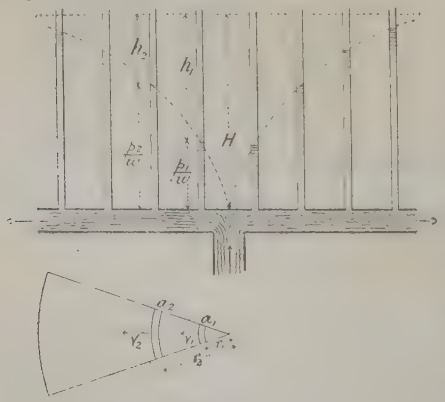


FIG. 644.

$$\text{H} - h_2 = \frac{p_2}{w}$$

Then, starting with a value for h_1, the h_2 for other positions is readily calculated and set down from the line above.

If a large number of radial segments were taken, they would form a complete cylinder of water, in which the water enters at the centre and escapes radially outwards. The distribution of pressure will be the same as in the radial segments, and the form of the water will be a solid of revolution formed by spinning the dotted line of pressures, known as Barlow's curve, round the axis.

The case in which this kind of vortex is most commonly met with is when water flows in radially to a central hole, and then escapes.

Forced Vortex.—If water be forced to revolve in and with a revolving vessel, the form taken up by the surface is readily found thus:

Let the vessel be rotating n times per second.

Any particle of water is acted upon by the following forces :—

(i.) The weight W acting vertically downwards.

(ii.) The centrifugal force $\frac{WV^2}{gr}$ acting horizontally, where V is its velocity in feet per second, and r its radius in feet.

(iii.) The fluid pressure, which is equal to the resultant of i. and ii.

From the figure, we have—

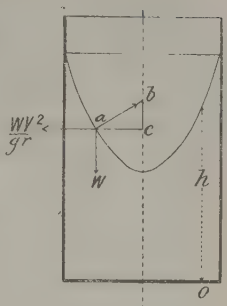


FIG. 645.

$$\frac{\frac{WV^2}{gr}}{W} = \frac{ac}{bc}$$

which may be written—

$$\frac{W2^2\pi^2r^2n^2}{Wgr} = \frac{ac}{bc}$$

But $\frac{2^2\pi^2}{g}$ is constant, say C;

$$\text{Then } Crn^2 = \frac{ac}{bc}$$

$$\text{But } ac = r$$

$$\text{therefore } Cn^2 = \frac{1}{bc}$$

And for any given number of revolutions per second n^2 does not vary; therefore bc, the subnormal, is constant, and the curve is therefore a parabola. If an orifice were made in the bottom of the vessel at o, the discharge would be due to the head h.

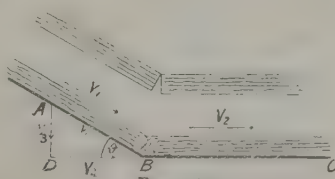


FIG. 646.

Loss of Energy due to Abrupt Change of Direction.—If a stream of water flow down an inclined surface AB with a velocity V_1 feet per second, when it reaches B the direction of flow is suddenly changed from AB to BC, and the

layers of water overtop one another, thus causing a breaking-up of the stream, and an eddying action which rapidly dissipates the energy of the stream by the frictional resistance of the particles of the water; this is sometimes termed the loss by shock. The velocity V_2 with which the water flows after passing the corner is given by the diagram of velocities ABD, from which we see that the component V_3, normal to BC, is wasted in eddying, and the energy wasted per pound of water is $\frac{V_3^2}{2g} = \frac{V_1^2 \sin^2 \theta}{2g}$.

As the angle ABD increases the loss of energy increases, and when it becomes a right angle the whole of the energy is wasted by shock (Fig. 646).

If the surface be a smooth curve (Fig. 647) in which there is no abrupt change of direction, there will be no loss due to

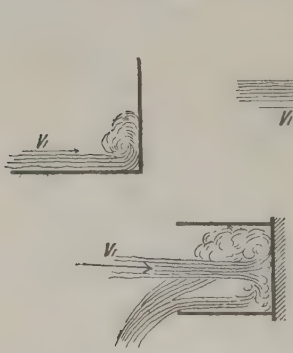


Fig. 646.

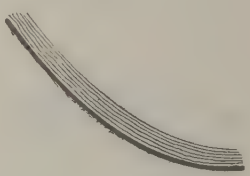


Fig. 647.

shock; hence the smooth easy curves that are adopted for the vanes of motors, etc.

If the surface against which the water strikes (normally) is moving in the same direction as the jet with a velocity $\frac{V_1}{n}$, then the striking velocity will be—

$$V_1 - \frac{V_1}{n} = V_0$$

and the loss of energy per pound of water will be—

$$\frac{V_0^2}{2g} = \frac{\left(V_1 - \frac{V_1}{n}\right)^2}{2g} = \frac{V_1^2}{2g}\left(1 - \frac{1}{n}\right)^2$$

When $n = 1$, no striking takes place, and consequently no loss of energy; when $n = \infty$, *i.e.* when the surface is stationary, the loss is $\frac{V_1^2}{2g}$, *i.e.* the whole energy of the jet is dissipated.

Loss of Energy due to Abrupt Change of Section.—When water flows along a pipe in which there is an abrupt change of section, as shown, we may regard it as a jet of water moving with a velocity V_1 striking against a surface (in this case a body of water) moving in the same direction, but with a velocity $\frac{V_1}{n}$; hence the loss of energy per pound of water is precisely the same as in the last paragraph, viz. $\frac{\left(V_1 - \frac{V_1}{n}\right)^2}{2g}$. The energy lost in this case is in eddying in the corners of the large section, as shown. As the water in the large section is moving $\frac{1}{n}$ as fast as in the small section, the area of the large section is n times the area of the small section. Then the loss of energy per pound of water, or the loss of head when a pipe suddenly enlarges n times, is—

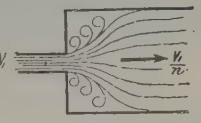


FIG. 648 (see also No. 3 facing p. 674).

$$\frac{V_1^2\left(1 - \frac{1}{n}\right)^2}{2g}$$

Or if we refer to the velocity in the large section as V_1, we have the velocity in the small section nV_1, and the loss of head—

$$\frac{(nV_1 - V_1)^2}{2g} = \frac{V_1^2}{2g}(n-1)^2$$

When the water flows in the opposite direction, *i.e.* from the large to the small section, the loss of head is due to the abrupt change of velocity from the contracted to the full section of the small stream. The contracted section in pipes under pressure is, according to some experiments made in the author's laboratory, from 0·62 to 0·66; hence, n^1 = from 1·61 to 1·5; then, the loss of head = $\frac{0{\cdot}3V_1^2}{2g}$.

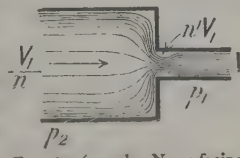


FIG. 649 (see also No. 4 facing p. 674).

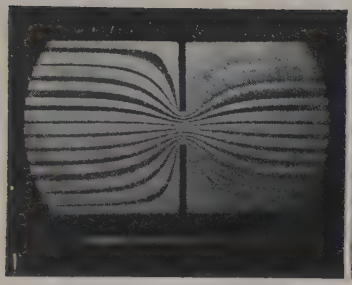
Total Loss of Energy due to a Sudden Enlargement and Contraction.—Let the section before the enlargement be termed 1, the enlarged section 2, and the section after the enlargement 3, with corresponding suffixes for velocities and pressures. Then for a horizontal pipe we have—

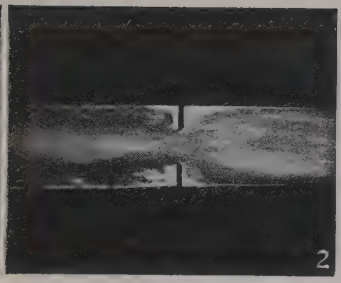
$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g} + \frac{V_1^2}{2g}\left(1 - \frac{1}{n}\right)^2$$
$$= \frac{p_3}{w} + \frac{V_3^2}{2g} + \frac{V_1^2}{2g}\left(1 - \frac{1}{n}\right)^2 + \frac{0{\cdot}3V_3^2}{2g}$$

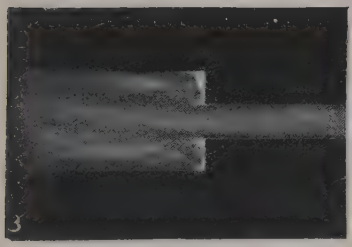
Gibson finds that the loss of energy is slightly greater than this expression gives. The author finds that the actual loss in some cases is nearly twice as great as the calculated.

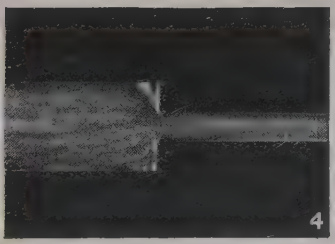
Experiments on the Character of Fluid Motion.—Some very beautiful experiments, by Professor Hele-Shaw, F.R.S., on the flow of fluids, enable us to study exactly the manner in which the flow takes place in channels of various forms. He takes two sheets of glass and fits them into a suitable frame, which holds them in position at about $\frac{1}{100}$ inch apart. Through this narrow space liquid is caused to flow under pressure, and in order to demonstrate the exact manner in which the flow takes place, bands of coloured liquid are injected at the inlet end. In the narrow sections of the channel, where the velocity of flow is greatest, the bands themselves are narrowest, and they widen out in that portion of the channel where the velocity is least. The perfect manner in which the bands converge and diverge as the liquid passes through a neck or a pierced diaphragm, is in itself an elegant demonstration of the behaviour of a perfect fluid (see Diagrams 1 and 5). The form and behaviour of the bands, moreover, exactly correspond with mathematical demonstrations of the mode of flow of perfect fluids. The author is indebted to Professor Hele-Shaw for the illustrations given, which are reproduced from his own photographs.

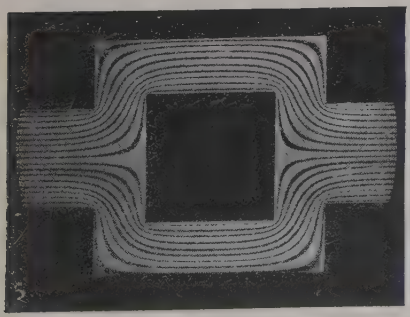
In the majority of cases, however, that occur in practice, we are unfortunately unable to secure such perfect stream-line motions as we have just described. We usually have to deal with water flowing in sinuous fashion with very complex eddyings, which is much more difficult to ocularly demonstrate than true stream-line motion. Professor Hele-Shaw's method of showing the tumultuous conditions under which the water is moving, is to inject fine bubbles of air into the water, which make the disturbances within quite evident. The diagrams 2,

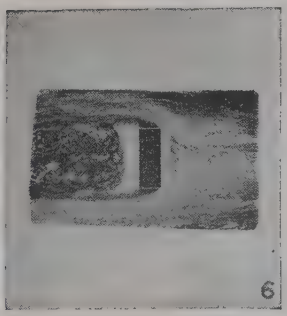












[*To face p.* 674.

FLOW OF WATER DIAGRAMS.

Kindly supplied by Professor Hele-Shaw, F.R.S.

3, 4, and 6, also reproduced from his photographs, clearly demonstrate the breaking up of the water when it encounters sudden enlargements and contractions, as predicted by theory. A careful study of these figures, in conjunction with the theoretical treatment of the subject, is of the greatest value in getting a clear idea of the turbulent action of flowing water.

Readers should refer to the original communications by Professor Hele-Shaw in the *Transactions of the Naval Architects*, 1897–98, also the engineering journals at that time.

Surface Friction.—When a body immersed in water is caused to move, or when water flows over a body, a certain resistance to motion is experienced; this resistance is termed the surface or fluid friction between the body and the water.

At very low velocities, only a thin film of the water actually in contact with the body appears to be affected, a mere skimming action; but as the velocity is increased, the moving body appears to carry more or less of the water with it, and to cause local eddying for some distance from the body. Experiments made by Professor Osborne Reynolds clearly demonstrate the difference between the two kinds of resistances—the surface resistance and the eddying resistance. Water is caused to flow through the glass pipe AB at a given velocity; a bent glass tube and funnel C is fixed in such a manner that a fine stream of deeply coloured dye is ejected. When the water flows through at a low velocity, the stream of dye runs right through like an unbroken thread; but as soon as the velocity is increased beyond a certain point, the thread breaks up and passes through in sinuous fashion, thus demonstrating that the water is not flowing through as a steady stream, as it did at the lower velocities.

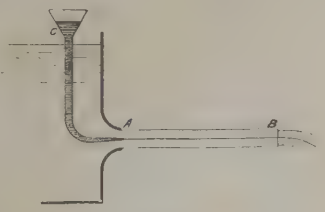


FIG. 650

Friction in Pipes.—*Continuous Flow.*—When the flow in a pipe is continuous, *i.e.* not of an eddying nature, the resistance to flow is entirely due to the viscosity of the fluid. On p. 314 we showed that the resistance to shearing a viscous fluid is—

$$F_r = \frac{AKS}{l}$$

where A is the wetted surface in square feet, K is the

coefficient of viscosity, S the speed of shearing (usually denoted by V in hydraulics) in feet per second, l is the thickness of the sheared element in feet, in the case of a pipe of radius R, $l = R$. Then, without going fully into the question, for which treatises on Hydraulics should be consulted—

Let P_1 = the initial pressure in pounds per sq. foot.
P_2 = the final pressure in pounds per sq. foot.
L = the length of the pipe in feet.
A_p = the area of the pipe in sq. feet.
$F_e = (P_1 - P_2)A_p = W_w(H_1 - H_2)A_p$.
$F_i = W_w h_c A_v$

where h_c is the loss of head in feet due to the resistance on a length of pipe L.

$$\text{Then } W_w h_c A_p = \frac{AKV}{R}$$

$$h_c = \frac{(2\pi RL)KV}{W_w(\pi R^2)R} = \frac{2LKV}{W_w R^3} = \frac{8LKV}{W_w D^2}$$

$$h_c = \frac{LV}{CD^2}$$

This expression only holds for stream line, or continuous flow, the critical velocity V_c at which the flow changes from continuous to sinuous is always much higher than the velocity V_{c1} at which the flow changes from sinuous to continuous. The critical velocity also largely depends upon the temperature of the water owing to a change in the viscosity.

Let V_c = the critical velocity, *i.e.* the velocity in feet per second at which the flow changes from continuous to sinuous.
V_{c1} = ditto at which the flow changes from sinuous to continuous.
n and n_1 = coefficients which depend upon the temperature of the water.
D = the diameter of the pipe in feet,

$$\text{then } V_c = \frac{n}{D}$$

$$\text{and } V_{c1} = \frac{n_1}{D}$$

Temperature Fahrenheit.	n.	n_1.	C.
32	0·25	0·040	52,500
40	0·21	0·034	61,000
60	0·15	0·025	83,500
80	0·12	0·019	107,000
100	0·10	0·016	134,000
120	0·08	0·013	164,000
140	0·065	0·011	204,000
160	0·055	0·009	240,000
180	0·047	0·008	278,000
200	0·040	0·006	328,000
212	0·037	0·006	350,000

Mr. E. C. Thrupp has, however, shown that the values

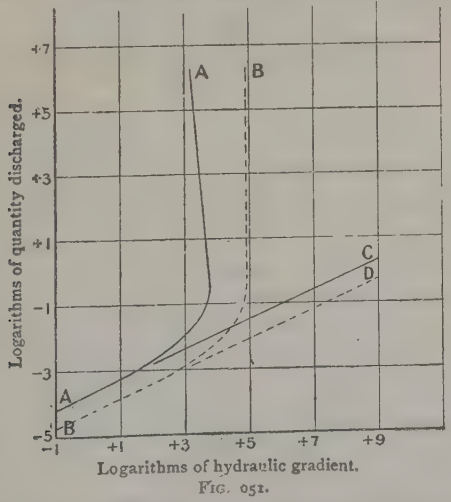


FIG. 051.

given in the above table only hold for very small pipes; in the case of large pipes, channels, and rivers the velocity at

which the water breaks up is very much greater than this expression gives. See a paper on "Hydraulics of the Resistance of Ships," read at the Engineering Congress in Glasgow, 1901; also *Engineering*, December 20, 1901, from which the curves in Fig. 651 have been taken. The Osborne Reynolds' law is represented by AC and BD, whereas Thrupp shows that experimental values lie somewhere between AA and BB.

The change points from continuous to sinuous flow are shown in Fig. 652. At low velocities of flow the loss of

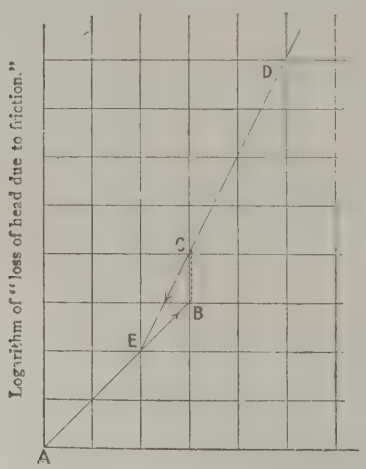


FIG. 652.

head varies simply as the velocity, therefore the slope of the line AB is 1 to 1. At B the flow suddenly changes to sinuous flow, and at higher velocities the loss of head varies approximately as the square of the velocity, hence the slope of the line CD is 2 to 1. When the velocity is decreased the loss of head continues to vary as the square until E is reached, and below that it returns to the state in which it varies simply as the velocity. The point B corresponds to V_o, and the point E to V_{o1}.

For further details of Reynolds' investigations, the reader is referred to the original papers in the *Philosophical Transactions* for 1884 and 1893; also to Gibson's "Hydraulics and

its Applications," and Turner and Brightmore's "Waterworks Engineering."

Sinuous Flow.—Experiments by Mr. Froude at Torquay (see *Brit. Ass. Proceedings*, 1874), on the frictional resistance of long planks, towed end-on through the water at various velocities, showed that the following laws appear to hold within narrow limits :—

(i.) The friction varies directly as the extent of the wetted surface.

(ii.) The friction varies directly as the roughness of the surface.

(iii.) The friction varies directly as the square of the velocity.

(iv.) The friction is independent of the pressure.

For fluids other than water, we should have to add—

(v.) The friction varies directly as the density and viscosity of the fluid.

Hence, if S = the wetted surface in square feet ;

f = a coefficient depending on the roughness of the surface ; *i.e.* the resistance per square foot at 1 foot per second in pounds ;

V = velocity of flow relatively to the surface in feet per second ;

R = frictional resistance in pounds ;

Then, $$R = Sf V^2$$

Some have endeavoured to prove from Mr. Froude's own figures that the first of the laws given above does not even approximately hold. The basis of their argument is that the frictional resistance of a plank, say 50 feet in length, is not ten times as great as the resistance of a plank 5 feet in length. This effect is, however, entirely due to the fact that the first portion of the plank meets with water at rest, and, therefore, if a plank be said to be moving at a speed of 10 feet a second, it simply means that this is the relative velocity of the plank and the still water. But the moving plank imparts a considerable velocity to the surrounding water by dragging it along with it, hence the relative velocity of the rear end of the plank and the water is less than 10 feet a second, and the friction is correspondingly reduced. In order to make this point clear the author has plotted the curves in Figs. 653 and 654, which are deduced from Mr. Froude's own figures. It is worthy of note that planks with rough surfaces drag the water along with them to a much greater extent than is the case with planks having

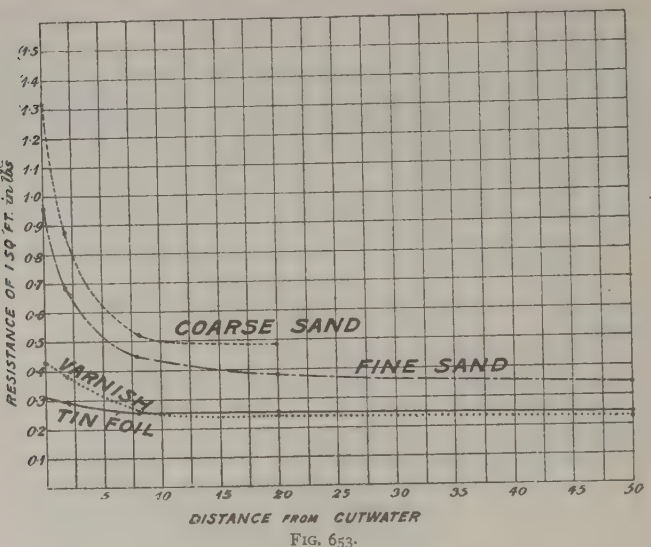


FIG. 653.

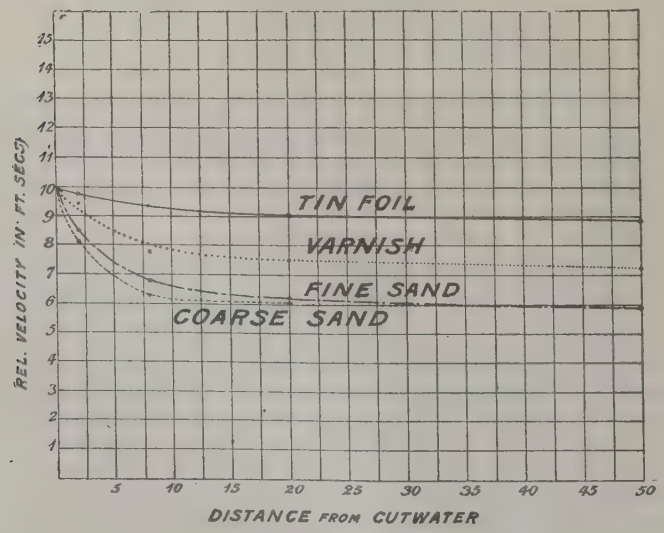


FIG. 654.

smooth surfaces, a result quite in accordance with what one might expect.

The value of f deduced from these experiments is—

Surface covered with coarse sand		0·0132 lb.
„ „ fine „		0·0096 „
„ „ varnish		0·0043 „
„ „ tinfoil		0·0031 „

Professor Unwin and others have also experimented on the friction of discs revolving in water, and have obtained results very closely in accord with those obtained by Mr. Froude.

Reducing the expression for the frictional resistance to a form suitable for application to pipes, we have, for any length of pipe L feet, the pressure P_1 in pounds per square foot at one end greater than the pressure P_2 at the other end, on account of the friction of the water. Then, if A be the area of the pipe in square feet, we have—

$$R = (P_1 - P_2)A$$

Then, putting $P_1 = h_1 W_w$ and $P_2 = h_2 W_w$, we have—

$$R = W_w A(h_1 - h_2) = W_w A h$$

where h is the loss of head due to friction on any length of pipe L; then—

$$W_w A h = S f V^2$$

$$\text{or } \frac{W_w \pi D^2 h}{4} = L \pi D f V^2$$

$$\text{hence } h = \frac{4f}{W_w} \cdot \frac{LV^2}{D} = \frac{1}{K} \cdot \frac{LV^2}{D}$$

$$\text{or } h = \frac{f}{W_w} \cdot \frac{LV^2}{R} = \frac{LV^2}{4KR}$$

where R = hydraulic mean depth (see p. 683).

The coefficient $\frac{1}{K}$ has to be obtained by experiment; according to D'Arcy—

$$\frac{1}{K} = \frac{1}{3200}\left(1 + \frac{1}{12D}\right)$$

where D is the diameter of the pipe in feet.

D'Arcy's experiments were made on pipes varying in diameter from $\frac{1}{2}$ inch up to 20 inches; for small pipes his coefficient appears to hold tolerably well, but it is certainly incorrect for large pipes.

The author has recently looked into this question, and finds that the following expression better fits the most recent published experiments for pipes of over 8 inch diameter (see a paper by Lawford, *Proceedings I.C.E.*, vol. cliii. p. 297):—

$$\frac{1}{K} = \frac{1}{5000}\left(1 + \frac{1}{2D}\right) \text{ for clean cast-iron pipes}$$

$$\frac{1}{K} = \frac{1}{2500}\left(1 + \frac{1}{2D}\right) \text{ for incrusted pipes}$$

But the above expression at the best is only a rough approximation, since the value of f varies very largely for different surfaces, and the resistance does not always vary as the square of the velocity, nor simply inversely as D.

The energy of motion of 1 lb. of water moving with a velocity V feet per second is $\frac{V^2}{2g}$; hence the whole energy of motion of the water is dissipated in friction when—

$$\frac{V^2}{2g} = \frac{LV^2}{KD}$$

Taking K = 2400 and putting in the numerical value for g, we get L = 37D. This value 37, of course, depends on the roughness of the pipe. We shall find this method of regarding frictional resistances exceedingly convenient when dealing with the resistances of T's, elbows, etc., in pipes.

Still adhering to the rough formula given above, we can calculate the discharge of any pipe thus:

$$\left.\begin{array}{l}\text{The quantity discharged in}\\ \text{cubic feet per second}\end{array}\right\} = Q = AV = \frac{\pi D^2 V}{4}$$

From the same formula, we have—

$$V = \sqrt{\frac{2400Dh}{L}}$$

Substituting this value, we have—

$$Q = 38{\cdot}5 D^{\frac{5}{2}} \sqrt{\frac{h}{L}} = 38{\cdot}5 D^2 \sqrt{\frac{Dh}{L}}$$

Thrupp's Formula for the Flow of Water.—All formulas for the flow of water are, or should be, constructed to fit experiments, and that which fits the widest range of experiments is of course the most reliable. Several investigators in recent years have collected together the results of published experiments, and have adjusted the older formulas or have constructed new ones to better accord with experiments. There is very little to choose between the best of recent formulas, but on the whole the author believes that this formula best fits the widest range of experiments; others are equally as good for smaller ranges. It is a modification of Hagen's formula, and was published in a paper read before the Society of Engineers in 1887.

Let V = velocity of flow in feet per second;
R = hydraulic mean radius in feet, *i.e.* the area of the stream divided by the wetted perimeter, and is $\frac{D}{4}$ for circular and square pipes:
L = length of pipe in feet;
h = loss of head due to friction in feet;
S = cosecant of angle of slope = $\frac{L}{h}$;
Q = quantity of water flowing in cubic feet per second.

$$\text{Then } V = \frac{R^x}{C \sqrt[n]{S}}$$

where x, C, n are coefficients depending on the nature of the surface of the pipe or channel.

For small values of R, more accurate results will be obtained by substituting for the index x the value $x + y\sqrt{\frac{z-R}{R}}$.

In this formula the effect of a change of temperature is not taken into account. The friction varies, roughly, inversely as the absolute temperature of the water.

Surface.	*n*.	C.	*x*.	*y*.	*z*.
Wrought-iron pipes	1·80	0·004787	0·65	0·018	0·07
Riveted sheet-iron pipes ...	1·825	0·005674	0·677	—	—
New cast-iron pipes	1·85	0·005347	0·67	—	—
	2·00	0·006752	0·63	—	—
Lead pipes	1·75	0·005224	0·62	—	—
Pure cement rendering ...	1·74	0·004000	0·67	—	—
	1·95	0·006429	0·61	—	—
Brickwork (smooth)	2·00	0·007746	0·61	—	—
,, (rough)	2·00	0·008845	0·625	0·01224	0·50
Unplaned plank	2·00	0·008451	0·615	0·03349	0·50
Small gravel in cement ...	2·00	0·01181	0·66	0·03938	0·60
Large ,, ,, ...	2·00	0·01415	0·705	0·07590	1·00
Hammer-dressed masonry ...	2·00	0·01117	0·66	0·07825	1·00
Earth (no vegetation) ...	2·00	0·01536	0·72	—	—
Rough stony earth	2·00	0·02144	0·78	—	—

If we take x as 0·62, and $n = 2$, C = 0·0067, we get—

$$Q = \frac{D^{2\cdot62}}{3\cdot01C\sqrt[2]{S}}$$

which reduces to—

$$Q = 49\cdot6D^{2\cdot62}\sqrt{\frac{h}{L}}$$

Similarly, for new cast-iron pipes—

$$h = \frac{LV^{1\cdot85}}{3200D^{1\cdot24}}$$

taking $n = 1\cdot85$, and $x = 0\cdot67$.

These expressions should be compared with the rougher ones given on pp. 681, 682.

Virtual Slope.—If two reservoirs at different levels be freely connected by a main through which water is flowing, the pressure in the main will diminish from a maximum at the upper reservoir to a minimum at the lower, and if glass pipes be inserted at intervals in the main, the height of the water in each will represent the pressure at the respective points, and the difference in height between any two points will represent the loss of head due to friction on that section. If a straight line be drawn from the surface of the water in the one reservoir to that in the other, it will touch the surface of the water in all the glass tubes in the case of a main of uniform diameter and roughness. The slope of this line is known as the "virtual

slope" of the main. If the lower end of the main be partially closed, it will reduce the virtual slope; and if it be closed altogether, the virtual slope will be *nil*, or the line will be horizontal, and, of course, no water will flow. The velocity of flow is proportional to the virtual slope, the tangent of the angle of slope is the $\frac{h}{L}$ in the expressions we use for the friction in pipes.

The above statement is only strictly true when there is no loss of head at entry into the main, and when the main is of uniform diameter and roughness throughout, and when there are no artificial resistances. When any such irregularities do exist, the construction of the virtual slope line offers, as a rule, no difficulties, but it is no longer straight.

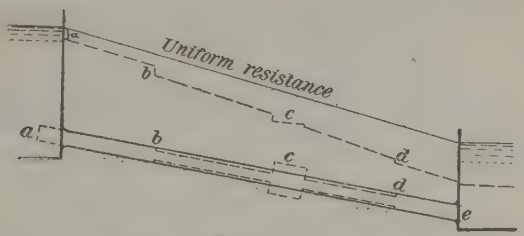


FIG. 655.

In the case of the pipe shown in full lines the resistance is uniform throughout, but in the case of the pipe shown in broken line, there is a loss at entry *a*, due to the pipe projecting into the top reservoir; the virtual slope line is then parallel to the upper line until it reaches *b*, when it drops, due to a sudden contraction in the main; from *b* to *c* its slope is steeper than from *a* to *b*, on account of the pipe being smaller in diameter; at *c* there is a drop due to a sudden enlargement and contraction, the slope from *c* to *d* is the same as from *b* to *c*, and at *d* there is a drop due to a sudden enlargement, then from *d* to *e* the line is parallel to the upper line. The amount of the drop at each resistance can be calculated by the methods already explained.

The pressure at every point in the main is proportional to the height of the virtual slope line above the main; hence, if the main at any point rises above the virtual slope line, the pressure will be negative, *i.e.* less than atmospheric, or there will be a partial vacuum at such a point. If the main rises more than 34 feet above the virtual slope line, the water will

break up, and may cause very serious trouble. In waterworks mains great pains are taken to keep them below the virtual slope line, but if it is impracticable to do so, air-cocks are placed at such summits to prevent the pressure falling below that of the atmosphere; the flow is then due to the virtual slope between the upper reservoir and this point. In certain cases it is better to put an artificial resistance in the shape of a pierced diaphragm or a valve on the outlet end of the pipe, in order to raise the virtual slope line sufficient to bring it above every point of the main, or the same result may be accomplished by using smaller pipes for the lower reaches.

Flow of Water down an Open Channel on a Steep Slope.—

Let u = the initial velocity of the water in feet per second.
V = the velocity of the water after running along a portion of the channel of length l (feet) measured on the slope.
θ = the angle of the slope to the horizontal.
H = the vertical fall of the channel in the length l then $H = l \sin \theta$.
h = the loss of head in feet due to friction while the water is flowing along the length l.
R = the hydraulic mean depth of the channel.
K = four times the constant in D'Arcy's formula for pipes (multiplied by 4 to make it applicable to channels, and using the hydraulic mean depth instead of the diameter).

The velocity of a particle of water running down a slope is the same as that of a particle falling freely through the same vertical height, if there is no friction. When there is friction we have—

$$V^2 = u^2 + 2g(H - h)$$

The loss of head due to friction is—

$$h = \frac{lV^2}{KR}$$

$$V^2 = u^2 + 2gH - \frac{2glV^2}{KR}$$

$$V = \sqrt{\frac{(u^2 + 2gH)KR}{KR + 2gl}} = \sqrt{\frac{(u^2 + 2gl \sin \theta)KR}{KR + 2gl}}$$

This expression is used by calculating in the first place the value for V, taking for H some small amount, say 10 feet. Then all the quantities under the root are constant, except u, hence we may write—

$$V_1 = \sqrt{\frac{(u^2 + n \sin \theta)m}{m + n}}$$

for the first 10 feet, then the u for the second 10 feet becomes V_1, hence—

$$V_2 = \sqrt{\frac{(V_1^2 + n \sin \theta)m}{m + n}}$$

and for the third 10 feet—

$$V_3 = \sqrt{\frac{(V_2^2 + n \sin \theta)m}{m + n}}$$

and so on for each succeeding 10 feet.

The following table shows a comparison between the results obtained by Mr. Hill's formula [1] and that given above—

$u = 15$ feet per second.
R = 1·53.
K = 15130 (deduced from Mr. Hill's constant).
$\theta = 12^\circ 40'$.
H = 10 feet. $l = 45{\cdot}7$ feet.

Fall reckoned from the top of the slope in feet.	Values of V.	
	Hill.	Author.
0	15·00	15·0
10	28·34	27·9
20	36·22	35·6
30	41·95	41·3
40	46·42	45·9
50	50·03	49·5
60	53·01	52·6
70	55·50	55·2
80	57·60	57·4
90	59·39	59·3
100	60·93	60·9

[1] *Proceedings Institution of Civil Engineers*, vol. clxi., p. 345.

If the slope varies from point to point the angle θ must be taken to suit: similarly, if the hydraulic mean depth varies the proper values of R must be inserted in the expression.

An interesting application of the above theory to the formation of ponds at the approach end of culverts will be found in a paper by the author, "The Flooding of the Approach End of a Culvert," *Proc. Inst. Civil Engineers*, vol. clxxxvi.

Flow through a Pipe with a Restricted Outlet.—When a pipe is fitted with a valve or nozzle at the outlet end, the kinetic energy of the escaping water is usually quite a large fraction of the potential energy of the water in the upper reservoir; but in the absence of such a restriction, the kinetic energy of the escaping water is quite a negligible quantity in the case of long pipes.

The velocity of the water can be found thus:

Let H = head of water above the valve in feet;
L = length of main in feet;
V = velocity of flow in the main in feet per second;
K = a constant depending on the roughness of the pipe (see p. 681);
D = diameter of the pipe in feet;
V_1 = velocity of flow through the valve opening;
n = the ratio of the valve opening to the area of the pipe, or $n = \frac{V}{V_1}$.

The total energy per pound of water is H foot-lbs. This is expended (i.) in overcoming friction, (ii.) in imparting kinetic energy to the water issuing from the valve; or—

$$H = \frac{LV^2}{KD} + \frac{V_1^2}{2g}$$

Substituting the value of $V_1 = \frac{V}{n}$, we have—

$$V = \sqrt{\frac{H}{\frac{L}{KD} + \frac{1}{2gn^2}}}$$

On p. 717 we give some diagrams to show how the velocity of the water in the main varies as the valve is closed; in all cases we have neglected the frictional resistance of the valve itself, which will vary with the type employed. In the case of a long pipe it will be noticed that the velocity of flow

in the pipe, and consequently the quantity of water flowing, is but very slightly affected by a considerable closing of the valve, *e.g.* by closing a fully opened valve on a pipe 1000 feet long to 0·3 of its full opening, the quantity of water has only been reduced to 0·9 of its full flow. But in the case of very short pipes the quantity passing varies very nearly in the same proportion as the opening of the valve.

Resistance of Knees, Bends, etc.—We have already shown that if the direction of a stream of water be abruptly changed through a right angle, the whole of its energy of motion is destroyed; a similar action occurs in a right-angled knee or elbow in a pipe, hence its resistance is at least equivalent to the friction in a length of pipe about 37 diameters long. In addition to this loss, the water overshoots the corner, as shown in Fig. 656, and causes a sudden contraction and enlargement of section with a further loss of head. The losses in sockets, sudden enlargements, etc., can be readily calculated; others have been obtained by experiment, and their values are given in the following table. When calculating the friction of systems of piping, the equivalent lengths as given should be added, and the friction calculated as though it were a length of straight pipe.

Fig. 656.

Nature of resistance.	Equivalent length of straight pipe expressed in diameters, on the basis of $L = 36D$.
Right-angled knee or elbow (experiments)	30–40 in plain pipe 50–90 with screwed elbow
Right-angled bends, exclusive of resistance of sockets at ends, radius of bend, = 4 diameters	3–15
Ditto including sockets (experiments) ...	22–30
Sockets (screwed) calculated from the sudden enlargement and contraction (average sizes)	24
Ditto by experiment	16–20
Sudden enlargement to a square-ended pipe, where $n = \frac{\text{large area}}{\text{small area}}$	$36\left(1 - \frac{1}{n}\right)^2$
Sudden contraction	12 approx.
Mushroom valves	120–400
Plug cock { handle 15° / turned 30° / through 45° } Unwin ...	27 200 1100
1½ inch check valve	700–1500
1½ inch ball check valve	2000–3000
Sluice and slide valves $n = \frac{\text{port area}}{\text{area of opening}}$	$100(n - 1)^2$
Pierced diaphragm $n = \frac{\text{area of pipe}}{\text{area of hole}}$ ...	$36(1{\cdot}6n - 1)^2$
Water entering a re-entrant pipe, such as a Borda's mouthpiece	18
Water entering a square-ended pipe flush with the side of the tank	9–12

Velocity of Water in Pipes.—Water is allowed to flow at about the velocities given below for the various purposes named :—

Pressure pipes for hydraulic purposes for long mains	3 to 4 feet per sec.
Ditto for short lengths [1]	Up to 25 ,,
Ditto through valve passages [1]	Up to 50 ,,
Pumping mains	3 to 5 ,,
Waterworks mains	2 to 3 ,,

[1] Such velocities are unfortunately common, but they should be avoided if possible.

CHAPTER XIX.

HYDRAULIC MOTORS AND MACHINES.

THE work done by raising water from a given datum to a receiver at a higher level is recoverable by utilizing it in one of three distinct types of motor.

1. Gravity machines, in which the weight of the water is utilized.

2. Pressure machines, in which the pressure of the water is utilized.

3. Velocity machines, in which the velocity of the water is utilized.

Gravity Machines.—In this type of machine the weight-energy of the water is utilized by causing the water to flow into the receivers of the machine at the higher level, then to descend with the receivers in either a straight or curved path to the lower level at which it is discharged. If W lbs. of water have descended through a height H feet, the work done = WH foot-lbs. Only a part, however, of this will be utilized by the motor, for reasons which we will now consider.

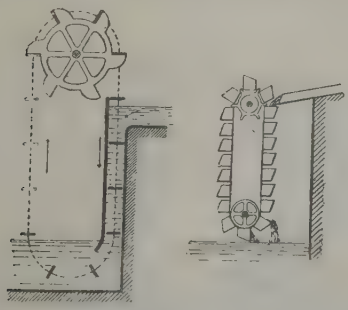


FIG. 657.

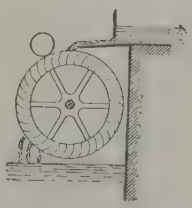


FIG. 658.

The illustrations, Figs. 657, 658, show various methods of

utilizing the weight-energy of water. Those shown in Fig. 657 are very rarely used, but they serve well to illustrate the principle involved. The ordinary overshot wheel shown in Fig. 658 will perhaps be the most instructive example to investigate as regards efficiency.

Although we have termed all of these machines *gravity* machines, they are not purely such, for they all derive a small portion of their power from the water striking the buckets on entry. Later on we shall show that, for motors which utilize the velocity of the water, the maximum efficiency occurs when the velocity of the jet is twice the velocity of the buckets or vanes.

In the case of an overshot water-wheel, it is necessary to keep down the linear velocity of the buckets, otherwise the centrifugal force acting on the water will cause much of it to be wasted by spilling over the buckets. If we decide that the inclination of the surface of the water in the buckets to the horizontal shall not exceed 1 in 8, we get the peripheral velocity of the wheel $V_w = 2\sqrt{R}$, where R is the radius of the wheel in feet.

Take, for example, a wheel required for a fall of 15 feet. The diameter of the wheel may be taken as a first approximation as 12 feet. Then the velocity of the rim should not exceed $2\sqrt{6}$ = say 5 feet per second. Then the velocity of the water issuing from the sluice should be 10 feet per second; the head h required to produce this velocity will be $h = \frac{V^2}{2g}$, or, introducing a coefficient to allow for the friction in the sluice, we may write it $h = \frac{1{\cdot}1V^2}{2g} = 1{\cdot}6$ foot. One-half of this head, we shall show later, is lost by shock. The depth of the shroud is usually from 0·75 to 1 foot; the distance from the middle of the stream to the c. of g. of the water in the bucket may be taken at about 1 foot, which is also a source of loss.

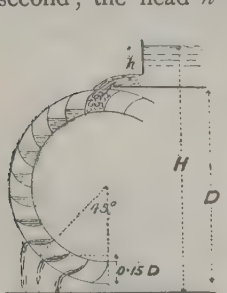


FIG. 659.

The next source of waste is due to the water leaving the wheel before it reaches the bottom. The exact position at which it leaves varies with the form of buckets adopted, but for our present purpose it may be taken that the mean discharge occurs at an angle of 45° as

shown. Then by measurement from the diagram, or by a simple calculation, we see that this loss is 0·15D. A clearance of about 0·5 foot is usually allowed between the wheel and the tail water. We can now find the diameter of the wheel, remembering that H = 15 feet, and taking the height from the surface of the water to the wheel as 2 feet. This together with the 0·5 foot clearance at the bottom gives us D = 12·5 feet.

Thus the losses with this wheel are—

Half the sluice head	= 0·8 foot
Drop from centre of stream to buckets	= 1·0 „
Water leaving wheel too early, 0·15 × 12·5 feet	= 1·9 „
Clearance at bottom	= 0·5 „
	4·2 feet

$$\text{Hydraulic efficiency of wheel} = \frac{15 - 4{\cdot}2}{15} = 72 \text{ per cent.}$$

The mechanical efficiency of the axle and one toothed wheel will be about 90 per cent., thus giving a total efficiency of the wheel of 65 per cent.

With greater falls this efficiency can be raised to 80 per cent.

The above calculations do not profess to be a complete treatment of the overshot wheel, but they fairly indicate the sort of losses such wheels are liable to.

The loss due to the water leaving too early can be largely avoided by arranging the wheel as shown in Fig. 660.

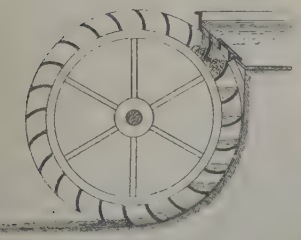


Fig. 660.

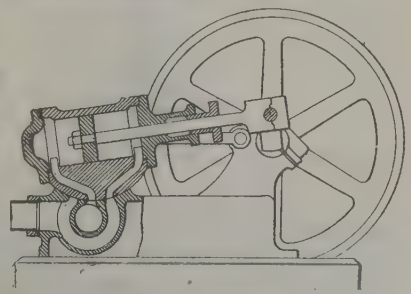


Fig. 661.

Pressure Machines.—In these machines the water at the higher level descends by a pipe to the lower level, from whence it passes to a closed vessel or a cylinder, and acts on a movable

piston in precisely the same manner as in a steam-engine. The work done is the same as before, viz. WH foot-lbs. for

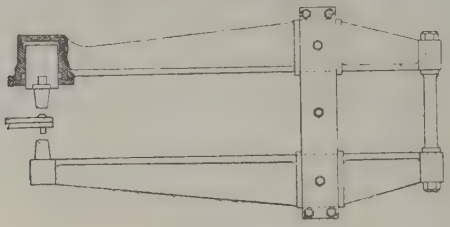


FIG. 662.

the pressure at the lower level is $W_w H$ lbs. per square foot; and the weight of water used per square foot of piston $= W_w L = W$,

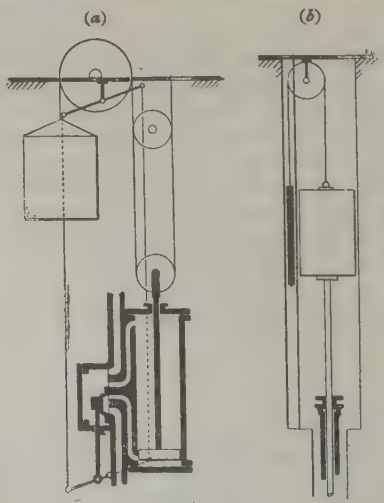


FIG. 663.

where L is the distance moved through by the piston in feet. Then the work done by the pressure water $= W_w LH = WH$ foot-lbs. Several examples of pressure machines are shown in Figs. 661, 662, 663, *a* and *b*. Fig. 661 is an oscillating cylinder pressure motor used largely on the continent. Fig. 662 is an ordinary hydraulic pressure riveter. Fig. 663 (*a*) is a passenger lift, with a wire-rope multiplying arrangement. Fig. 663 (*b*) is an ordinary ram lift. For details the reader is referred

to special books on hydraulic machines, such as Blaine[1] or Robinson.[2]

The chief sources of loss in efficiency in these motors are—

1. Friction of the water in the mains and passages.
2. Losses by shock through abrupt changes in velocity of water.
3. Friction of mechanism.
4. Waste of water due to the same quantity being used when running under light loads as when running with the full load.

The friction and shock losses may be reduced to a minimum by careful attention to the design of the ports and passages; re-entrant angles, abrupt changes of section of ports and passages, high velocities of flow, and other sources of loss given in the chapter on hydraulics should be carefully avoided.

By far the most serious loss in most motors of this type is that mentioned in No. 4 above. Many very ingenious devices have been tried with the object of overcoming this loss.

Amongst the most promising of those tried are devices for automatically regulating the length of the stroke in proportion to the resistance overcome by the motor. Perhaps the best known of these devices is that of the Hastie engine, a full description of which will be found in Professor Unwin's article on Hydromechanics in the "Encyclopædia Britannica."

In an experiment on this engine, the following results were obtained:—

Weight in pounds lifted 22 feet	chain only	427	633	745	857	969	1081	1193
Water used in gallons at 80 lbs. per square inch	7·5	10	14	16	17	20	21	22
Efficiency per cent. (actual)	—	51	54	56	60	58	61	65
Efficiency per cent. if stroke were of fixed length ...	—	23	34	40	46	53	59	65

The efficiency in lines 3 and 4 has been deduced from the other figures by the author, on the assumption that the motor was working full stroke at the highest load given.

The great increase in the efficiency at low loads due to the compensating gear is very clear.

Cranes and elevators are often fitted with two cylinders of different sizes, or one cylinder and a differential piston. When lightly loaded, the smaller cylinder is used, and the larger one

[1] "Hydraulic Machinery" (Spon).
[2] "Hydraulic Power and Machinery" (Griffin).

only for full loads. The valves for changing over the conditions are usually worked by hand, but it is very often found that the man in charge does not take advantage of the smaller cylinder. In order to place it beyond his control, the extremely ingenious device shown in Fig. 664 is sometimes used.

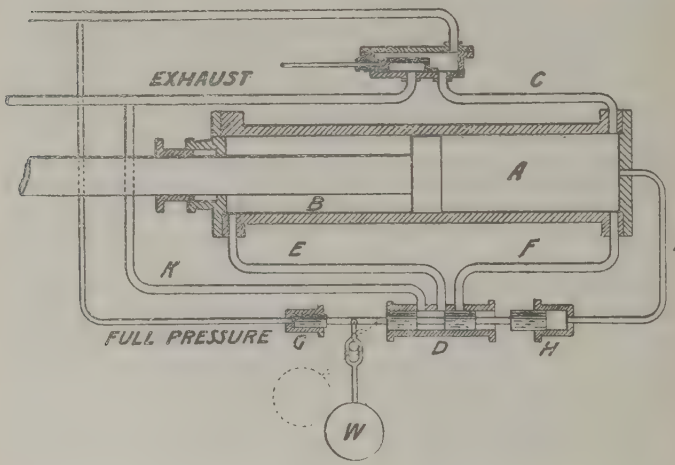


Fig. 664.

The author is indebted to Mr. R. H. Thorp, of New York, the inventor, for the drawings and particulars from which the following account is taken. The working cylinder is shown at AB. When working at full power, the valve D is in the position shown in full lines, which allows the water from B to escape freely by means of the exhaust pipes E and K; then the quantity of water used is given by the volume A. But when working at half-power, the valve D is in the position shown in dotted lines; the water in B then returns *viâ* the pipe E, the valve D, and the pipe F to the A side of the piston. Under such conditions it will be seen that the quantity of high-pressure water used is the volume A *minus* the volume B, which is usually one-half of the former quantity. The position of the valve D, which determines the conditions of full or half power, is generally controlled by hand. The action of the automatic device shown depends upon the fact that the pressure of the water in the cylinder is proportional to the load lifted, for if the pressure were in excess of that required to steadily raise a light

load, the piston would be accelerated, and the pressure would be reduced, due to the high velocity in the ports. In general, the man in charge of the crane throttles the water at the inlet valve in order to prevent any such acceleration. In Mr. Thorp's arrangement, the valve D is worked automatically.

In the position shown, the crane is working at full power; but if the crane be only lightly loaded, the piston will be accelerated and the pressure of the water will be reduced by friction in passing through the pipe C, until the total pressure on the plunger H will be less than the total full water-pressure on the plunger G, with the result that the valve D will be forced over to the right, thus establishing communication between B and A, through the pipes E and F, and thereby putting the crane at half-power. As soon as the pressure is raised in A, the valve D returns to its full-power position, due to the area of H being greater than that of G, and to the pendulum weight W.

It very rarely happens that a natural supply of high-pressure water can be obtained, consequently a power-driven pump has to be resorted to as a means of raising the water to a sufficiently high pressure. In certain simple operations the water may be used direct from the pump, but nearly always some method of storing the power is necessary. If a tank could be conveniently placed at a sufficient height, the pump might be arranged to deliver into it, from whence the hydraulic installation would draw its supply of high-pressure water. In the absence of such a convenience, which, however, is seldom met with, a hydraulic accumulator (Fig. 665) is used. It consists essentially of a vertical cylinder, provided with a long-stroke plunger, which is weighted to give the required pressure, usually from 700 to 1000 lbs. per square inch. With such a means of storing energy, a very large amount of power—far in excess

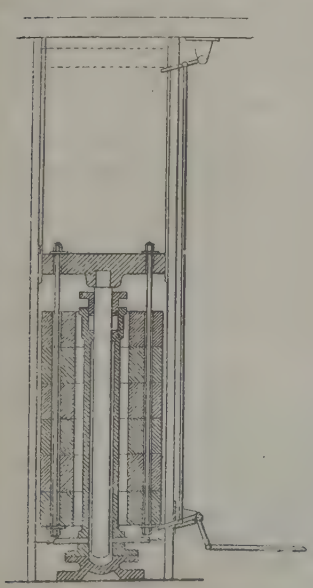


FIG. 665.

of that of the pump—may be obtained for short periods. In fact, this is one of the greatest points in favour of hydraulic methods of transmitting power. The levers shown at the side are for the purpose of automatically stopping and starting the pumps when the accumulator weights get to the top or bottom of the stroke.

Energy stored in an Accumulator.—

If s = the stroke of the accumulator in feet;
d = the diameter of the ram in inches;
p = the pressure in pounds per square inch.

Then the work stored in foot-lbs. $= 0{\cdot}785 d^2 p s$

Work stored per cubic foot of water in foot-lbs. $= 144p$

Work stored per gallon of water $= \frac{144p}{6{\cdot}25} = 23{\cdot}04p$

Number of gallons required per minute at the pressure p per horse-power $= \frac{33{,}000}{23{\cdot}04p} = \frac{1432}{p}$

Number of cubic feet required per minute at the pressure p $= \frac{33{,}000}{144p} = \frac{229{\cdot}2}{p}$

Effects of Inertia of Water in Pressure Systems.— In nearly all pressure motors and machines, the inertia of the water seriously modifies the pressures actually obtained in the cylinders and mains. For this reason such machines have to be run at comparatively low piston speeds, seldom exceeding 100 feet per minute. In the case of free piston machines, such as hydraulic riveters, the pressure on the rivet due to this cause is frequently twice as great as would be given by the steady accumulator pressure.

In the case of a water-pressure motor, the water in the mains moves along with the piston, and may be regarded as a part of the reciprocating parts. The pressure set up in the pipes, due to bringing it to rest, may be arrived at in the same manner as the "Inertia pressure," discussed in Chapter VI.

Let w = weight of a column of water 1 square inch in section, whose length L in feet is that of the main along which the water is flowing to the motor $= 0{\cdot}434\mathrm{L}$;

m = the ratio $\frac{\text{area of plunger or piston}}{\text{area of section of water main}}$

p = the pressure in pounds per square inch set up in the pipe, due to bringing the water to rest at the end of the stroke (with no air-vessel);

N = the number of revolutions per minute of the motor;

R = the radius of the crank in feet.

Then, remembering that the pressure varies directly as the velocity of the moving masses, we have, from pp. 182, 187—

$$p = 0{\cdot}00034m(0{\cdot}434\text{L})\text{RN}^2\left(1 \pm \frac{1}{n}\right)$$

$$p = 0{\cdot}00015m\text{LRN}^2\left(1 \pm \frac{1}{n}\right)$$

Relief valves are frequently placed on long lines of piping, in order to relieve any dangerous pressure that may be set up by this cause.

Pressure due to Shock.—If water flows along a long pipe with a velocity V feet per second, and a valve at the outlet end is suddenly closed, the kinetic energy of the water will be expended in compressing the water and in stretching the walls of the pipe. If the water and the pipe were both materials of an unyielding character, the whole of the water would be instantly brought to rest, and the pressure set up would be infinitely great. Both the water and the pipe, however, do yield considerably under pressure. Hence, even after the valve is closed, water continues to enter at the inlet end with undiminished velocity for a period of t seconds, until the whole of the water in the pipe is compressed, thus producing a momentary pressure greater than the static pressure of the water. The compressed water then expands, and the distended pipe contracts, thus setting up a return-wave, and thereby causing the water-pressure to fall below the static pressure.

Let K = the modulus of elasticity of bulk of water = 300,000 lbs. per square inch (see p. 405);

x = the amount the column of water is shortened, due to the compression of the water and to the distention of the pipe, in feet;

f = the compressive stress or pressure in pounds per square inch due to shock;

w = the weight of a unit column of water, *i.e.* 1 sq. inch section, 1 foot long, = 0·434 lb.;

L = the length of the column of flowing water in feet;

d = the diameter of the pipe in inches;
T = the thickness of the pipe in inches;
f_t = the tensile stress in the pipe (considered thin) due to the increased internal pressure f;
E = Young's modulus of elasticity for the pipe material.

$$\text{Then } f_t = \frac{fd}{2T}$$

$$\left.\begin{array}{l}\text{The increase in diameter due to the}\\ \text{increased pressure}\end{array}\right\} = \frac{fd^2}{2TE}$$

$$\text{The increase in cross-section} = \frac{fd^2}{2TE} \times \frac{\pi d}{2}$$

$$\left.\begin{array}{l}\text{The increase in volume of the pipe per}\\ \text{square inch of cross-section}\end{array}\right\} = \frac{Lfd}{TE}$$

Let a portion of the pipe in question be represented by Fig. 666. Consider a plane section of the pipe, *ab*, distant L from the valve at the instant the valve is suddenly closed. On account of the yielding of the pipe and the compression of the water, the plane *ab* still continues to move forward until the spring of the water and the pipe is a maximum, *i.e.* when the position *a'b'* is reached, let the distance between them be x; then, due to the elastic compression of the water, the plane *ab* moves forward by an amount $x_w = \frac{fL}{K}$ (see p. 374), and a further amount due to the distention of the pipe of $x_p = \frac{fLd}{TE}$, hence—

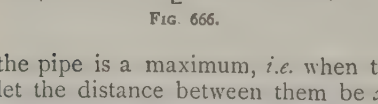


FIG. 666.

$$x = fL\left(\frac{1}{K} + \frac{d}{TE}\right)$$

$$\text{and } f = \frac{x}{L\left(\frac{1}{K} + \frac{d}{TE}\right)}$$

But since x is proportional to L in an elastic medium, the pressure f is therefore independent of the length of the pipe.

At the instant of closing the valve the pressure in the immediate neighbourhood rises above the static pressure by an amount f and a wave of pressure starting at the valve is transmitted along the pipe until it reaches the open end, the velocity of which V_e is constant. The time t_n taken by the

wave in traversing a distance $\frac{L}{n}$ is $\frac{L}{nV_e}$ and the distance x_n which the plane *ab* traverses in this time is $\frac{x}{n}$, but

$$x = f\left(\frac{1}{K} + \frac{d}{TE}\right)L = mL.$$

Hence $$\frac{x_n}{t_n} = \frac{x}{t} = \frac{mL}{n} \div \frac{L}{nV_e} = mV_e$$

Thus the velocity with which the plane *ab* travels is constant. Let the velocity of the water at the instant of closing the valve be V, and since the velocity of *ab* is constant the water continues to enter the pipe at the velocity V for a period of t seconds after closing the valve, *i.e.* until the pressure wave reaches the open end of the pipe, hence $V = \frac{x}{t}$

The change of momentum in the time t = mass of the water × change of velocity in the time t

$$ft = \frac{0{\cdot}434L}{g} \times \frac{x}{t}$$

Substituting the value of x, we have—

$$\frac{L}{t} = \sqrt{\frac{g}{w\left(\frac{1}{K} + \frac{d}{TE}\right)}}$$

and when the elasticity of the pipe is neglected—

$$\frac{L}{t} = \sqrt{\frac{gK}{w}}$$

The quantity $\frac{L}{t}$ is the velocity with which the compression wave traverses the pipe, or the velocity of pulsation. Inserting numerical values for the symbols under the root, we get the velocity of pulsation 4720 feet per second, *i.e.* the velocity of sound in water when the elasticity of the pipe is neglected.

The kinetic energy of the column of water per sq. inch of section $= \frac{0{\cdot}434LV^2}{2g}$

The work done in compressing the water and in the distention of the pipe $= \frac{f(x_w + x_p)}{2}$

$$= \frac{f^2L}{2}\left(\frac{1}{K} + \frac{d}{TE}\right)$$

$$\text{Then } \frac{0{\cdot}434LV^2}{2g} = \frac{f^2L}{2}\left(\frac{1}{K}+\frac{d}{TE}\right)$$

$$\text{and } f = \frac{V}{8{\cdot}6\sqrt{\frac{1}{K}+\frac{d}{TE}}}$$

When the elasticity of the pipe is neglected—

$$f_1 = 63{\cdot}5V$$

A comparison between calculated and experimental results are given below. The experimental values are taken from Gibson's "Hydraulics and its Applications," p. 217.

SUDDEN CLOSING OF VALVE.

Velocity in feet per second . .	0·6	2·0	3·0	7·5
Observed pressure lb. sq. inch .	43	113	173	426
Calculated $p = 63{\cdot}4v$	38	127	190	476
Calculated allowing for elasticity of pipe	35	116	175	436

When the valve is closed uniformly in a given time, the manner in which the pressure varies at each instant can be readily obtained by constructing (i.) a velocity-time curve; (ii.) a retardation or pressure curve, as explained on p. 140. But the pressure set up cannot exceed that due to a suddenly closed valve, although it may closely approach it.

When the pressure wave reaches the open end of the pipe, the whole column of water is under compression to its full extent, it then expands, and when it reaches its unstrained volume the water at the open end is travelling outwards with a velocity V (very nearly, there is a small reduction due to molecular friction) and overshoots the mark, thus producing a negative pressure, *i.e.* below the static pressure in the pipe, to be followed by a pressure wave and so on. The time during which the initial pressure is maintained is therefore the time taken by a compression wave in traversing the pipe and returning, viz. $\frac{2L}{V_s}$ seconds. Hence, if the time occupied in

closing the valve is not greater than this, the pressure set up at the instant of closing will be approximately that given by the above expression for f, but the length of time during which the pressure is maintained at its full amount will be correspondingly reduced. If the period of closing be greater than $\frac{2L}{V_0}$ seconds, the pressure set up at the instant of closing will be less than f, but a rigid solution of the problem then becomes somewhat complex.

Maximum Power transmitted by a Water-Main.—We showed on p. 682 that the quantity of water that can be passed through a pipe with a given loss of head is—

$$Q = 38{\cdot}5D^{\frac{5}{2}}\sqrt{\frac{h}{L}}$$

Each cubic foot of water falling per second through a height of one foot gives—

$$\frac{62{\cdot}5 \times 60}{33{,}000} = 0{\cdot}1135 \text{ horse-power}$$

$$\text{hence } \overline{H.P.} = 0{\cdot}1135QH$$

where H is the fall in feet, and Q the quantity of water in cubic feet per second.

Then, if h be the loss of head due to friction, the horse-power delivered at the far end of the main L feet away is—

$$\overline{H.P.} = 0{\cdot}1135Q(H - h)$$

Substituting the value of Q from above, we have—

$$\overline{H.P.} = 0{\cdot}1135 \times 38{\cdot}5D^{\frac{5}{2}}(H - h)\sqrt{\frac{h}{L}}$$

Let $h = nH$. Then, by substitution and reduction, we get the power delivered at the far end—

$$\overline{H.P.} = 4{\cdot}37\,(1 - n)\sqrt{\frac{nH^3D^5}{L}}$$

$$\text{or } L = \frac{19{\cdot}1nH^3D^5(1 - n)^2}{H.P.^2}$$

These equations give us the horse-power that can be

transmitted with any given fraction of the head lost in friction; also the permissible length of main for any given loss when transmitting a certain amount of power.

The power that can be transmitted through a pipe depends on (i.) the quantity of water that can be passed; (ii.) the effective head, *i.e.* the total head *minus* the friction head.

$$\text{Power transmitted } P = Q(H - h) \times \text{a constant}$$

$$\text{or } P = AV\left(H - \frac{LV^2}{2400D}\right) \times \text{a constant}$$

$$P = \left(AHV - \frac{ALV^3}{2400D}\right) \times \text{a constant}$$

Then—

$$\frac{dP}{dV} = \left(AH - \frac{3ALV^2}{2400D}\right) \times \text{a constant}$$

When the power is a maximum this becomes zero; then—

$$\frac{LV^2}{2400D} = \frac{H}{3}$$

$$\text{or } h = \frac{H}{3}$$

hence the maximum power is transmitted when $\frac{1}{3}$ of the head is wasted in friction.

Those not familiar with the differential method can arrive at the same result by calculating out several values of $V - V^3$, until a maximum is found.

Whence the maximum horse-power that can be transmitted through any given pipe is—

$$\text{H.P. } (max.) = 1{\cdot}67\sqrt{\frac{H^3D^5}{L}}$$

obtained by inserting $n = \frac{1}{3}$ in the equation above.

N.B.—H, D, and L are all expressed in feet.

Velocity Machines.—In these machines, the water, having descended from the higher to the lower level by a pipe, is allowed to flow freely and to acquire velocity due to its head. The whole of its energy then exists as energy of motion. The energy is utilized by causing the water to impinge on moving vanes, which change its direction of flow, and more or less reduce its velocity. If it left the vanes with

no velocity relative to the earth, the whole of the energy would be utilized, a condition of affairs which is never attained in practice.

The velocity with which the water issues, apart from friction, is given by—

$$V = \sqrt{2gH}$$

where H is the head of water above the outlet. When friction is taken into account—

$$V = \sqrt{2g(H - h)}$$

where h is the head lost in friction.

Relative and Absolute Velocities of Streams.—We shall always use the term "absolute velocity," as the velocity relative to the earth.

Let the tank shown in Fig. 667 be mounted on wheels, or otherwise arranged so that it can be moved along horizontally

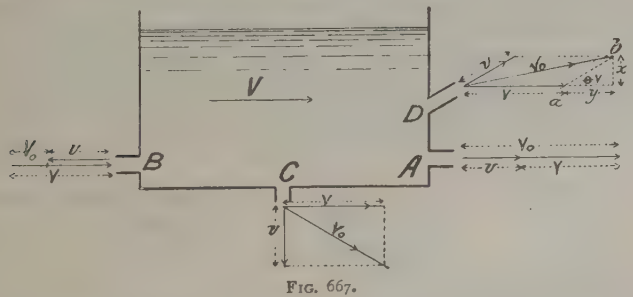


FIG. 667.

at a velocity V in the direction indicated by the arrow, and let water issue from the various nozzles as shown. In every case let the water issue from the tank with a velocity v at an angle θ with the direction of motion of the tank; then we have—

Nozzle.	Velocity rel. to tank v.	Velocity rel. to ground V_0.	θ.	Cos θ.
A	v	$V + v$	0	1
B	v	$V - v$	180°	−1
C	v	$\sqrt{V^2 + v^2}$	90°	0
D	v	$\sqrt{V^2 + v^2 + 2vV \cos \theta}$	θ	cos θ

The velocity V_0 will be clear from the diagram. The expression for D is arrived at thus:

$$y = ab \cos\theta = v \,.\, \cos\theta$$
$$\text{and } x = v \,.\, \sin\theta$$
$$V_0 = \sqrt{(V + y)^2 + x^2}$$

Substituting the values of x and y, and remembering that $\cos^2\theta + \sin^2\theta = 1$, we get the expression given above. If the value of $\cos\theta$ for A, B, and C be inserted in the general expression D, the same results will be obtained as those given.

Now, suppose a jet of water to be moving, as shown by the arrow, with a velocity V_0 relative to the ground; also the tank to be moving with a velocity V relative to the ground; then it is obvious that the velocity of the water relatively to the tank is given by ab or v. We shall be constantly making use of this construction when considering turbines.

Pressure on a Surface due to an Impinging Jet.—When a body of mass M, moving with a velocity V, receives an impulse due to a force P for a space of time t, the velocity will be increased to V_1, and the energy of motion of the body will also be increased; but, as no other force has acted on the body during the interval, this increase of energy must be equal to the work expended on the body, or—

$$\left.\begin{array}{l}\text{The work done on}\\ \text{the body}\end{array}\right\} = \text{impulse} \times \left\{\begin{array}{l}\text{distance through which}\\ \text{it is exerted}\end{array}\right.$$
$$= \text{increase in kinetic energy}$$
$$\left.\begin{array}{l}\text{The kinetic energy}\\ \text{before the impulse}\end{array}\right\} = \frac{MV^2}{2}$$
$$\left.\begin{array}{l}\text{The kinetic energy}\\ \text{after the impulse}\end{array}\right\} = \frac{MV_1^2}{2}$$
$$\left.\begin{array}{l}\text{Increase in kinetic}\\ \text{energy}\end{array}\right\} = \frac{M}{2}(V_1^2 - V^2)$$

The distance through which the impulse is exerted is—

$$\frac{V_1 + V}{2} \,.\, t$$

$$\text{hence } \frac{M}{2}(V_1^2 - V^2) = Pt\frac{V_1 + V}{2}$$
$$\text{or } Pt = M(V_1 - V)$$

or impulse in time t = change of momentum in time t

Let a jet of water moving with a velocity V feet per second impinge on a plate, as shown. After impinging, its velocity in its original direction is zero, hence its change of velocity on striking is V, and therefore—

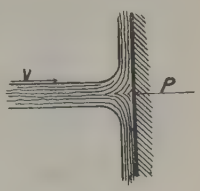


FIG. 668.

$$Pt = MV$$

$$\text{or } P = \frac{M}{t}V$$

But $\frac{M}{t}$ is the mass of water delivered per second.

Let W = weight of water *delivered per second.*

$$\text{Then } \frac{W}{g} = \frac{M}{t}$$

$$\text{and } P = \frac{WV}{g}$$

For another method of arriving at the same result, see p. 593.

It should be noticed that the pressure due to an impinging jet is just twice as great as the pressure due to the head of water corresponding to the same velocity. This can be shown thus:

$$h = \frac{V^2}{2g}$$

$$p = wh = \frac{wV^2}{2g}$$

where w = the weight of a unit column of water.

We have $W = wV$. Substituting this value of wV—

$$p = \frac{WV}{2g}$$

The impinging jet corresponds to a dynamic load, and a column of water to a steady load (see p. 627).

In this connection it is interesting to note that, in the case of a sea-wave, the pressure due to a wave of oscillation is approximately equal to that of a head of water of the same height as the wave, and, in the case of a wave of translation, to twice that amount.

Pressure on a Moving Surface due to an Impinging Jet.—Let the plate shown in the Fig. 669 be one of a series on which the jet impinges at very short intervals. The reason for making this stipulation will be seen shortly.

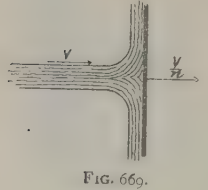


FIG. 669.

Let the weight of water delivered per second be W lbs. as before; then, if the plates succeed one another very rapidly as in many types of water-wheels, the quantity impinging on the plates will also be sensibly equal to W. The impinging velocity is $V - \frac{V}{n}$, or $V\left(1 - \frac{1}{n}\right)$; hence the pressure in pounds' weight on the plates is—

$$P = \frac{WV\left(1 - \frac{1}{n}\right)}{g}$$

And the work done per second on the plates in foot-lbs.—

$$P\frac{V}{n} = \frac{WV^2\left(1 - \frac{1}{n}\right)}{ng} \quad \ldots\ldots\ldots \text{(i.)}$$

and the energy of the jet is—

$$\frac{WV^2}{2g} \quad \ldots\ldots\ldots \text{(ii.)}$$

hence the efficiency of the jet $= \frac{\text{i.}}{\text{ii.}} = \frac{2}{n}\left(1 - \frac{1}{n}\right)$

The value of n for maximum efficiency can be obtained by plotting or by differentiation.[1] It will be found that $n = 2$. The efficiency is then 50 per cent., which is the highest that can be obtained with a jet impinging on flat vanes. A common example of a motor working in this manner is the ordinary

[1] Efficiency $= \eta = \frac{2}{n}\left(1 - \frac{1}{n}\right)$

$$\eta = \frac{2}{n} - \frac{2}{n^2} = 2n^{-1} - 2n^{-2}$$

$$\frac{d\eta}{dn} = -2n^{-2} + 4n^{-3} = 0, \text{ when } \eta \text{ is a maximum}$$

or $2n^{-2} = 4n^{-3}$

$$\frac{2}{n^2} = \frac{4}{n^3}, \text{ whence } n = 2$$

undershot water-wheel; but, due to leakage past the floats, axle friction, etc., the efficiency is rarely over 30 per cent.

If the jet had been impinging on only one plate instead of a large number, the quantity of water that reached the plate per second would only have been $W\left(1 - \frac{1}{n}\right)$; then, substituting this value for W in the equation above, it will be seen that the efficiency of the jet $= \frac{2}{n}\left(1 - \frac{1}{n}\right)^2$, and the maximum efficiency occurs when $n = 3$, and is equal to about 30 per cent.

Pressure on an Oblique Surface due to an Impinging Jet.—The jet impinges obliquely at an angle θ to the plate, and splits up into two streams. The velocity V may be resolved into V_1 normal and V_0 parallel to the plate. After impinging, the water has no velocity normal to the plate, therefore the normal pressure—

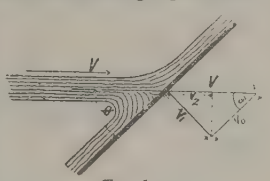
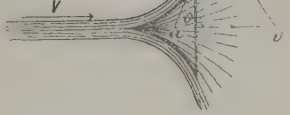


FIG. 670.

$$P = \frac{WV_1}{g} = \frac{WV \sin \theta}{g}$$

Pressure on a Smooth Curved Surface due to an



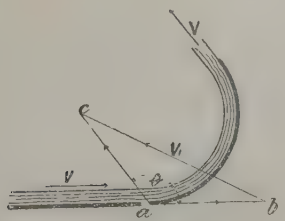


FIG. 671.

Impinging Jet.—We will first consider the case in which the surface is stationary and the water slides on it without shock;

how to secure this latter condition we will consider shortly. We show three forms of surface (Fig. 671), to all of which the following reasoning applies.

Draw *ab* to represent the initial velocity V of the jet in magnitude and direction; then, neglecting friction, the final velocity of the water on leaving the surface will be V, and its direction will be tangential to the last tip of the surface. Draw *ac* parallel to the final direction and equal to *ab*, then *bc* represents the change of velocity V_1; hence the resultant pressure on the surface in *the direction of cb* is—

$$P = \frac{WV_1}{g}$$

Then, reproducing the diagram of velocities above, we have—

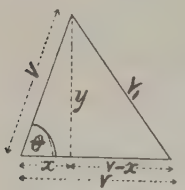


FIG. 672.

$$y = V \sin \theta$$
$$x = V \cos \theta$$
$$V_1^2 = (V - x)^2 + y^2$$

Then, substituting the values of x and y and reducing, we have—

$$V_1 = V\sqrt{2(1 - \cos \theta)}$$

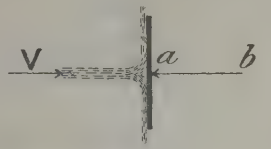
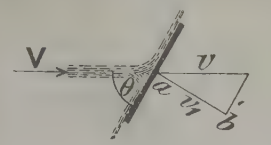
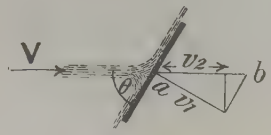
The component parallel to the jet is $V - x = V(1 - \cos \theta)$. Thus in all the three cases given above we have the pressure parallel to the jet—

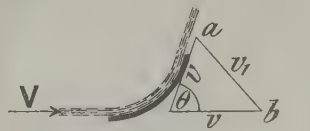
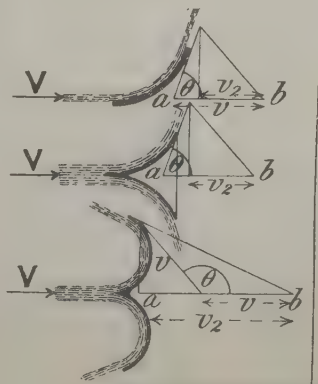
$$P_0 = \frac{WV(1 - \cos \theta)}{g}$$

[1] The effect of friction is to reduce the length *ac* (Fig. 671), hence when θ is less than 90°, the pressure is greater, and when θ is greater than 90°, it is less than P_0. The following results were obtained in the author's laboratory. The calculated value being taken as unity.

θ.	Cone (45°).	Hollowed cone (55°).	Flat.	Approx. hemisphere (171°).	Pelton bucket (162°).
P_0 by experiment .	1·4	1·2	1·0	0·7	0·8

Velocity of jet = V ft. per second. W lbs. of water delivered per second.

	Pressure exerted in the direction *ab*.		Velocity of moving plate in the direction *ab*.	Work done in foot-lbs. per second on the plate.	Efficiency of the jet.
	When stationary.	When moving in the direction *ab*.			
	$\frac{WV}{g}$	$\frac{WV}{g}\left(1-\frac{1}{n}\right)$	$\frac{V}{n}$	$\frac{WV^2}{ng}\left(1-\frac{1}{n}\right)$	$\frac{2}{n}\left(1-\frac{1}{n}\right)$
	$\frac{WV_1}{g}$ or— $\frac{WV}{g}\sin\theta$	$\frac{WV_1}{g}\left(1-\frac{1}{n}\right)$ or— $\frac{WV}{g}\sin\theta\left(1-\frac{1}{n}\right)$	$\frac{V_1}{n}$ or— $\frac{V\sin\theta}{n}$	$\frac{WV_1^2}{ng}\left(1-\frac{1}{n}\right)$ or— $\frac{WV^2\sin^2\theta}{ng}\left(1-\frac{1}{n}\right)$	$\frac{2}{n}\sin^2\theta\left(1-\frac{1}{n}\right)$
	$\frac{WV_2}{g}$ or— $\frac{WV}{g}\sin^2\theta$	$\frac{WV_2}{g}\left(1-\frac{1}{n}\right)$ or— $\frac{WV\sin^2\theta}{g}\left(1-\frac{1}{n}\right)$	$\frac{V_2}{n}$ or— $\frac{V\sin^2\theta}{n}$	$\frac{WV_2^2}{ng}\left(1-\frac{1}{n}\right)$ or— $\frac{WV^2\sin^4\theta}{ng}\left(1-\frac{1}{n}\right)$	$\frac{2}{n}\sin^4\theta\left(1-\frac{1}{n}\right)$

Velocity of jet = V ft. per second.			W lbs. of water delivered per second.		
	Pressure exerted in the direction *ab*.		Velocity of moving plate in the direction *ab*.	Work done in foot-lbs. per second on the plate.	Efficiency of the jet.
	When stationary.	When moving in the direction *ab*.			
	$\frac{WV_1}{g}$ or— $\frac{WV}{g}\sqrt{2(1-\cos\theta)}$	$\frac{WV_1}{g}\left(1-\frac{1}{n}\right)$ or— $\frac{WV}{g}\sqrt{2(1-\cos\theta)}\left(1-\frac{1}{n}\right)$	$\frac{V_1}{n}$ or— $\frac{V}{n}\sqrt{2(1-\cos\theta)}$	$\frac{WV_1^2}{ng}\left(1-\frac{1}{n}\right)$ or— $\frac{WV^2}{ng}2(1-\cos\theta)\left(1-\frac{1}{n}\right)$	$\frac{4}{n}\left(1-\frac{1}{n}\right)(1-\cos\theta)$
	$\frac{WV_2}{g}$ or— $\frac{WV}{g}(1-\cos\theta)$	$\frac{WV_2}{g}\left(1-\frac{1}{n}\right)$ or— $\frac{WV}{g}(1-\cos\theta)\left(1-\frac{1}{n}\right)$	$\frac{V}{n}$	$\frac{WV^2}{ng}(1-\cos\theta)\left(1-\frac{1}{n}\right)$	$\frac{2}{n}\left(1-\frac{1}{n}\right)(1-\cos\theta)$

Pelton or Tangent Wheel Vanes.—The double vane shown in section is usually known as the Pelton Wheel Vane; but whether Pelton should have the credit of the invention or not is a disputed point. In this type of vane the angle θ approaches 180°, then $1 - \cos\theta = 2$, and the resultant pressure on such a vane is twice as great as that on a flat vane, and the theoretical efficiency is 100 per cent. when $n = 2$; but for various reasons such an efficiency is never reached, although it sometimes exceeds 80 per cent., including the friction of the axle.

A general view of such a wheel is shown in Fig. 673.

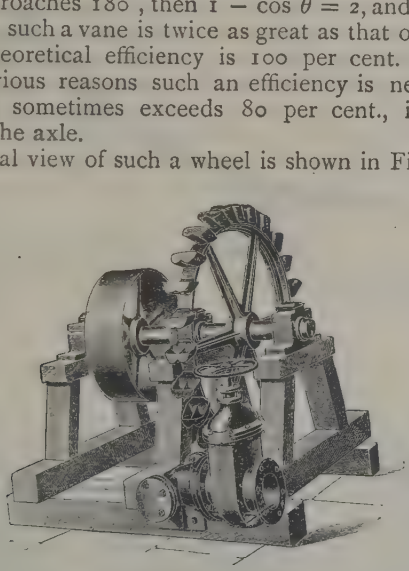


Fig. 673.

It is very instructive to examine the action of the jet of water on the vanes in wheels of this type, and thereby to see why the theoretical efficiency is never reached.

(1) There is always some loss of head in the nozzle itself; but this may be reduced to an exceedingly small amount by carefully proportioning the internal curves of the nozzle.

(2) The vanes are usually designed to give the best effect when the jet plays fairly in the centre of the vanes; but in other positions the effect is often very poor, and, consequently, as each vane enters and leaves the jet, serious losses by shock very frequently occur. In order to avoid the loss at entry, Mr. Doble, of San Francisco, after a very careful study of the matter, has shown that the shape of vane as usually used is

[1] Reproduced by the kind permission of Messrs. Gilbert Gilkes and Co., Kendal.

very faulty, since the water after striking the outer lip is abruptly changed in direction at the corners *a* and *b*, where much of its energy is dissipated in eddying; then, further, on

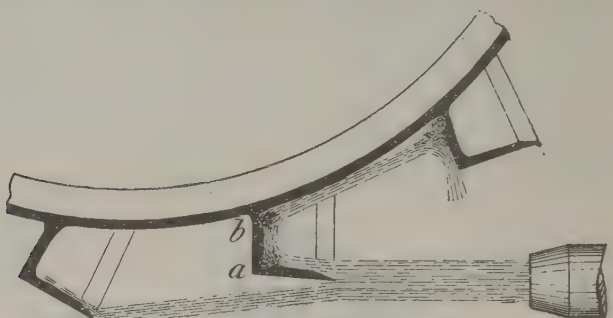


FIG 674.

leaving the vane it strikes the back of the approaching vane, and thereby produces a back pressure on the wheel with a

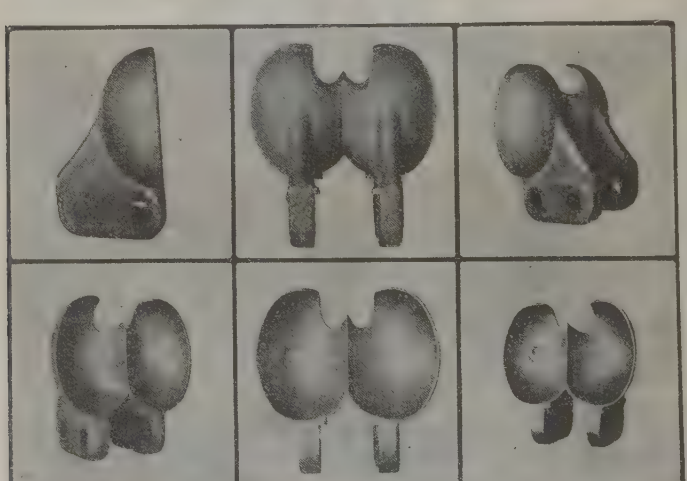


FIG. 675.—Doble "Tangent Wheel" buckets.

consequent loss in efficiency. This action is shown in Fig. 674. The outer lip is not only unnecessary, but is distinctly wrong in theory and practice. In the Doble vane (Fig. 675)

the outer lip is dispensed with, and only the central rib retained for parting the water sideways, with the result that the efficiency of the Doble wheel is materially higher than that obtained from wheels made in the usual form.

(3) The angle θ cannot practically be made so great as 180°, because the water on leaving the sides of the vanes would strike the back edges of the vanes which immediately follow; hence for clearance purposes this angle must be made somewhat less than 180°, with a corresponding loss in efficiency.

(4) Some of the energy of the jet is wasted in overcoming the friction of the axle.

In an actual wheel the maximum efficiency does not occur

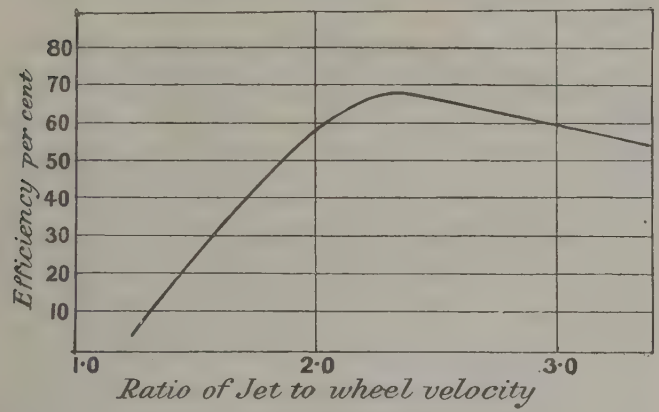


FIG. 676.

when the velocity of the jet is twice that of the vanes, but when the ratio is about 2·2.

The curve shown in Fig. 676 shows how the efficiency varies with a variation in speed ratio. The results were obtained from a small Pelton or tangent wheel in the author's laboratory; the available water pressure is about 30 lbs. per square inch. Probably much better results would be obtained with a higher water pressure.

This form of wheel possesses so many great advantages over the ordinary type of impulse turbine that it is rapidly coming into very general use for driving electrical and other installations; hence the question of accurately governing it is one of great importance. In cases in which a waste of water is immaterial, excellent results with small wheels can be

obtained by the Cassel governor, in which the two halves of the vanes are mounted on separate wheels. When the wheel is working at its full power the two halves are kept together, and thus form an ordinary Pelton wheel; when, however, the speed increases, the governor causes the two wheels to partially separate, and thus allows some of the water to escape between the central rib of the vanes. For much larger wheels Doble obtains the same result by affixing the jet nozzle to the end of a pivoted pipe in such a manner that the jet plays centrally on the vanes for full power, and when the speed increases, the governor deflects the nozzle to such an extent that the jet partially or fully misses the tips of the vanes, and so allows some of the water to escape without performing any work on the wheel.

But by far the most elegant and satisfactory device for regulating motors of this type is the conical expanding nozzle, which effects the desired regulation without allowing any waste of water. The nozzle is fitted with an internal cone of special construction, which can be advanced or withdrawn, and thereby it reduces or enlarges the area of the annular stream of water. Many have attempted to use a similar device, but have failed to get the jet to perfectly coalesce after it leaves the point of the cone. The cone in the Doble [1] arrangement is balanced as regards shifting along the axis of the nozzle; therefore the governor only has to overcome a very small resistance in altering the area of the jet. Many other devices have been tried for varying the area of the jet in order to produce the desired regulation of speed, but not always with marked success. Another method in common use for governing and for regulating the power supplied to large wheels of this type is to employ several jets, any number of which can be brought to play on the vanes at will, but the arrangement is not altogether satisfactory, as the efficiency of the wheel decreases materially as the number of jets increases. In some tests made in California the following results were obtained:—

Number of jets.	Total horse-power.	Horse-power per jet.
1	155	155
2	285	130
3	390	105
4	480	90

[1] A similar device is used by Messrs. Gilbert Gilkes & Co., Kendal.

The problem of governing water-wheels of this type, even when a perfect expanding nozzle can be produced, is one of considerable difficulty, and those who have experimented upon such motors have often obtained curious results which have greatly puzzled them. The theoretical treatment which follows is believed to throw much light on many hitherto unexplained phenomena, such as (i.) It has frequently been noticed that the speed of a water motor decreases when the area of the jet is increased, the head of water, and the load on the motor,

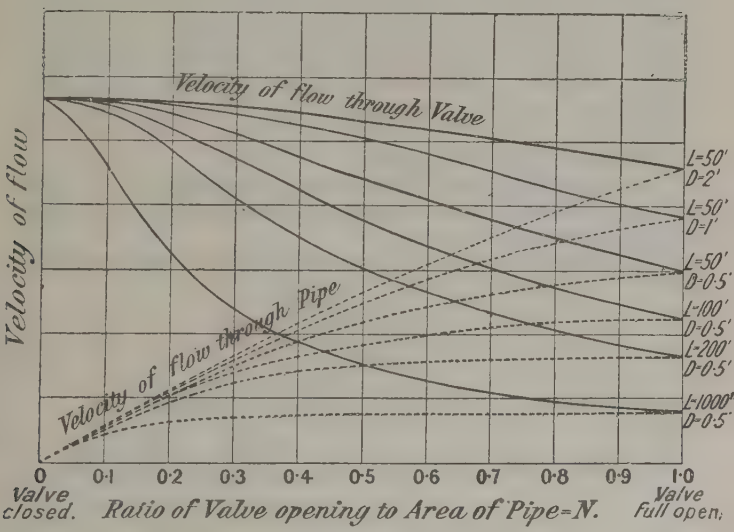


FIG. 677.

remaining the same, and *vice versâ*, when the area of the jet is decreased the speed increases. If the area of the jet is regulated by means of a governor, the motor under such circumstances will hunt in a most extraordinary manner, and the governor itself is blamed; but, generally speaking, the fault is not in the governor at all, but in the proportions of the pipe and jet. (ii.) A governor which controls the speed admirably in the case of a given water motor when working under certain conditions, may entirely fail in the case of a similar water motor when the conditions are only slightly altered, such as an alteration in the length or diameter of the supply pipe.

On p. 688 we showed that the velocity (V) of flow at any instant in a pipe is given by the expression—

$$V = \sqrt{\frac{H}{\frac{L}{KD} + \frac{1}{2gn^2}}}$$

In Fig. 677 we give a series of curves to show the manner in which V varies with the ratio of the area of the jet to the area of the cross-section of the pipe, viz. n. From these curves it will be seen that the velocity of flow falls off very slowly at first, as the area of the jet is diminished, and afterwards, as the "shut" position of the nozzle is approached, the velocity falls very rapidly. The velocity of efflux $V_1 = \frac{V}{n}$ of the jet itself is also shown by full lined curves.

The quantity of water passing any cross-section of the main per second, or through the nozzle, is AV cubic feet per second, or 62·4AV lbs. per second.

The kinetic energy of the stream issuing from the nozzle is—

$$\frac{62{\cdot}4AV^3}{2gn^2}$$

Inserting the value of V, and reducing, we get—

$$\text{The kinetic energy of the stream} = \frac{0{\cdot}76D^2}{n^2}\left(\frac{H}{\frac{L}{KD} + \frac{1}{2gn^2}}\right)^{\frac{3}{2}}$$

Which may be written—

$$K_e = \frac{C}{n^2}\left(\frac{1}{B + \frac{1}{n^2}}\right)^{\frac{3}{2}} \quad \text{where } B = \frac{2gL}{KD}$$

$$= \frac{C}{(n^{\frac{4}{3}})^{\frac{3}{2}}} \frac{1}{\left(B + \frac{1}{n^2}\right)^{\frac{3}{2}}} = \frac{C}{(Bn^{\frac{4}{3}} + n^{-\frac{2}{3}})^{\frac{3}{2}}}$$

$$= C(Bn^{\frac{4}{3}} + n^{-\frac{2}{3}})^{-\frac{3}{2}}$$

$$\frac{dK_e}{dn} = C\left\{-\tfrac{3}{2}(Bn^{\frac{4}{3}} + n^{-\frac{2}{3}})^{-\frac{5}{2}}(\tfrac{4}{3}Bn^{\frac{1}{3}} - \tfrac{2}{3}n^{-\frac{5}{3}})\right\}$$

which may be written—

$$C\left\{-\tfrac{3}{2}n^{\frac{1}{3}}(Bn^2+1)^{-\frac{1}{3}}\times\tfrac{2}{3}n^{-\frac{5}{3}}(2Bn^2-1)\right\}$$

and $$C\left\{-(Bn^2+1)^{-\frac{5}{2}}\times(2Bn^2-1)\right\}=0$$

when it has its maximum value, but (Bn^2+1) is greater than unity, hence $2Bn^2-1=0$

$$\text{or } n^2=\frac{1}{2B}=\frac{KD}{4gL}$$

$$\text{and } n=\sqrt{\frac{KD}{4gL}}=4{\cdot}4\sqrt{\frac{D}{L}}$$

taking an average value for K.

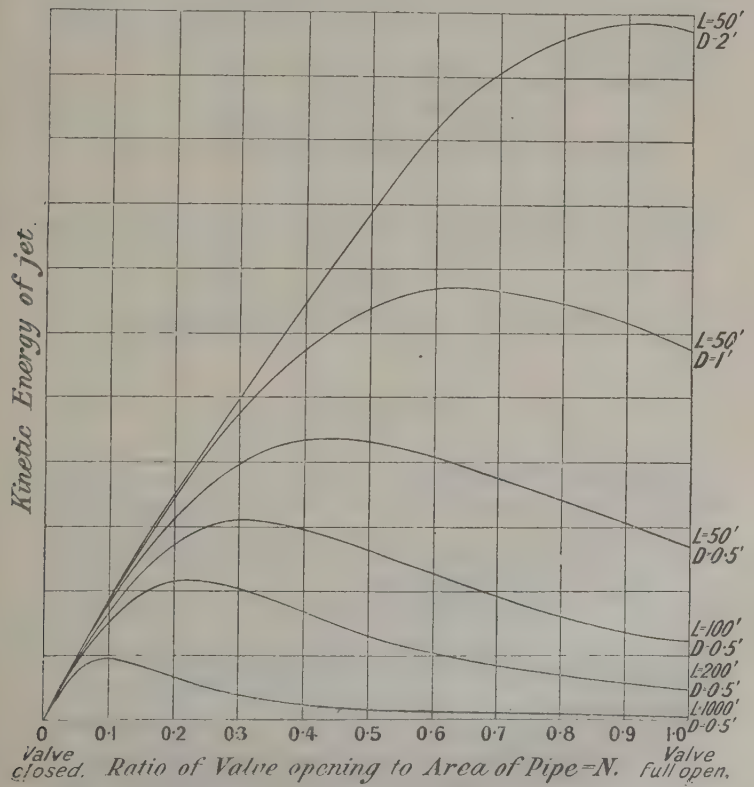


FIG. 678.

Curves showing how the kinetic energy of the stream varies with n are given in Fig. 678. Starting from a fully opened nozzle, the kinetic energy increases as the area of the jet is decreased up to a certain point, where it reaches its maximum value, and then it decreases as the area of the jet is further decreased. The increase in the kinetic energy, as the area of the jet decreases, will account for the curious action mentioned above, in which the speed of the motor was found to increase when the area of the jet was decreased, and *vice versâ*. The speed necessarily increases when the kinetic energy increases, if the load on the motor remains constant. If, however, the area of the jet be small compared with the area of the pipe, the kinetic energy varies directly as the area of the jet, or nearly so. Such a state is, of course, the only one consistent with good governing.

We have shown above that—

$$\text{the kinetic energy is a maximum when } n = 4{\cdot}4\sqrt{\frac{D}{L}}$$

hence for good governing the area of the jet must be less than—

$$4{\cdot}4 \text{ (area of the cross-section of the pipe)}\sqrt{\frac{D}{L}}$$

$$\text{or, } 3{\cdot}45\sqrt{\frac{D^5}{L}}$$

Poncelot Water-wheel Vanes.—In this wheel a thin stream of water having a velocity V feet per second glides up curved vanes having a velocity $\frac{V}{n}$ in the same direction as the stream. The water moves up the vane with the velocity $V - \frac{V}{n}$ relatively to it, and attains a height corresponding to this velocity, when at its uppermost point it is at rest relatively to the vane; it then falls, attaining the velocity $-\left(V - \frac{V}{n}\right)$, neglecting friction; the negative sign is used because it is in the reverse direction to which it entered. Hence, as the water is moving

forward with the wheel with a velocity $\frac{V}{n}$, and backward relatively to the wheel with a velocity $-\left(V - \frac{V}{n}\right)$, the absolute velocity of the water on leaving the vanes is—

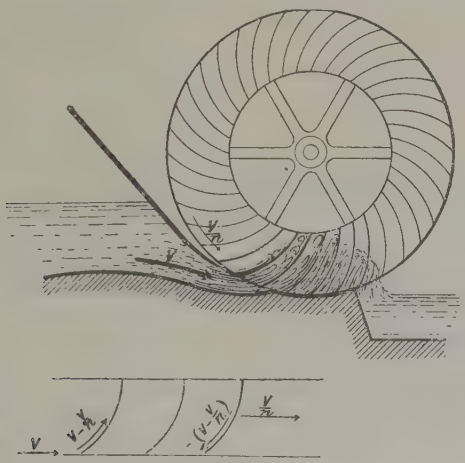


FIG. 679.

$$\frac{V}{n} - \left(V - \frac{V}{n}\right) = V\left(\frac{2}{n} - 1\right)$$

hence, when $n = 2$ the absolute velocity of discharge is zero, or all the energy of the stream is utilized. The efficiency may also be arrived at thus—

Let α = the angle between the direction of the entering stream and a tangent to the wheel at the point of entry. Then the component of V along the tangent is V cos α.

The pressure exerted in the direction of the tangent $= \frac{2W}{g}\left(V \cos \alpha - \frac{V}{n}\right)$

The work done per second on the plate $= \frac{2WV^2}{gn}\left(\cos \alpha - \frac{1}{n}\right)$

And the hydraulic efficiency of the wheel $= \frac{4}{n}\left(\cos \alpha - \frac{1}{n}\right)$

The efficiency of these wheels varies from 65 to 75 per cent., including the friction of the axle.

Form of Vane to prevent Shock.—In order that the water may glide gently on to the vanes of any motor, the tangent to the entering tip of the vanes must be in the same direction as the path of the water relative to the tip of the wheel; thus, in the figure, if *ab* represents the velocity of the entering stream, *ad* the velocity of the vane, then *db* represents the relative path of the water, and the entering tip must be parallel to it. The stream then gently glides on the vane without shock.

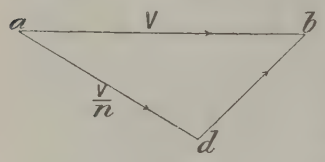


FIG. 680.

Turbines.—Turbines may be conveniently divided into two classes: (1) Those in which the whole energy of the water is converted into energy of motion in the form of free jets or streams which are delivered on to suitably shaped vanes in order to reduce the absolute velocity of the water on leaving to zero or nearly so. Such a turbine wheel receives its impulse from the direct action of impinging jets or streams; and is known as an "impulse" turbine. When the admission only takes place over a small portion of the circumference, it is known as a "partial admission" turbine. The jets of water proceeding from the guide-blades are perfectly free, and after impinging on the wheel-vanes the water at once escapes into the air above the tail-race.

(2) Those in which some of the energy is converted into pressure energy, and some into energy of motion. The water is therefore under pressure in both the guide-blades and in the wheel passages, consequently they must always be full, and there must always be a pressure in the clearance space between the wheel and the guides, which is not the case in impulse turbines. Such are known as "reaction" turbines, because the wheel derives its impulse from the reaction of the water as it leaves the wheel passages. There is often some little difficulty in realizing the pressure effects in reaction turbines. Probably the best way of making it clear is to refer for one moment to the simple reaction wheel shown in Fig. 681, in which water runs into the central chamber and is discharged at opposite sides by two curved horizontal pipes as shown; the reaction of the jets on the horizontal pipes causes the whole to revolve. Now, instead of allowing the central chamber to

revolve with the horizontal pipes, we may fix the central chamber, as in Fig. 682, and allow the arms only to revolve; we shall get a crude form of a reaction turbine. It will be clear that a water-tight joint must be made between the arms and the chamber, because there is pressure in the clearance space between. It will also be seen that the admission of water must take place over

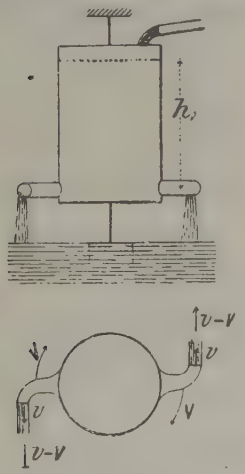


Fig. 681.

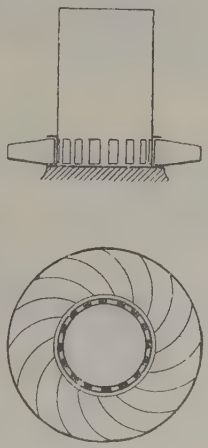


Fig. 682.

the whole circumference, and, further, that the passages must always be full of water. A typical case of such a turbine is shown in Fig. 689. These turbines may either discharge into the air above the tail-water, or the revolving wheel may discharge into a casing which is fitted with a long suction pipe, and a partial vacuum is thereby formed into which the water discharges.

In addition to the above distinctions, turbines are termed parallel flow, inward flow, outward flow, and mixed-flow turbines, according as the water passes through the wheel parallel to the axis, from the circumference inwards towards the axis, from the axis outwards towards the circumference, or both parallel to the axis and either inwards or outwards.

We may tabulate the special features of the two forms of turbine thus:

Impulse.	Reaction.
All the energy of the water is converted into kinetic energy before being utilized.	Some of the energy of the water is converted into kinetic energy, and some into pressure energy.
The water impinges on curved wheel-vanes in free jets or streams, consequently the wheel passages must not be filled.	The water is under pressure in both the guide and wheel passages, also in the clearance space; hence the wheel passages are always full.
The water is discharged freely into the atmosphere above the tail-water; hence the turbine must be at the foot of the fall.	As the wheel passages are always full, it will work equally well when discharging into the atmosphere or into water, *i.e.* above or below the tail-water, or into suction pipes. The turbine may be placed 30 feet above the foot of the fall.
Water may be admitted on a portion or on the whole circumference of the wheel.	Water must be admitted on the whole circumference of the wheel.
Power easily regulated without much loss.	Power difficult to regulate without loss.

In any form of turbine, it is quite impossible to so arrange it that the water leaves with no velocity, otherwise the wheel would not clear itself. From 5 to 8 per cent. of the head is often required for this purpose, and is rejected in the tail-race.

Form of Blades for Impulse Turbine.—The form of blades required for the guide passages and wheel of a turbine are most easily arrived at by a graphical method. The main points to be borne in mind are—the water must enter the guide and wheel passages without shock. To avoid losses through sudden changes of direction, the vanes must be smooth easy curves, and the changes of section of the passages gradual (for reaction turbines specially). The absolute velocity of the water on leaving must be as small as is consistent with making the wheel to clear itself.

For simplicity we shall treat the wheel as being of infinite radius, and after designing the blades on that basis we shall, by a special construction, bend them round to the required radius. In all the diagrams given the water is represented as entering the guides in a direction normal to that in which the wheel is moving. Let the *velocity* of the water be reduced 6 per cent. by friction in passing over the guide-blades, and let 7 per cent. of the *head* be rejected in the tail-race.

The water enters the guides vertically, hence the first tip of the guide-blade must be vertical as shown. In order to

find the direction of the final tip, we proceed thus: We have decided that the water shall enter the wheel with a velocity of

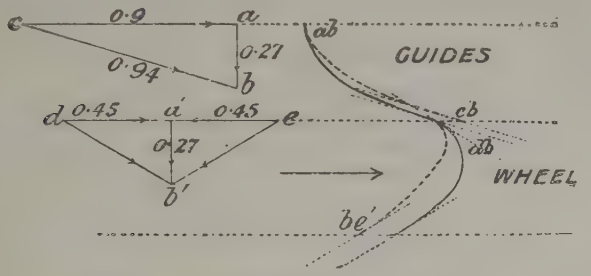


FIG. 683.

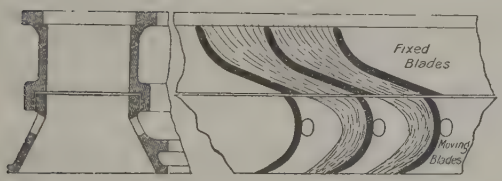


FIG. 684.

94 per cent. of that due to the head, since 6 per cent. is lost in friction, whence—

$$V_0 = 0{\cdot}94\sqrt{2gH}$$

also the velocity of rejection—

$$V_r = \sqrt{\frac{2g \times 7H}{100}} = 0{\cdot}27\sqrt{2gH}$$

We now set down *ab* to represent the vertical velocity with which the water passes through the turbine wheel, and from *b* we set off *bc* to represent V_0; then *ac* gives us the horizontal component of the velocity of the water, and $= \sqrt{0{\cdot}94^2 - 0{\cdot}27^2} = 0{\cdot}9$[1] $= V_1$: *cb* gives us the direction in which the water leaves the guide-vanes; hence a tangent drawn to the last tip of the

[1] We omit $\sqrt{2gH}$ to save constant repetition.

guide-vanes must be parallel to *cb*. We are now able to construct the guide-vanes, having given the first and last tangents by joining them up with a smooth curve as shown.

Let the velocity of the wheel be one-half the horizontal velocity of the entering stream, or $V_w = \frac{0{\cdot}9}{2} = 0{\cdot}45$; hence the horizontal velocity of the water relative to the wheel is also 0·45. Set off $a'b'$ as before = 0·27, and $a'd$ horizontal and = 0·45: we get db' representing the velocity of the water relative to the vane; hence, in order that there may be no shock, a tangent drawn to the first tip of the wheel-vane must be parallel to db'; but, as we want the water to leave the vanes with no absolute horizontal velocity, we must deflect it during its passage through the wheel, so that it has a backward velocity relative to the wheel of −0·45, and as it moves forward with it, the absolute horizontal velocity will be $-0{\cdot}45 + 0{\cdot}45 = 0$. To accomplish this, set off $a'e = 0{\cdot}45$. Then $b'e$ gives us the final velocity of the water relative to the wheel; hence the tangent to the last tip of the wheel-vane must be parallel to $b'e$. Then, joining up the two tangents with a smooth curve, we get the required form of vane.

It will be seen that an infinite number of guides and vanes could be put in to satisfy the conditions of the initial and final tangents, such as the dotted ones shown. The guides are, for frictional reasons, usually made as short as is consistent with a smooth easy-connecting curve, in order to reduce the surface to a minimum. The wheel-vanes should be so arranged that the absolute path of the water through the wheel is a smooth curve without a sharp bend.

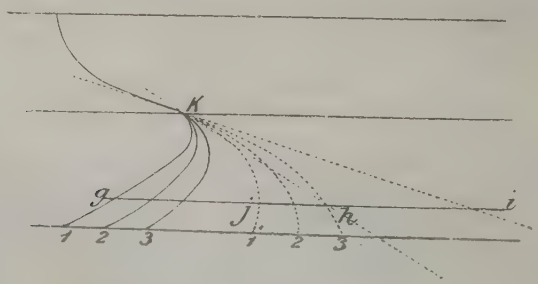


FIG. 685.

The water would move along the absolute path Ki and along the path Kh relative to the wheel if there were no vanes

to deflect it, where *hi* is the distance moved by the vane while the water is travelling from *k* to *i*; but the wheel-vanes deflect it through a horizontal distance *hg*, hence a particle of water at *g* has been deflected through the distance *gj* by the vanes, where $gh = ij$. The absolute paths of the water corresponding to the three vanes, 1, 2, 3, are shown in the broken lines bearing the same numbers. In order to let the water get away very freely, and to prevent any possibility of them choking, the sides of the wheel-passages are usually provided with ventilation holes, and the wheel is flared out. The efficiency of the turbine is readily found thus:

The whole of the horizontal component of the velocity of the water has been imparted to the turbine wheel, hence—

$$\left.\begin{array}{l}\text{the work done per pound of water}\\ \text{the energy per pound of the water on entering}\end{array}\right\} = \frac{V_1^2}{2g} = \frac{(0{\cdot}9V)^2}{2g} = H = \frac{V^2}{2g}$$

$$\text{the hydraulic efficiency} = \frac{V_1^2}{V^2} = 0{\cdot}9^2 = 81 \text{ per cent.}$$

The losses assumed in this example are larger than is usual in well-designed turbines in which the velocity of rejection $V_r = 0{\cdot}125\sqrt{2gH}$; hence a higher hydraulic efficiency than that found above may readily be obtained. The total or overall efficiency is necessarily lower than the hydraulic efficiency on account of the axle friction and other losses. Under the most favourable conditions an overall efficiency of 80 per cent. may be obtained; but statements as to much higher values than this must be regarded with suspicion.

In some instances an analytical method for obtaining the blade angles is more convenient than the graphical. Take the case of an outward-flow turbine, and let, say, 5 per cent. of the *head* be wasted in friction when passing over the guide-blades, and the velocity of flow through the guides be $0{\cdot}125\sqrt{2gH}$. Then the last tip of the guide-blades will make an angle θ_1 with a tangent to the outer guide-blade circle, where $\sin\theta_1 = \frac{0{\cdot}125}{0{\cdot}975} = 0{\cdot}128$, and $\theta_1 = 7^\circ\ 22'$. The horizontal component of the velocity $V_1 = 0{\cdot}125 \cot\theta = 0{\cdot}967$.

The circumferential velocity of the inner periphery of the wheel $V_w = 0{\cdot}483$. The inlet tip of the wheel-vanes makes an angle θ_2 with a tangent to the inner periphery of the wheel;

or, to what is the same thing, the outer guide-blade circle, where $\tan \theta_2 = \frac{0{\cdot}125}{0{\cdot}483} = 0{\cdot}259$, and $\theta_2 = 14^\circ\ 31'$.

Let the velocity of flow through the wheel be reduced to $0{\cdot}08\sqrt{2gH}$ at the outer periphery of the wheel, due to widening or flaring out the wheel-vanes, and let the outer diameter of

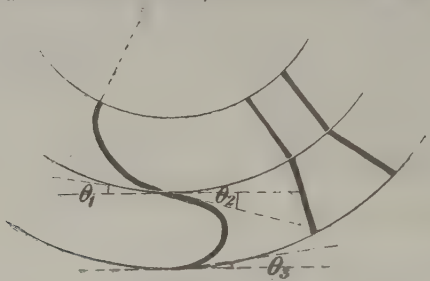
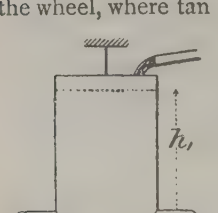


FIG. 686.

the wheel be 1·3 times the inner diameter; then the circumferential velocity of the outer periphery of the wheel will be $0{\cdot}483 \times 1{\cdot}3 = 0{\cdot}628$, and the outlet tip of the wheel-vanes will make an angle θ_3 with a tangent to the outer periphery of the wheel, where $\tan \theta_3 = \frac{0{\cdot}08}{0{\cdot}628} = 0{\cdot}127$, and $\theta_3 = 7^\circ\ 16'$.



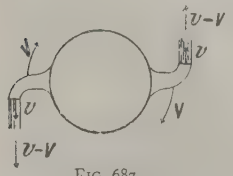


FIG. 687.

Pressure Turbine.—Before proceeding to consider the vanes for a pressure turbine, we will briefly look at its forerunner, the simple reaction wheel. Let the speed of the orifices be V; then, if the water were simply left behind as the wheel revolved, the velocity of the water relative to the orifices would be V, and the head required to produce this velocity—

$$h = \frac{V^2}{2g}$$

Let h_1 = the height of the surface of the water or the head above the orifices.

Then the total head producing flow, $H = h + h_1 = \frac{V^2}{2g} + h_1$

Let v = the relative velocity with which the water leaves the orifices.

$$\text{Then } v^2 = 2gH = V^2 + 2gh_1$$

The velocity of the water relative to the ground $= v - V$. If the jet impinged on a plate, the pressure would be $\frac{v - V}{g}$ per pound of water; but the reaction on the orifices is equal to this pressure, therefore the reaction—

$$R = \frac{v - V}{g}$$

and the work done per second by the jets in foot-lbs. $\Big\}$ $RV = \frac{V(v - V)}{g}$ . . . (i.)

energy wasted in discharge water per pound in foot-lbs. $\Big\}$ $= \frac{(v - V)^2}{2g}$ . . . (ii.)

total energy per pound of water $\Big\}$ i. + ii. $= \frac{v^2 - V^2}{2g}$

hydraulic efficiency $\frac{\text{i.}}{\text{i.} + \text{ii.}} = \frac{2V}{v + V}$

As the value of V approaches v, the efficiency approaches 100 per cent., but for various reasons such a high efficiency

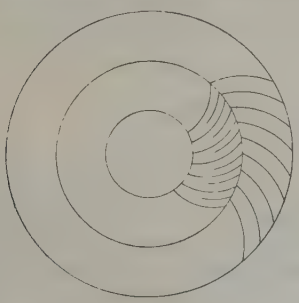


FIG. 688.

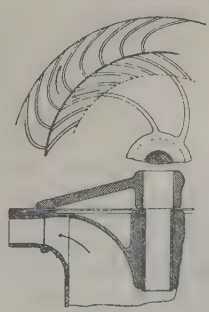


FIG. 689.

can never be reached. The hydraulic efficiency may reach 65 per cent., and the total 60 per cent. The loss is due to the water leaving with a velocity of whirl $v - V$. In order to reduce this loss, Fourneyron, by means of guide-blades, gave the water an initial whirl in the opposite direction before it entered the wheel, and thus caused the water to leave with

little or no velocity of whirl, and a corresponding increase in efficiency.

The method of arranging such guides is shown in Fig. 688; they are simply placed in the central chamber of such a wheel as that shown in Fig. 682. Sometimes, however, the guides are outside the wheel, and sometimes above, according to the type of turbine.

Form of Blades for Pressure Turbine.—As in the impulse turbine, let, say, 7 per cent. of the *head* be rejected in the tail-race, and say 13 per cent. is wasted in friction. Then we get 20 per cent. wasted, and 80 per cent. utilized.

Some of the head may be converted into pressure energy, and some into kinetic energy; the relation between them is optional as far as the efficiency is concerned, but it is convenient to remember that the speed of the turbine increases as the ratio $\frac{V_v}{V_h}$ increases; hence within the limit of the head of water at disposal any desired speed of the turbine wheel can be obtained, but for practical reasons it is not usual to make the above-mentioned ratio greater than 1·6. Care must always be taken to ensure that the pressure in the clearance space between the guides and the wheel is not below that of the atmosphere, or air may leak in and interfere with the smooth working of the turbine. In this case say one-half is converted into pressure energy, and one-half into kinetic energy. If 80 per cent. of the *head* be utilized, the corresponding velocity will be—

$$V = \sqrt{2g\frac{H \times 80}{100}} = 0{\cdot}89\sqrt{2gH}$$

Thus 89 per cent. of the velocity will be utilized. To find the corresponding vertical or pressure component V_v and the horizontal or velocity component V_h, we draw the triangle of velocities as shown (Fig. 690), and find that each is $0{\cdot}63\sqrt{2gH}$.

We will determine the velocity of the wheel by the principle of momentum. Water enters the wheel with a horizontal velocity $V_h = 0{\cdot}63$, and leaves with no horizontal momentum.

$$\text{Horizontal pressure per pound of water} = \frac{V_h}{g} \text{ lbs.}$$

$$\left.\begin{array}{l}\text{useful work done in foot-lbs. per second per}\\ \text{pound of water}\end{array}\right\} = \frac{V_h V_w}{g} = 0{\cdot}81H$$

since we are going to utilize 80 per cent. of the head.

$$\text{Hence } V_w = \frac{0{\cdot}8g\text{H}}{0{\cdot}63\sqrt{2g\text{H}}} = \frac{0{\cdot}8\sqrt{g\text{H}}}{0{\cdot}63\sqrt{2}}$$

$$V_w = 0{\cdot}64\sqrt{2g\text{H}}$$

We now have all the necessary data to enable us to determine the form of the vanes. Set down *ab* to represent the velocity of flow through the wheel, and *ac* the horizontal velocity. Completing the triangle, we find *cb* = 0·69, which gives us the direction in which the water enters the wheel or leaves the guides. The guide-blade is then put in by the method explained for the impulse wheel.

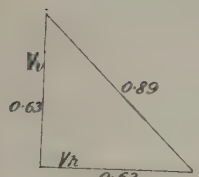


FIG. 690.

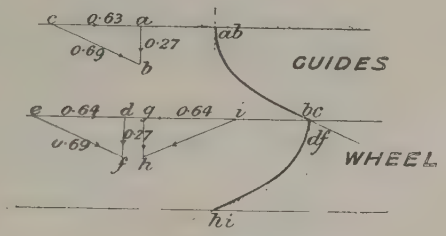


FIG. 691.

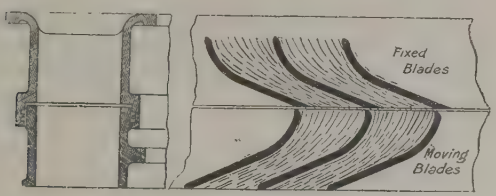


FIG. 692.

To obtain the form of a wheel-vane. Set off *ed* to represent the horizontal velocity of the wheel, and *ef* parallel to *cb* to represent the velocity of the water on leaving the guides; then *df* represents the velocity of the water relative to the wheel, which gives us the direction of the tangent to the first tip of the wheel-vanes. Then, in order that the water shall leave with no horizontal velocity, we must deflect it during its passage through the wheel so that it has a backward velocity relative to the wheel of − 0·64. Then, setting down *gh* = 0·27, and *gi* = − 0·64, we get *hi* as the final velocity of the water, which gives us the direction of the tangent to the last tip of the vane.

In Figs. 693, 694, 695 we show the form taken by the

wheel-vanes for various proportions between the pressure and velocity energy.

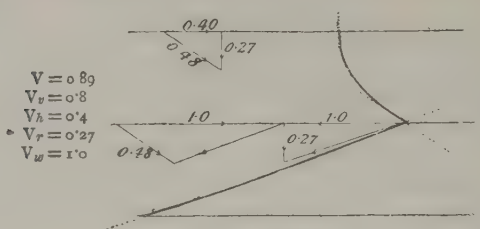


FIG. 693.

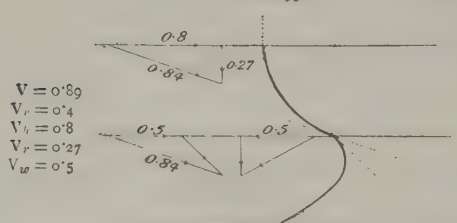


FIG. 694.

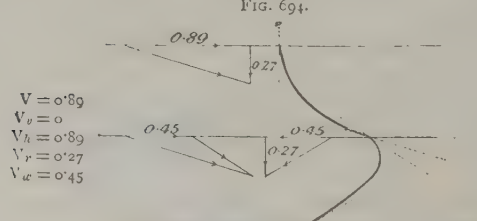


FIG. 695.

In the case in which $V_v = o$, the whole of the energy is converted into kinetic energy; then $V_h = o{\cdot}89$, and—

$$V_w = \frac{o{\cdot}8g H}{o{\cdot}89\sqrt{2gH}}$$

$$V_w = o{\cdot}45\sqrt{2gH}$$

Or the velocity of the wheel is one-half the horizontal velocity of the water, as in the impulse wheel. The form of blades in this case is precisely the same as in Fig. 683, but they are arrived at in a slightly different manner.

For the sake of clearness all these diagrams are drawn with

assumed losses much higher than those usually found in practice.

Centrifugal Head in Turbines.—It was pointed out some years ago by Professor James Thompson that the centrifugal force acting on the water which is passing through an inward-flow turbine may be utilized in securing steady running, and in making it partially self-governing, whereas in an outward-flow turbine it has just the opposite effect.

Let R_e = external radius of the turbine wheel;
R_i = internal " " "
V_e = velocity of the outer periphery of the wheel;
V_i = " " inner " "
ω = angular velocity of the wheel;
w = weight of a unit column of water.

Consider a ring of rotating water of radius r and thickness dr, moving with a velocity v.

The mass of a portion of the ring of area a measured normal to a radius $= \frac{wa}{g}dr$

The centrifugal force acting on the mass $= \frac{wav^2}{gr}dr$

$$= \frac{wa\omega^2}{g} r \,.\, dr$$

The centrifugal force acting on all the masses lying between the radius R_i and R_e $= \frac{wa\omega^2}{g} \int_{R_i}^{R_e} r \,.\, dr$

or—

$$\text{The centrifugal head} = \frac{wa\omega^2}{g}\left(\frac{R_e^2 - R_i^2}{2}\right)$$

$$= \frac{wa}{2g}(V_e^2 - V_i^2)$$

The centrifugal head per square inch per pound of water $= \frac{V_e^2 - V_i^2}{2g}$

This expression gives us the head which tends to produce outward radial flow of the water through the wheel due to centrifugal force—it increases as the velocity of rotation increases; hence in the case of an inward-flow turbine, when an increase of speed occurs through a reduction in the external load, the centrifugal head also increases, which thereby reduces the effective head producing flow, and thus tends to reduce

the quantity of water flowing through the turbine, and thereby to keep the speed of the wheel within reasonable limits. On the other hand, the centrifugal head tends to increase the flow through the wheel in the case of an outward-flow turbine when the speed increases, and thus to still further augment the speed instead of checking it.

The following results of experiments made in the author's laboratory will serve to show how the speed affects the quantity of water passing through the wheel of an inward flow turbine :—

GILKES' VORTEX TURBINE.

External diameter of wheel ... ... 0·75 foot
Internal ,, ,, ... 0·375 ,,
Static head (H_s) of water above turbine... ... 24 feet
Guide blades ... ... Half open

Revolutions per minute.	Quantity of water passing in cubic feet per second.	Centrifugal head H_c.	$K_d A \sqrt{2g(H_s - H_c)}$
0	0·68	0	0·68
100	0·68	0·18	0·68
200	0·67	0·72	0·67
300	0·66	1·6	0·66
400	0·64	2·9	0·64
500	0·62	4·5	0·61
600	0·59	6·5	0·58
700	0·55	8·8	0·54
800	0·50	11·5	0·49
900	0·42	14·5	0·43

The last column gives the quantity of water that will flow through the turbine due to the head $H_s - H_c$. The coefficient of discharge K_d is obtained from the known quantity that passed through the turbine when the centrifugal head was zero, *i.e.* when the turbine was standing.

Dimensions of Turbines.—The general leading dimensions of a turbine for a given power can be arrived at thus—

Let B = breadth of the water-inlet passages in the turbine wheel;
R = mean radius of the inlet passage where the water enters the wheel;
V_f = velocity of flow through the turbine wheel;
H = available head of water above the turbine;
Q = quantity of water passing through the turbine in cubic feet per second;
P = horse-power of the turbine;
η = the efficiency of the turbine;
N = revolutions of wheel per minute.

Then, if B be made proportional to the mean radius of the water opening, say—

$$\left.\begin{aligned} B &= aR \\ \text{and } V_f &= b\sqrt{2gH} \\ V &= c\sqrt{2gH} \end{aligned}\right\} \text{where } a, b, \text{ and } c \text{ are constants}$$

then $Q = 2\pi RBV_f = 2ab\pi R^2\sqrt{2gH}$, neglecting the thickness of the blades

$$P = \frac{62{\cdot}4QH\eta}{550} = 0{\cdot}71ab R^2\eta H\sqrt{2gH}$$

$$R = \sqrt{\frac{P}{0{\cdot}71ab\eta H\sqrt{2gH}}}$$

$$V_h = \sqrt{2gH(c^2 - b^2)}$$

$$V_w = m\sqrt{2gH(c^2 - b^2)}, \text{ where } m = \frac{V_w}{V_h} = \tfrac{1}{2} \text{ for impulse turbines}$$

The velocity of the wheel V_w is to be measured at the mean radius of the water inlet. Then—

$$2\pi RN = 60V_w = 60m\sqrt{2gH(c^2 - b^2)}$$

$$N = \frac{60m\sqrt{2gH(c^2 - b^2)}}{2\pi R}$$

By substituting the value of R and reducing, we get—

$$N = \frac{182mH^{\frac{5}{4}}\sqrt{ab\eta(c^2 - b^2)}}{\sqrt{P}}$$

These equations enable us to find the necessary inlet area for the water, and the speed at which the turbine must run in order to develop the required power, having given the available head and quantity of water. The constants can all be determined by the methods already described.

Projection of Turbine Blades.—In all the above cases we have constructed the vanes for a turbine of infinite radius, sometimes known as a "turbine rod." We shall now proceed to give a construction for bending the rod round to a turbine of small radius.

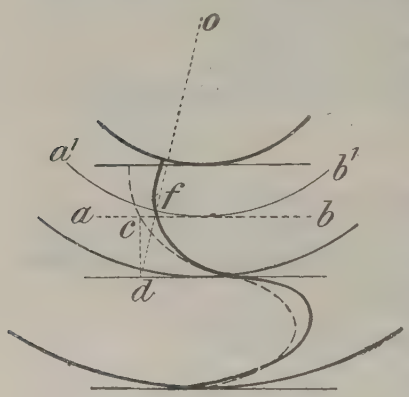


FIG. 696.

The blades for the straight turbine being given, draw a series of lines across as shown; in this case only one is shown for sake of clearness, viz. *ab*, which cuts the blade in the point *c*. Project this point on to the base-line, viz. *d*. From the centre *o* describe a circle *a'b'* touching the line *ab*. Join *od*, cutting the circle *a'b'* in the point *f*. This point *f* on the circular turbine blade corresponds to the point *c* on the straight blade. Other points are found in the same manner, and a smooth curve is drawn through them.

The blades for the straight turbine are shown dotted, and those for the circular turbine in full lines.

Efficiency of Turbines.—The following figures are taken from some curves given by Professor Unwin in a lecture delivered at the Institution of Civil Engineers in the Hydromechanics course in 1884–5 :—

Type of turbine.	Efficiency per cent. at various sluice-openings.						
	Full.	0·9.	0·8.	0·7.	0·6.	0·5.	0·4.
Impulse (Girard)	80	80	80	80	81	81	—
Pressure (Jonval) (throttle-valve) ...	71	59	46	35	25	16	—
Hercules	82	82	80	75	68	63	55

Losses in Turbines.—The various losses in turbines of course depend largely on the care with which they are designed and manufactured, but the following values taken from the source mentioned above will give a good idea of the magnitude of the losses.

Loss due to surface friction, eddying, etc., in the turbine	10 to 14 per cent.
Loss due to energy rejected in tail-race .	3 to 7 ,,
,, shaft friction . . .	2 to 3 ,,

CHAPTER XX.

PUMPS.

NEARLY all water-motors, when suitably arranged, can be made reversible—that is to say, that if sufficient power be supplied to drive a water-motor backwards, it will raise water from the tail-race and deliver it into the head-race, or, in other words, it will act as a pump.

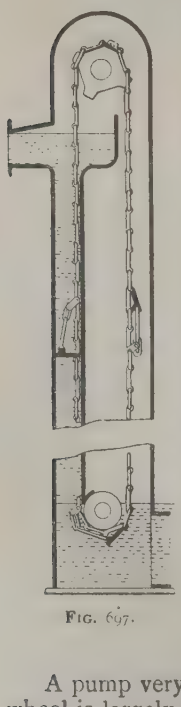


FIG. 697.

The only type of motor that cannot, for practical purposes, be reversed is an impulse motor, which derives its energy from a free jet or stream of water, such as a Pelton wheel.

We shall consider one or two typical cases of reversed motors.

Reversed Gravity Motors: Bucket Pumps, Chain Pump, Dredgers, Scoop Wheels, etc.—The two gravity motors shown in Figs. 657, 658, will act perfectly as pumps if reversed; an example of a chain pump is shown in Fig. 697. The floats are usually spaced about 10 feet apart, and the slip or the leakage past the floats is about 20 per cent. They are suitable for lifts up to 60 feet. The chain speed varies from 200 to 300 feet per minute, and the efficiency is about 63 per cent.

The ordinary dredger is also another pump of the same type.

Reversed overshot water-wheels have been used as pumps, but they do not readily lend themselves to such work.

A pump very similar to the reversed undershot or breast wheel is largely used for low lifts, and gives remarkably good

results; such a pump is known as a "scoop wheel" (see Fig. 698).

The circumferential speed is from 6 to 10 feet per second. The slip varies from 5 per cent. in well-fitted wheels to 20 per cent. in badly fitted wheels.

The diameter varies from 20 to 50 feet, and the width from 1 to 5 feet; the paddles are pitched at about 18 inches. The total efficiency, including the engine, varies from 50 to 70 per cent.

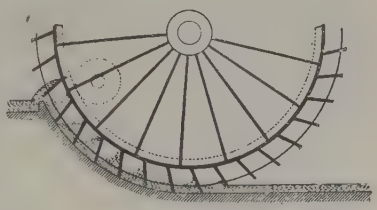


FIG. 698.

Reversed Pressure Motors, or Reciprocating Pumps.—If a pressure motor be driven from some external source the feed pipe becomes a suction pipe, and the exhaust a delivery pipe; such a reversed motor is termed a plunger, bucket, or

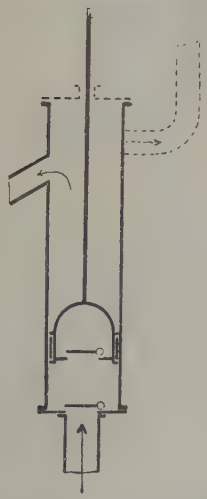


FIG. 699.

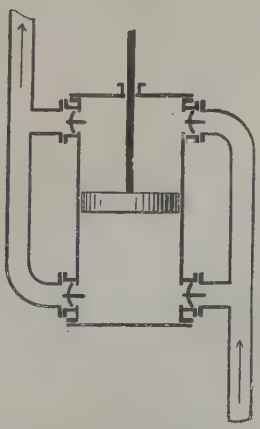


FIG. 700.

piston pump. They are termed double or single acting according as they deliver water at every or at alternate strokes of the piston or plunger. In Figs. 669, 700, 701, and 702 we show typical examples of various forms of reciprocating pumps.

Fig. 699 is a bucket pump, single acting, and gives an intermittent discharge. It is only suitable for low lifts. Sometimes this form of pump is modified as shown in dotted lines, when it is required to force water to a height.

Fig. 700 is a double-acting force or piston pump. When such pumps are made single acting, the upper set of valves are dispensed with. They can be used for high lifts. The manner in which the flow fluctuates will be dealt with in a future paragraph.

Fig. 701 is a plunger pump. It is single acting, and is the

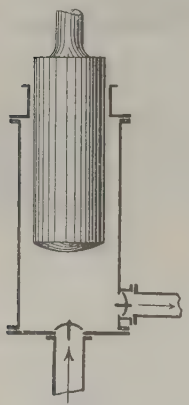


FIG. 701.

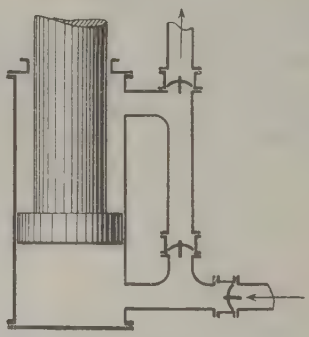


FIG. 702.

form usually adopted for very high pressures. The flow is intermittent.

Fig. 702 is a combined bucket-and-plunger pump. It is double acting, but has only one inlet and one delivery valve. The flow is similar to that of Fig. 700.

There is no need to enter into a detailed description of the manner in which these pumps work; it will be obvious from the diagrams. It may, however, be well to point out that if the velocity past the valves be excessive, the frictional resistance becomes very great, and the work done by the pump greatly exceeds the work done in simply lifting the water. Provided a pump is dealing with water only, and not air and water, the amount of clearance at each end of the stroke is a matter of no importance.

Fluctuation of Delivery.—In all forms of reciprocating

pumps there is more or less fluctuation in the delivery, both during the stroke and, in the case of single-acting pumps, between the delivery strokes. The fluctuation during the stroke is largely due to the variation in the speed of the piston, bucket, or plunger. As this fluctuation is in some instances a serious matter, *e.g.* on long lengths of mains, we shall carefully consider the matter.

When dealing with the steam-engine mechanism in Chapter VI. we gave a construction for finding the velocity of the piston at every part of the stroke. We repeat it in Fig. 703, showing the construction lines for only one or two points. The flow varies directly as the velocity, hence the velocity diagram is also a flow diagram.

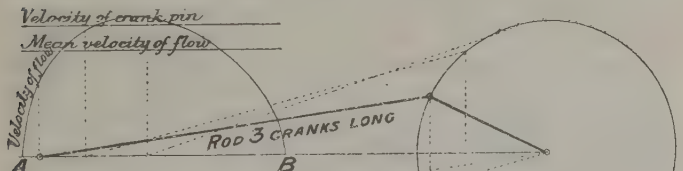


FIG. 703.

We give some typical flow diagrams below for the case of pumps having no stand-pipe or air-vessel. The vertical height of the diagram represents the quantity of water being delivered at that particular part of the stroke. In all cases we have assumed that the crank revolves at a constant speed, and have adhered to the proportion of the connecting-rod = 3 cranks. The letters A and B refer to the particular end of the stroke, corresponding to *d* and *e*, Fig. 206.

Single-acting Pump.—One barrel.

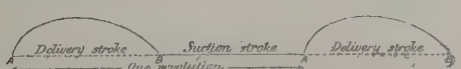


FIG. 704.

Single-acting Pump.—Two barrels, cranks Nos. 1 and 2 opposite one another or at 180°.

Each stroke is precisely the same as in the case above.

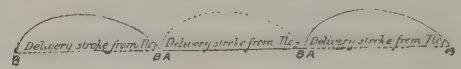


FIG. 705.

Single-acting Pump.—Two barrels, cranks Nos. 1 and 2 at 90°.

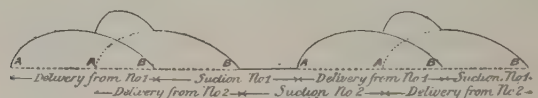


FIG. 706.

Where the two curves overlap, the ordinates are added. It will be observed that the fluctuation is very much greater than before. It should be noted that the second barrel begins to deliver just after the middle of the stroke.

Single-acting Pump.—Three barrels, Nos. 1, 2, 3 cranks at 120°.

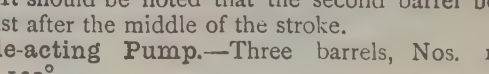


FIG. 707.

It will be observed that the flow is much more constant than in any of the previous cases.

Similar diagrams are easily constructed for double-acting pumps with one or more barrels; it should, however, be noticed that the diagram for the return stroke should be reversed end for end, thus—

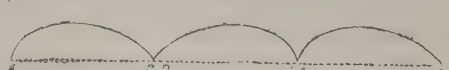


FIG. 708.

We shall shortly see the highly injurious effects that sudden changes of flow have on a long pipe. In order to partially mitigate them and to equalize the flow, air-vessels are usually placed on the delivery pipe close to the pump. Their function

on a pump is very similar to that of a flywheel on an engine. When the pump delivers more than its mean quantity of water the surplus finds its way into the air-vessel, and compresses the air in the upper portion; then when the delivery falls below the mean, the compressed air forces the stored water into the main, and thereby tends to equalize the pressure and the flow. The method of arriving at the size of air-vessel required to keep the flow within given limits is of a similar character to that adopted for determining the size of flywheel required for an engine.

For three-throw pumps the ratio of the volume of the air-vessel to the volume displaced by the pump plunger per stroke is from 1 to 2, in duplex pumps from 1·5 to 5, and in certain instances of fast-running, single-acting pumps it gets up to 30. Great care must be taken to ensure that air-vessels do contain the intended quantity of air. They are very liable to get water-logged through the absorption of the air by the water.

Speed of Pumps.—The term "speed of pump" is always used for the mean speed of the piston or plunger. The speed has to be kept down to moderate limits, or the resistance of the water in passing the valves becomes serious, and the shock due to the inertia of the water causes mischief by bursting the pipes or parts of the pump. The following are common maximum speeds for pumps:—

Large pumping engines and mining pumps	100 to 300 ft. min
Exceptional cases with controlled valves ...	up to 600 ,,
Fire-engines	150 to 250 ,,
Slow-running pumps	30 to 50 ,,
Force pumps on locomotives	up to 900 ,,

The high speed mentioned above for the loco. force-pump would not be possible unless the pipes were very short and the valves very carefully designed; the valves in such cases are often 4-inch diameter with only $\frac{1}{8}$-inch lift. If a greater lift be given, the valves batter themselves to pieces in a very short time; but in spite of all precautions of this kind, the pressures due to the inertia of the water are very excessive. One or two instances will be given shortly.

Suction.—We have shown that the pressure of the atmosphere is equivalent to a head of water of about 34 feet. No pump will, however, lift water to so great a height by suction, partly due to the leakage of air into the suction pipes, to the resistance of the suction valves, and to the fact that the water

gives off vapour at very low pressures and destroys the vacuum that would otherwise be obtained. Under exceptionally favourable circumstances, a pump will lift by suction through a height of 30 feet, but it is rarely safe to reckon on more than 25 feet for pumps of average quality.

Inertia Effects in Pumps.—The hydraulic ramming action in reciprocating pumps due to the inertia of the water has to be treated in the same way as the inertia effects in water-pressure motors (see p. 699). In a pump the length of either the suction *or* the delivery pipe corresponds to the length of main, L.

As an illustration of the very serious effects of the inertia of the water in pumps, the following extreme case, which came under the author's notice, may be of interest. The force-pumps on the locomotives of one of the main English lines of railway were constantly giving trouble through bursting; an indicator was therefore attached in order to ascertain the pressures set up. After smashing more than one instrument through the extreme pressure, one was ultimately got to work successfully. The steam-pressure in the boiler was 140 lbs., but that in the pump sometimes amounted to 3500 lbs. per square inch; the velocity of the water was about 28 feet per second through the pipes, and still higher through the valves; the air-vessel was of the same capacity as the pump. After greatly increasing the areas through the valves, enlarging the pipes and air-vessels to five times the capacity of the pump, the pressure was reduced to 900 lbs. per sq. inch, but further enlargements failed to materially reduce it below this amount. The friction through the valve pipes and passages will account for about 80 lbs. per sq. inch, and the boiler pressure 140, or 220 lbs. per sq. inch due to both; the remaining 680 lbs. per sq. inch are due to inertia of the water in the pipes, etc.

Volume of Water delivered.—In the case of slow-speed pumps, if there were no leakage past the valves and piston, and if the valves opened and closed instantly, the volume of water delivered would be the volume displaced by the piston. This, however, is never the case; generally speaking, the quantity delivered is less than that displaced by the piston, sometimes it only reaches 90 per cent. No figures that will apply to all cases can possibly be given, as it varies with every pump and its speed of working. As might be expected, the leakage is generally greater at very slow than at moderate speeds, and is greater with high than with low pressures. This deficiency in the quantity delivered is termed the "slip" of the pump.

With a long suction pipe and a low delivery pressure, it is often found that both small and large pumps deliver more water than the displacement of the piston will account for; such an effect is due to the momentum of the water. During the suction stroke the delivery valves are supposed to be closed, but just before the end of the stroke, when the piston is coming to rest, there may be a considerable pressure in the barrel due to the inertia of the water (see p. 746), which, if sufficiently great, will force open the delivery valves and allow the water to pass into the delivery pipes until the pressure in the barrel becomes equal to that in the uptake.

When a pump is running so slowly that the inertia effects on the water are practically *nil*, the indicator diagram from the pump barrel is rectangular in form; the suction line will be slightly below but practically parallel to the atmospheric line. When, however, the speed increases, the inertia effect on the water in the suction pipe begins to be felt, and the suction line of the diagram is of the same form as the inertia pressure line $a_o i_o h_o b_o$ in Fig. 206. Theoretical and actual pump diagrams are shown in Fig. 709. In this case the inertia pressure at the end of the stroke was less than the delivery pressure in the uptake of the pump. If the speed be still further increased until the inertia pressure exceeds that in the delivery pipe, the pressure in the suction pipe will force open the delivery valves before the end of the suction stroke, and such diagrams as those shown in Fig. 710 will be obtained. A comparison of these diagrams with the delivery-valve lift diagram taken at the same time is of interest in showing that the delivery valve actually does open at the instant that theory indicates.

Apart from friction, the whole of the work done in accelerating the water in the suction pipe during the early portion of the stroke is given back by the retarded water during the latter portion of the stroke.

In Fig. 710 the line $a'b'$ represents the distribution of pressure due to the inertia at all parts of the stroke. From a' to c' the pressure is negative, because the pump is accelerating the water in the suction pipe. At c' the inertia pressure becomes zero, and in consequence of the water being retarded after that point is reached, it actually exerts a driving effort on the plunger. When the plunger reaches d', the inertia pressure becomes equal to the delivery pressure, and it immediately forces open the delivery valve, causing the water to pass up the delivery pipe before the completion of the suction stroke. Since the ordinates of the curve $a'b'$ represent the water-pressure on the plunger,

and abscissæ the horizontal distances the plunger has moved, the area $d'd'k'$ represents the work done by the retarded water in forcing the plunger forward, and the area $d'b'f$ represents the work done in delivering the extra water during the suction stroke; and the work done under normal conditions is pro-

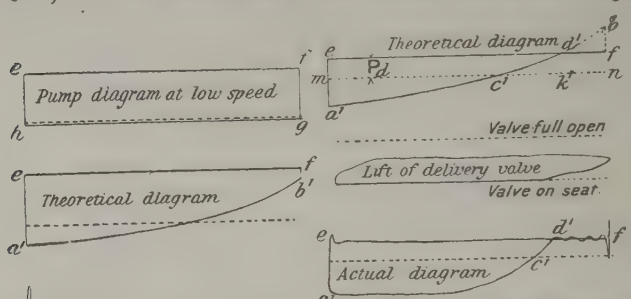


FIG. 709. FIG. 710.—Pressure due to inertia of wate

portional to the area $efnm$; hence the discharge of the pump under these conditions is greater in the ratio $1 + \frac{\text{area } d'b'f}{\text{area } efnm}$ to 1. This ratio is known as the "discharge coefficient" of the pump. This quantity can be readily calculated for the case of a long connecting-rod thus—

Let R = the radius of the crank in feet;

L^1 = the length of connecting-rod in feet;

$n = \frac{L^1}{R}$;

N = revolutions of the pump per minute;

w = the weight of a unit column of water 1 foot high and 1 sq. inch section = 0·434 lb.;

W = weight of the reciprocating parts per square inch of plunger in pounds; in this case the weight of a column of water of 1 sq. inch sectional area and whose length is equal to that of the suction pipe, *i.e.* $0{\cdot}434L_s = wL_s$;

P = the "inertia pressure" at the end of the stroke, *i.e.* the pressure required to accelerate and retard the column of water at the beginning and end of the stroke.

Then, if the suction pipe be of the same diameter as the pump plunger, we have $P = 0{\cdot}00034WRN^2$, but if the area of the suction pipe be A_s, and that of the plunger A_p, we have, substituting the value of W given above—

$$P = 0{\cdot}00034 \times 0{\cdot}434L_sRN^2\frac{A_p}{A_s}$$

L is the length of the suction pipe in feet.

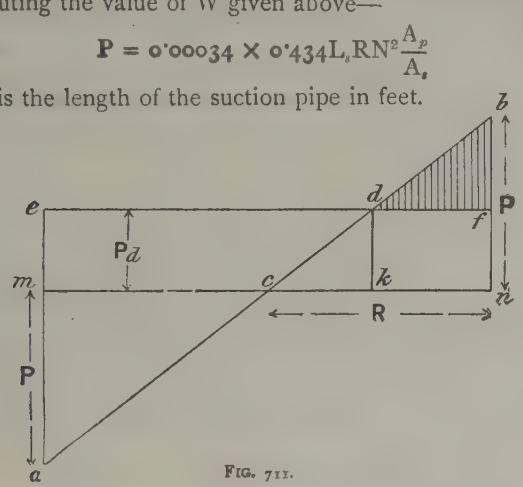


FIG. 711.

The area *dbf* (Fig. 711) $= (P - P_d)\frac{df}{2} = \frac{R(P - P_d)^2}{2P}$

the area *efnm* $= 2P_dR$

the discharge coefficient $= 1 + \frac{(P - P_d)^2}{4PP_d}$

This value will not differ greatly from that found for a pump having a short connecting-rod when running at the same speed.

The discharge coefficients found by experiment agree quite closely with the calculated values, provided the pump is running under normal conditions, *i.e.* with the plunger always in contact with the water in the barrel.

Cavitation in Reciprocating Pumps.—During the suction stroke of a pump the water follows the plunger only so long as the absolute external pressure acting on the water is greater than that in the pump barrel; the velocity with which the water enters the barrel is due to the excess of external pressure over the internal, hence the velocity of flow into the barrel can never exceed that due to a perfect vacuum in the pump barrel *plus* the head of water in the suction sump *above* the barrel, or *minus* if it be *below* the barrel. In the event of the velocity of the plunger being greater than the velocity of

the surface speed of the water, the plunger leaves the water, and thereby forms a cavity between itself and the water. Later in the stroke the water catches up the plunger, and when the two meet a violent bang is produced, and the water-ram pressure then set up in the suction pipe and barrel of the pump is far higher than theory can at present account for. The "discharge coefficient," when cavitation is taking place, is also very much greater than the foregoing theory indicates.

The speed of the pump at which cavitation takes place is readily arrived at, thus—

Adhering to the notation given above, and further—

Let h_s = the suction head below the pump, *i.e.* the height of the surface of the water in the sump below the bottom of the pump barrel; if it be above, this quantity must be given the negative sign;

h_f = the loss of head due to friction in the passages and pipes. Then the pressure required to accelerate the moving water at the beginning of the suction stroke, in this case when the plunger is at the bottom of the stroke, is as before—

$$P = 0{\cdot}00034 \times 0{\cdot}434 LRN^2 \left(1 - \frac{1}{n}\right) \frac{A_p}{A_s}$$

The height of the water-barometer may be taken as 34 feet, then the effective pressure driving the water into the pump barrel is $(34 - h_s - h_f)w$. Separation occurs when this quantity is less than P; equating these two quantities, we get the maximum speed N, at which the pump can run without separation taking place, and reducing we get—

$$N = 54{\cdot}5\sqrt{\frac{(34 - h_s - h_f)A_s}{LR\left(1 - \frac{1}{n}\right)A_p}}$$

That the theory and experiment agree fairly well will be seen from the following results :—

Suction head.	Loss of head due to friction.	Length of pipe.	Ratio of cylinder area to pipe area.	Radius of crank.	Speed at which separation occurs.	
					By calculation.	By experiment.
feet.	feet.	feet.		feet.	Revolutions per minute.	
0·07	8·5	63	1·83	0·25	60·5	between 56 and 62
0·10	8·8	36	1·83	0·25	77·9	,, 73 and 78

The manner in which the "discharge coefficient" varies with the speed is clearly shown by Fig. 712, which is one of the series of curves obtained by the author, and published in the *Proceedings of the Institution of Mechanical Engineers*, 1903.

The dimensions of the pump were—

Diameter of plunger	4 inches.
Stroke	6 ,,
Length of connecting-rod	12 ,,
Length of suction and delivery pipes	63 feet each.
Diameter of suction pipe	3 inches.

The manner in which the water ram in the suction pipe is dependent upon the delivery pressure is shown in Fig. 713.

Direct-acting Steam-pumps. — The term "direct-acting" is applied to those steam-pumps which have no crank or flywheel, and in which the water-piston or plunger is on the same rod as the steam-piston; or, in other words, the steam and water ends are in one straight line. They are usually made double acting, with two steam and two water cylinders. The relative advantages and disadvantages are perhaps best shown thus:

Advantages.	Disadvantages.
Compact.	Steam cannot be used expansively, except with special arrangements, hence—
Small number of working parts.	Wasteful in steam.
No flywheel or crank-shaft.	Liable to run short strokes.
Small fluctuation in the discharge.	
Almost entire avoidance of shock in the pipes.	
Less liability to cavitation troubles than flywheel pumps.	Liable to stick when the steam pressure is low.

We have seen that, when a pump is driven by a uniformly revolving crank, the velocity of the piston, and consequently the flow, varies between very wide limits, and, provided there is a heavy flywheel on the crank-shaft, the velocity of the piston (within fairly narrow limits) is not affected by the resistance it has to overcome; hence the serious ramming effects in the pipes and pump chambers. In a direct-acting pump, however, the pistons are free, hence their velocity depends entirely on the water-resistance to be overcome, provided the steam-pressure is constant throughout the stroke; therefore the water is very gradually put into motion, and kept flowing much more steadily than is possible with a piston which moves practically irrespectively of the resistance it has to overcome. A diagram

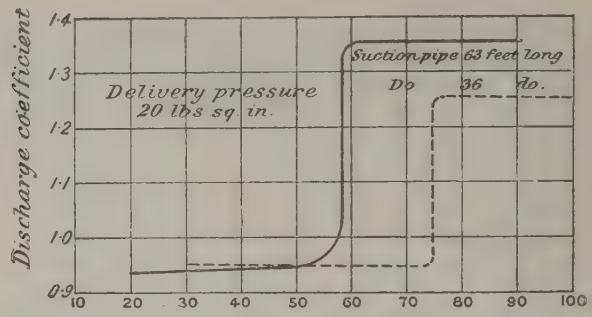


FIG 712.

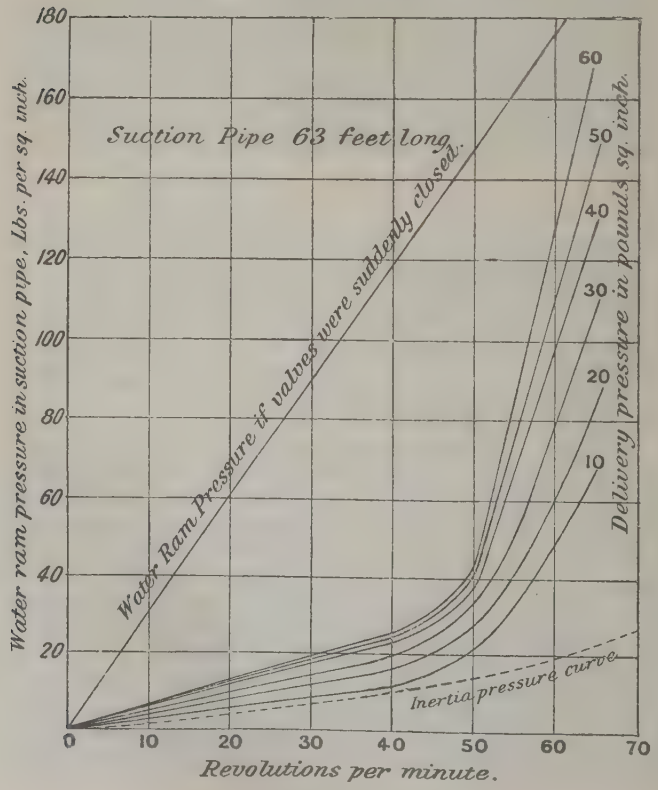


FIG. 713.

of flow from a pump of this character is shown in Fig. 714; it is intended to show the regularity of flow from a Worthington pump, which owes much of its smoothness of running to the fact that the piston pauses at the end of each stroke.[1]

We will now look at some of the disadvantages of direct-acting pumps, and see how they can be avoided. The reason why steam cannot be used expansively in pumps of this type is because the water-pressure in the barrel is practically constant, hence the steam-pressure must at all parts of the stroke be sufficiently high to overcome the water-pressure; thus, if the steam were used expansively, it would be too high at the beginning of the stroke and too low at the end of the stroke. The economy, however, resulting from the expansive use of steam is so great, that this feature of a direct-acting pump is considered to be a very great drawback, especially for large sizes. Many ingenious devices have been tried with

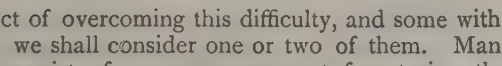


FIG. 714.

the object of overcoming this difficulty, and some with marked success; we shall consider one or two of them. Many of the devices consist of some arrangement for storing the excess energy during the first part of the stroke, and restoring it during the second part, when there is a deficiency of energy. It need hardly be pointed out that the work done in the steam-cylinder is equal to that done in the water-cylinder together with the friction work of the pump (Fig. 715). It is easy enough to see how this is accomplished in the case of a pump fitted with a crank and heavy flywheel (see Chapter VI.), since energy is stored in the wheel during the first part of the stroke, and returned during the latter part. Now, instead of using a rotating body such as a flywheel to store the energy, a reciprocating body such as a very heavy piston may be used; then the excess work during the early part of the stroke is absorbed in accelerating this heavy mass, and the stored energy is given back during the latter part of the stroke while the mass is being retarded, and a very nearly even driving pressure throughout the stroke can be obtained.

The heavy piston is, however, not a practical success for

[1] Taken from a paper on the Worthington Pump, *Proc. Inst. of Civil Engineers*, vol. lxxxvi. p. 293.

many reasons. A far better arrangement is D'Auria's pendulum pump, which is quite successful (Fig. 716). It is in principle

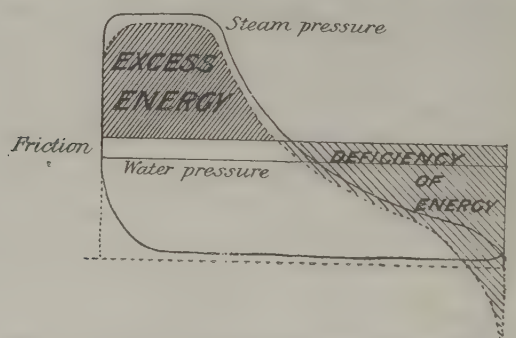


FIG. 715.—The line of real pressure has been put in as in Fig. 200.

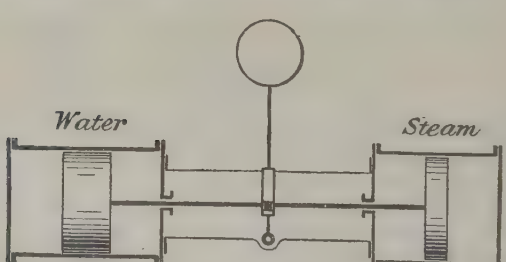


FIG. 716.

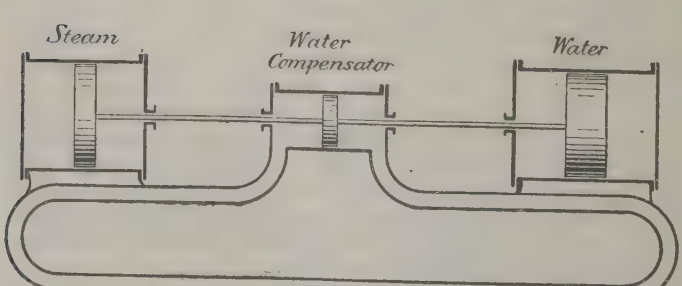


FIG. 717.

the same as the heavy piston, only a much smaller reciprocating (or swinging) mass moving at a higher velocity is used. By far

the most elegant arrangement of this kind[1] is D'Auria's water-compensator, shown in Fig. 717. It consists of ordinary steam and water ends, with an intermediate water-compensator

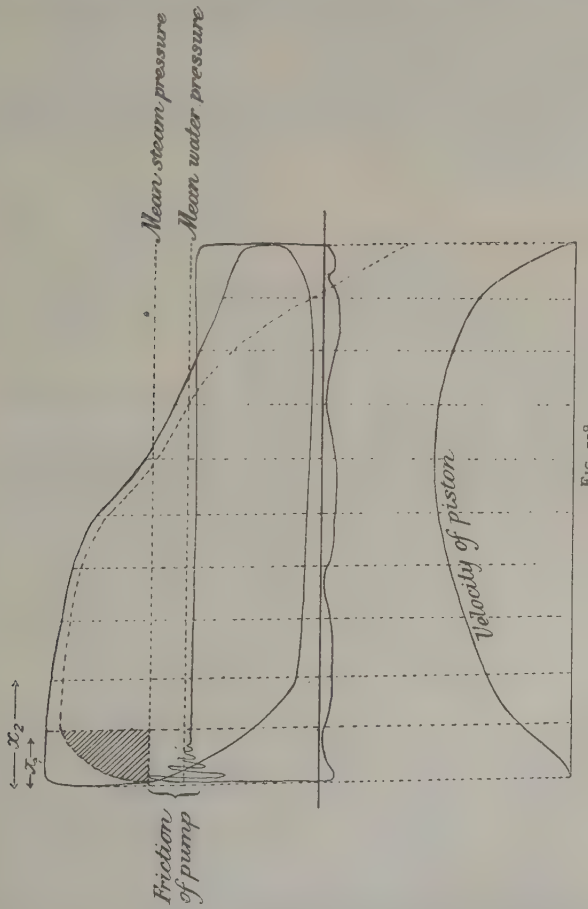


FIG. 718.

cylinder, which, together with the curved pipe below, is kept full of water. The piston in this cylinder simply causes the

[1] The author is indebted to Messrs. Thorp and Platt, of New York, for the particulars of this pump.

water to pass to and fro through the pipe, and thus transfers the water from one end of the cylinder to the other. Its action is precisely similar to that of a heavy piston, or rather the pendulum arrangement shown in Fig. 716, for the area of the pipe is smaller than that of the cylinder, hence the water moves with a higher velocity than the piston, and consequently a smaller quantity is required.

The indicator diagrams in Fig. 718 were taken from a pump of this type. The mean steam-pressure line has been

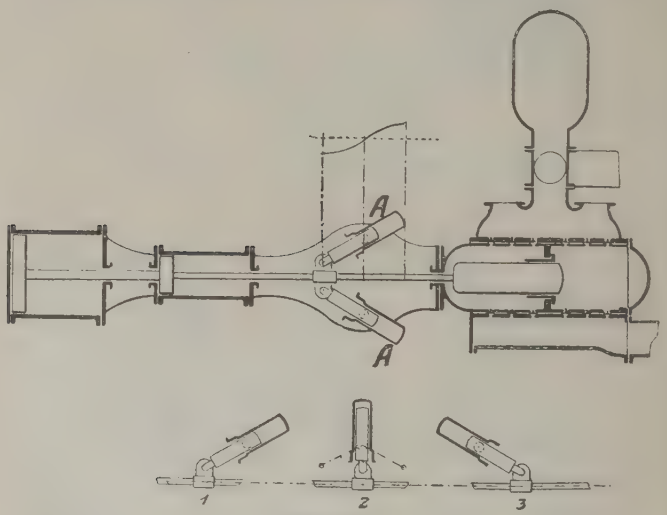


FIG. 719.

added to represent the work lost in friction, and the velocity curve has been arrived at thus:

Let M = the moving mass of the pistons and water in the compensator, etc., per square inch of piston, each reduced to its equivalent velocity;

V = velocity of piston in feet per second.

Then the energy stored in the mass at any instant is $\frac{MV^2}{2}$; but this work is equal to that done by the steam over and above that required to overcome friction and to pump the water, hence the shaded area $px_1 = \frac{MV_1^2}{2}$, and likewise $px_2 = \frac{MV_2^2}{2}$,

where p is the mean pressure acting during the interval x; but $\frac{M}{2}$ is a constant for any given case, hence $V = \sqrt{px} \times$ constant. When the steam-pressure falls below the mean steam-pressure line the areas are reckoned as negative. Sufficient data for the pump in question are not known, hence no scale can be assigned.

Another very ingenious device for the same purpose is the compensator cylinders of the Worthington high duty pump (see Fig. 719). The high and low pressure steam-cylinders are shown to the left, and the water-plunger to the right. Midway between, the compensator cylinders A, A are shown; they are kept full of water, but they communicate with a small high-pressure air-vessel not shown in the figure.

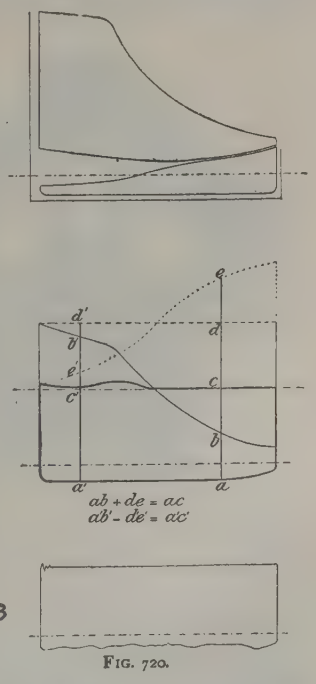


FIG. 720.

When the steam-pistons are at the beginning of the "out" stroke, and the steam-pressure is higher than that necessary to overcome the water-pressure, the compensator cylinder is in the position 1, and as the pistons move forward the water is forced from the compensator cylinder into the air-vessel, and thus by compressing the air stores up energy; at half-stroke the compensator is in position 2, but immediately the half-stroke is passed, the compensator plunger is forced out by the compressed air in the vessel, thus giving back the stored energy when the steam-pressure is lower than that necessary to overcome the water-pressure; at the end of the stroke the compensator is in position 3. A diagram of the effective compensator pressure on the piston-rods is shown to a small scale above the compensators, and complete indicator diagrams are shown in Fig. 720. No. 1 shows the two steam diagrams reduced to a common scale. No. 3 is the water diagram. In No. 2 the dotted curved line shows the compensator pressures

at all parts of the stroke; the light full line gives the combined diagram due to the high and low pressure cylinders, and the dark line the combined steam and compensator pressure diagram, from which it will be seen that the pressure due to the combination is practically constant and similar to the water-pressure diagram in spite of the variation of the steam-pressure.

We must not leave this question without reference to another most ingenious arrangement for using steam expansively in a pump—the Davy differential pump. It is not, strictly speaking, a direct-acting pump, but on the other hand it is not a flywheel pump. We show the arrangement in Fig. 721.[1] The discs

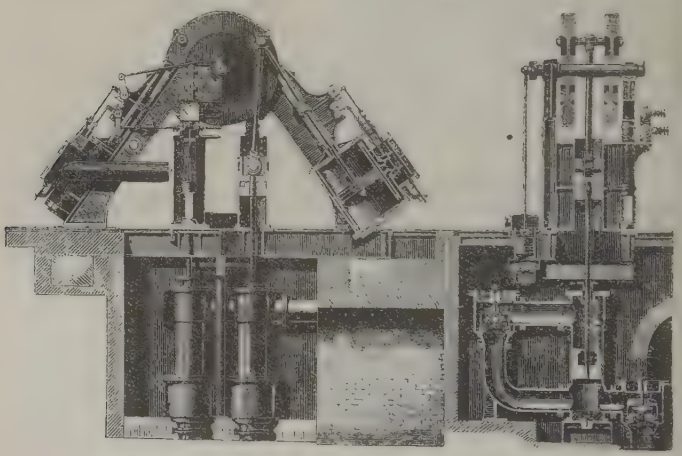


FIG. 721.

move to and fro through an angle rather less than a right angle.

In Fig. 722 we show a diagram which roughly indicates the manner in which the varying pressure in the steam-cylinder produces a tolerably uniform pressure in the water-cylinder. When the full pressure of steam is on the piston it moves slowly, and at the same time the water-piston moves rapidly; then when the steam expands, the steam-piston moves rapidly and the water-piston slowly: thus in any given period of time the work done in each cylinder is approximately the same. We

[1] Reproduced by the kind permission of the Editor of the *Engineer* and the makers, Messrs Hathorn, Davy & Co., Leeds.

have moved the crank-pin through equal spaces, and shaded alternate strips of the water and steam diagrams. The corresponding positions of the steam-piston have been found by simple construction, and the corresponding strips of the steam diagram have also been shaded; they will be found to be equal to the corresponding water-diagram strips (neglecting friction). It will be observed that by this very simple arrangement the variable steam-pressure on the piston is very nearly balanced at each portion of the stroke by the constant water-pressure in the pump barrel. The pressure in the pump barrel is indicated by the horizontal width of the strips. We

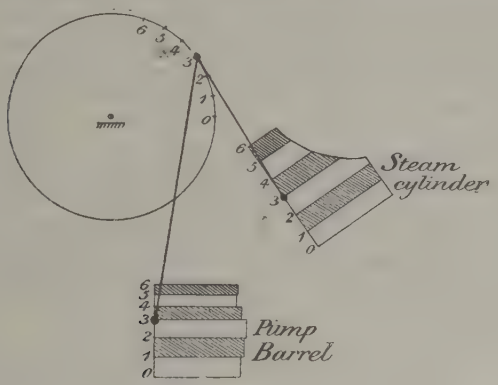


FIG. 722.

have neglected friction and the inertia of the moving masses, but our diagram will suffice to show the principle involved.

Reversed Reaction Wheel.—If a reaction wheel, such as that shown in Fig. 681, be placed with the nozzles in water, and the wheel be revolved in the opposite direction from some external source of power, the water will enter the nozzles and be forced up the vertical pipe; the arrangement would, however, be very faulty, and a very poor efficiency obtained. A far better result would be, and indeed has been, obtained by turning it upside down with the nozzles revolving in the same sense as in a motor, and with the bottom of the central chamber dipping in water. The pipes have to be primed, *i.e.* filled with water before starting; then the water is delivered from the nozzles in the same manner as it would be from a motor. The arrangement is inconvenient in many respects.

A far more convenient and equally efficient form of pump will be dealt with in the next paragraph.

Reversed Turbine or Centrifugal Pump.—If an ordinary turbine were driven backwards from an external source of power, it would act as a pump, but the efficiency would probably be very low; if, however, the guides and blades were suitably curved, and the casing was adapted to the new conditions of flow, a highly efficient pump might be produced. Such pumps, known as turbine-pumps, are already on the market, and give considerably better results than have ever been obtained from the old type of centrifugal pump, and, moreover, they will raise water to heights hitherto considered impossible for pumps other than reciprocating. In most cases turbine-pumps are made in sections, each of which used singly is only suitable for a moderate lift; but when it is required to deal with high lifts, several of such sections are bolted together in series.

In the Parsons centrifugal augmentor pump the flow is axial, and both the wheel and guide blades are similar in appearance to the well-known Parsons steam-turbine blades; but, of course, their exact angles and form are arranged to suit the pressure and flow conditions of the water. In a test of a pump of this type made by the author, the following results were obtained:—

Total lift in feet.	Millions of gallons pumped per day.	Water horse-power.	Speed in revolutions per minute.	Efficiency of pump.
281	3·53	209	2385	From 52 to 56 per cent.
329	3·88	269	2664	
383	3·17	255	2625	
425	2·54	227	2555	
458	2·14	206	2499	

See *Engineering*, vol. ii. pp. 9, 32, 86 (1903).

The form of centrifugal pump usually adopted is the

reversed outward-flow turbine, or, more accurately, a reversed mixed-outward-flow turbine, since the water usually flows parallel to the axle before entering the "eye" of the pump, and when in the "eye" it partakes of the rotary motion of the vanes which give it a small velocity of whirl, the magnitude of which is a somewhat uncertain quantity. In order to determine the most efficient form of blades a knowledge of this velocity is necessary, but unfortunately it is a very difficult quantity to measure, and practically no data exist on the question; therefore in most cases an estimate is made, and the blades are designed accordingly; but in some special cases guide-blades are inserted to direct the water in the desired path. Such a refinement is, however, never adopted in any but very large pumps, since the loss at entry into the wheel is not, as a rule, very serious. The manner in which the shape of the blades is determined will be dealt with shortly. The water on leaving the wheel possesses a considerable amount of kinetic energy in virtue of its velocity of whirl and radial velocity. In some types of pump no attempt is made to utilize this energy, and consequently their efficiency is low; but in other types great care is taken to convert it into useful or pressure energy, and thus to materially raise the hydraulic efficiency of the pump. Almost all improvements in centrifugal pumps consist of some method of converting the useless kinetic energy of the water on leaving the runner into useful pressure, or head energy.

Form of Vanes for Centrifugal Pump.—Let the figure represent a small portion of the disc and vanes, sometimes termed the "runner," of a centrifugal pump. Symbols with the suffix 1 refer to the inner edge of the vanes.

Let the water have a radial velocity of flow V_r;

" " velocity of whirl V_{w1}, in the "eye" of the pump just as it enters the vanes;

Let the water have a velocity of whirl V_w as it leaves the vanes and enters the casing.

N.B.—It should be noted that this is not the velocity with which the water circulates in the casing around the wheel.

Let the velocity of the tips of the vanes be V and V_1.

Then, setting off $ab = V_{r1}$ radially and $ad = V_{w1}$ tangentially, on completing the parallelogram we get ac representing the velocity of entry of the water; then, setting off $ae = V_1$ also tangentially, and completing the figure, we get ec representing the velocity of the water relative to the disc. Hence, in order that there shall be no shock on entry, the tangent to the first

tip of the vane must be parallel to *cc*. In a similar manner, if we decide upon the velocity of whirl V_w on leaving the vanes, we can find the direction of the tangent to the outer tip of the vanes; the angle that this makes to the tangent we term θ. In some cases we shall decide upon the angle θ beforehand, and

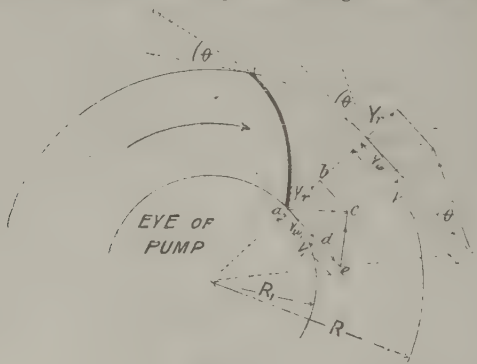


FIG. 723.

obtain the velocity of whirl by working backwards. Having found the first and last tangents, we connect them by a smooth curve. The radial velocity V_r is usually made $\frac{1}{8}\sqrt{2gH}$, about the same as in a turbine.

In order to keep this velocity constant as the water passes

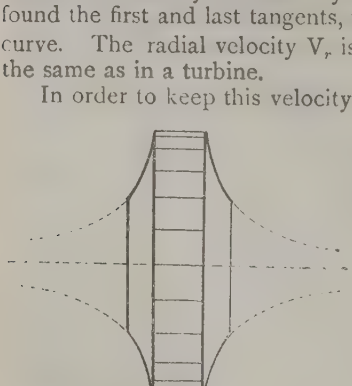


FIG. 724.

FIG. 725.

through the fan, the side discs are generally made hyperboloidal (Fig. 724). This is secured by making the product of the width of the waterway between the discs at any radius and that radius a constant.

If the width at the outer radius is one-half the inner, and the

radius of the "eye" is one-half the radius of the outer rim, the area of the waterway will be constant. The area must be regulated to give the velocity of flow mentioned above.

The net area of the waterway on the circumference of the runner in square feet, multiplied by the velocity of flow V_r in feet per second, gives the quantity of water that is passing through the pump in cubic feet per second.

If the pump does not run at the velocity for which it was designed, there will be a loss due to shock on entry; if the first tip of the vanes were made tangential to *fa* (Fig. 725), and for the altered conditions of running it should have been tangential to *ga*, the loss of head due to shock will be $\frac{(fg)^2}{2g}$, where *fa* and *ga* are proportional to the respective velocities.

Hydraulic Efficiency of Centrifugal Pumps. *General Case in which the Speed and Lift of the Pump are known.*—In all cases we shall neglect friction losses.

Let H = the total head against which the pump is lifting the water, including the suction, delivery, and head due to friction in the pipes.

N.B.—When fixing a centrifugal pump it should be placed close to the water, with as little suction head as possible, *e.g.* a pump required to lift water through a total height of, say, 20 feet will work more efficiently with 1-foot suction and 19-feet delivery head than with 5-feet suction and 15-feet delivery head; it is liable to give trouble if the length of the delivery pipe is less than the suction. All high-speed centrifugal pumps should have a very ample suction pipe and passages, and the entry resistances reduced to a minimum. If these precautions are not attended to, serious troubles due to cavitation in the runner will be liable to arise.

Let W lbs. of water pass through the pump per second, and let the runner or vanes be acted upon by a twisting moment T derived from the driving belt or other source of power. And further, let the angular velocity of the *water* be ω. Then, adhering to the symbols used in the last paragraph—

The change of momentum of the water per second in passing through the runner $= \frac{W}{g}(V_w - V_{w1})$

The change in the moment of the momentum of the water per second, or, the change of angular momentum per second $= \frac{W}{g}(V_w R - V_{w1} R_1)$

This change in the angular momentum of the water is due to the twisting moment which is acting on the runner or vanes.

Hence—

$$T = \frac{W}{g}(V_w R - V_{w1} R_1)$$

The work done per second on the water by the twisting moment T $= T\omega = \frac{W}{g}(V_w R\omega - V_{w1} R_1 \omega)$

The useful work done by the pump per second $= WH$

Hence—

The hydraulic efficiency of the pump $\eta = \dfrac{WH}{\frac{W}{g}(V_w R\omega - V_{w1} R_1 \omega)}$

In some instances guide-blades are inserted in the eye of the pump to give the water an initial velocity whirl of V_{w1}, but usually the water enters the eye radially; in that case V_{w1} is zero, and putting $R\omega = V$, we get—

$$\eta = \frac{gH}{VV_w}$$

In designing the vanes we assumed an initial velocity of whirl for the water on entering the vanes proper. This velocity is imparted to the water by the revolving arms of the runner after it has entered the eye, but before it has entered the vanes; hence the energy expended in imparting this velocity to the water is derived from the same source as that from which the runner is driven, and therefore, as far as the efficiency of the pump is concerned, the initial velocity of whirl is zero.

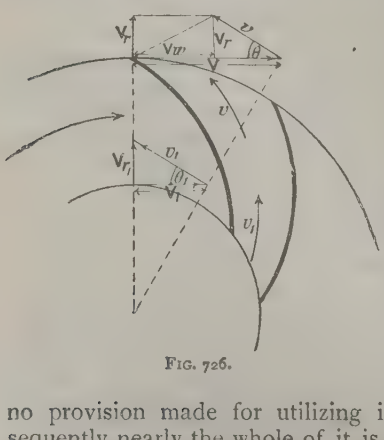


FIG. 726.

Investigation of the Efficiency of Various Types of Centrifugal Pumps. CASE I. *Pump with no Volute.*—In this case the water is discharged from the runner at a high velocity into a chamber in which there is no provision made for utilizing its energy of motion, consequently nearly the whole of it is dissipated in eddies before

it reaches the discharge pipe. A small portion of the velocity of whirl V_w may be utilized, but it is so small that we shall neglect it.

The whole of the energy due to the radial component of the velocity V_r is necessarily wasted in this form of pump.

In the figure the water flows through the vanes with a relative velocity v_1 at entry and v on leaving; then, by Bernoullis' theorem, we have the difference of pressure-head due to any change in the section of the passages—

$$\frac{p - p_1}{w} = \frac{v_1^2 - v^2}{2g}$$

and the difference of pressure-head due to centrifugal force $= \dfrac{V^2 - V_1^2}{2g}$ (see p. 733)

If the water flows radially on entering the vanes, we have—

$$v_1^2 = V_1^2 + V_{r1}^2$$

and on leaving—

$$v = V_r \operatorname{cosec} \theta$$

The increase of pressure-head due to both causes $= H = \dfrac{v_1^2 - v^2 + V^2 - V_1^2}{2g}$

By substitution and reduction, we find—

$$H = \frac{V_{r1}^2 - V_r^2 \operatorname{cosec}^2 \theta + V^2}{2g}$$

The total head that has to be maintained by the pump is due to the direct lift H, and to that required to generate the velocity of flow V_{r1} in the eye of the pump.

Whence $$H + \frac{V_{r1}^2}{2g} = \frac{V_{r1}^2 - V_r^2 \operatorname{cosec}^2 \theta + V^2}{2g}$$

and $$H = \frac{V^2 - V_r^2 \operatorname{cosec}^2 \theta}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

also $$V = \sqrt{2gH + V_r^2 \operatorname{cosec}^2 \theta} = K\sqrt{2gH} \; . \quad \text{(ii.)}$$

Substituting the value of H in the expression for the efficiency, and putting $V_w = V - V_r \cot \theta$, we have—

$$\eta = \frac{V^2 - V_r^2 \operatorname{cosec}^2 \theta}{2V(V - V_r \cot \theta)} \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

which reduces to—

$$\eta = \frac{1}{2K\left(K - \frac{\cot\theta}{4}\right)}$$

when $V_r = \frac{1}{4}\sqrt{2gH}$.

When $\theta = 90°$, *i.e.* with radial blades—

$$\eta = \frac{V^2 - V_r^2}{2V^2} \quad \text{(iv.)}$$

and when θ is very small—

$$V_w = V - v \text{ (nearly)}$$

$$\text{Then } \eta = \frac{V^2 - v^2}{2VV_w} = \frac{V + v}{2V}$$

$$\eta = \frac{V + V_r \operatorname{cosec}\theta}{2V} \text{ (nearly)} \quad \text{(v.)}$$

The following table shows how the efficiency and the peripheral velocity of the runner depend upon the vane angle for the case of a pump *having no volute*, and no means of utilizing the kinetic energy of the water on leaving the runner. In arriving at the efficiency η, the skin friction on the disc has been neglected, and, since it varies as the cube of the speed, it will be readily seen that the actual efficiency does not continue to increase as the angle is reduced, because the speed also increases. According to Professor Unwin, *Proc. I.C.E.*, vol. lii., the best results are obtained when θ lies between 30° and 40°, or when V = about $1{\cdot}1\sqrt{2gH}$.

θ	$V_r = \frac{1}{4}\sqrt{2gH}$	
	η	V
90°	0·47	$1{\cdot}03\sqrt{2gH}$
45°	0·58	$1{\cdot}06\sqrt{2gH}$
20°	0·73	$1{\cdot}24\sqrt{2gH}$

NOTE.—*If the reader will take various values of* V_r *in terms of* $\sqrt{2gH}$, *and plot a curve showing how the efficiency varies, he will find that the best results are obtained when the coefficient is between one-fourth and one-third. It falls off slightly if these values be greatly departed from.*

It appears from the table given above that the efficiency of a pump having radial blades is very low, but it must not be forgotten that this low efficiency is due to the fact that about one half the energy of the water leaving a radial-bladed runner is dissipated in eddies in the pump casing and at discharge; but if suitable means, such as gradually enlarging guide passages, a whirlpool chamber, or a bell mouth, be adopted, a radial-bladed pump runner may be made to give excellent results (see a paper by Mons. Gerard Lavergne, *Le Genie Civil*, April 21, 1894).

Many large pumps are fitted with guide blades as shown in Fig. 727, the guides are arranged parallel to $V_{/0}$, Fig.

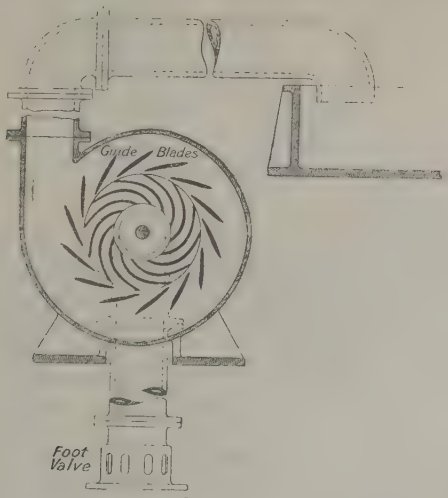


FIG. 727.

729. Readers should refer to a paper by Gibson on "The Design of Volute Chambers and of Guide Passages for Centrifugal Pumps." *Proceedings Institution Mechanical Engineers*, March, 1913.

In very high-speed centrifugal pumps, driven electrically or by steam turbines, runners of very small diameter are used, and the angle θ is often as small as 15°. The friction is reduced as much as possible by careful polishing of the discs, and the efficiency is said to be very high, even when a single pump is employed for lifting water to a height of 300 feet. For

greater lifts two or more pumps are placed in series, each of which raises the pressure through a like amount.

Case II. *Pump with a Volute* (Fig. 728).—The mean velocity of the water over any section of the volute is constant, hence the area of the volute at any section must be proportional to the quantity of water passing that section in any given time. To secure a uniform velocity of flow, the form of the volute is arranged as shown in Fig. 728.

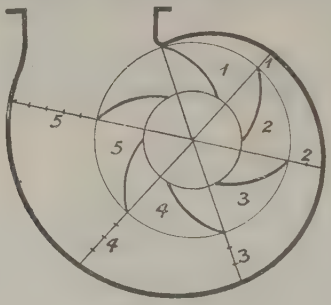


Fig. 728.

The volute is shown in radial sections. The first section has to carry off the water from section 1 of the runner, the second section from sections 1 and 2 of the runner; therefore at the radius 2 it is made twice as wide as at 1; likewise at 3 it is three times as wide as at 1, and so on. As these radii enclose equal angles, it will be seen that the form of the casing is an Archimedian spiral if the breadth be made constant.

Let V_v be the velocity of whirl in the volute, or the velocity with which the water circulates round the spiral casing; it is generally less than V_w.

The water leaves the vanes with an absolute velocity of flow V_f, which is changed to V_v. If the change be abrupt, the *loss* of head due to shock is—

$$\frac{V_{fo}^2}{2g} = \frac{V_f^2 + V_v^2 - 2V_w V_v}{2g}$$

But if the change be gradual, the *gain* of head due to the decreased velocity is—

$$\frac{V_f^2 - V_v^2}{2g}$$

And the net *gain* of head due to the change is—

$$H_g = \frac{V_f^2 - V_v^2 - V_{fo}^2}{2g} = \frac{2(V_v V_w - V_v^2)}{2g}$$

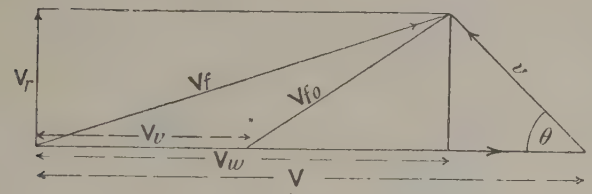


FIG. 729.

Differentiating, we get—

$$\frac{dH_g}{dV_v} = \frac{V_w - 2V_v}{g}$$

Hence the maximum possible gain of head due to this cause is when—

$$V_v = \frac{V_w}{2}$$

whence—

$$\text{The maximum gain of head} = \frac{2V_v^2}{2g} \quad . \quad . \quad . \quad \text{(vi.)}$$

Adding this to the value found for H (equation i.), we have—

$$H = \frac{V^2}{2g} - \frac{V_r^2 \operatorname{cosec}^2 \theta}{2g} + \frac{2V_v^2}{2g} \quad . \quad . \quad . \quad \text{(vii.)}$$

$$\text{and } V = \sqrt{2gH + V_r^2 \operatorname{cosec}^2 \theta - 2V_v^2} \quad . \quad . \quad \text{(viii.)}$$

and the efficiency (with a volute casing)—

$$\eta = \frac{V^2 - V_r^2 \operatorname{cosec}^2 \theta + 2V_v^2}{2V(V - V_r \cot \theta)} \quad . \quad . \quad . \quad \text{(ix.)}$$

the value of $V_v = \frac{V - V_r \cot \theta}{2}$

The following table of efficiencies and velocity of the vanes should be compared with the similar figures given for the pump with no volute :—

θ	$V_r = \frac{1}{2}\sqrt{2gH}$	
	η	V
90°	0·60	$0{\cdot}84\sqrt{2gH}$
45°	0·78	$0{\cdot}94\sqrt{2gH}$
20°	0·82	$1{\cdot}00\sqrt{2gH}$

When the pump is running at a speed below that necessary for delivery, the water stands in the delivery pipe at a height corresponding to the speed. The quantities V_r and V_v are then zero, and $V = \sqrt{2gH}$. The curve given in Fig. 730 shows how nearly experiment and theory agree, but it should be noticed that the speed rises above $\sqrt{2gH}$ at the high lifts.

Case III. *Pump fitted with a Bell Mouth on the Discharge Pipe.*—The area of the discharge pipe is usually equal to that of the volute at its largest section, hence the velocity of flow in the discharge pipe is equal to that in the spiral casing. Consequently, the kinetic energy of the water rejected at discharge is $\frac{V_v^2}{2g}$. If a suitable bell mouth be fitted to the end of the discharge pipe, this velocity may be materially reduced. Let the area of the bell mouth be n times that of the pipe. Then, provided there is no breaking up of the stream (see p. 653), the velocity of discharge will be $\frac{V_v}{n}$, and the gain of head due to the bell mouth is $\frac{V_v^2}{2g}\left(1 - \frac{1}{n^2}\right)$. Even under the most favourable circumstances this is not a very large quantity, since V_v seldom exceeds 15 feet a second, and is more generally about one-half that amount. Hence the gain of head due to a bell mouth cannot well be greater than 3 feet, and is therefore of no great importance, except in the case of very low lifts. The gain in efficiency is readily obtained from the expressions

given above, by adding to the expression for H (Equations i. and vii.) the gain due to the bell-mouth.

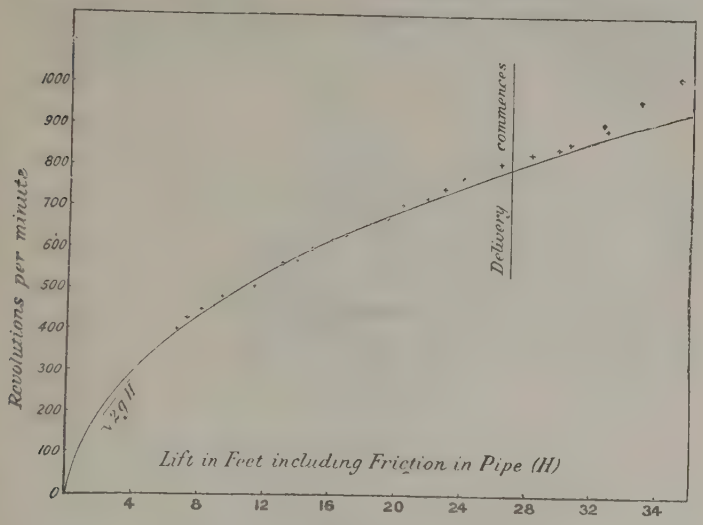


FIG. 730.

CASE IV. *Pump fitted with a Whirlpool Chamber.*—If a whirlpool chamber, such as that shown in Fig. 731, be fitted to a pump, a free vortex will be formed in the chamber, and a corresponding gain in head will be effected, amounting to—

$$\frac{V_f^2}{2g}\left(1 - \frac{r_i^2}{r_e^2}\right)$$

for a pump with no volute,

$$\text{or } \frac{V_{f_0}^2}{2g}\left(1 - \frac{r_i^2}{r_e^2}\right)$$

for a pump with a volute (see p. 767).

This gain of head for well-designed pumps is usually small, and rarely amounts to more than 5 per cent. of the total head, and is often as low as $1\frac{1}{2}$ per cent. The cost of making a pump with a whirlpool chamber is considerably greater than that of providing a bell mouth for the discharge pipe, hence many makers have discarded the whirlpool chamber in favour of the bell-mouth. There is, of course, no need to use both a bell-mouth and a whirlpool chamber. Some very interesting matter bearing on this question will be found in a paper by

Dr. Stanton in the *Proceedings of the Institution of Mechanical Engineers*, October, 1903.

Skin Friction of the Rotating Discs in a Centrifugal Pump.—The skin friction of the rotating discs is a very important factor in the actual efficiency of centrifugal pumps.

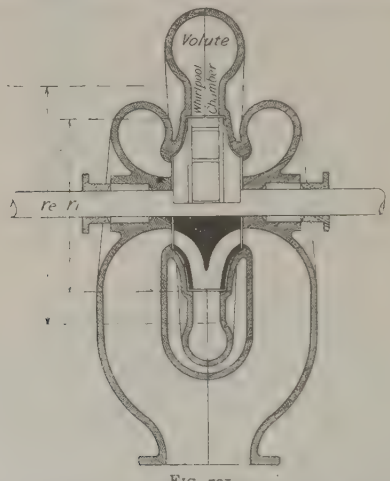


FIG. 731.

Let f = the coefficient of friction (see p. 681).

Consider a ring of radius r feet and thickness dr rotating with a linear velocity u feet per second, relatively to the water in the casing; then—

$$u = \frac{Vr}{R}$$

The skin resistance of the ring $= fu^2 2\pi r dr$

$$= f\frac{V^2}{R^2} 2\pi r^3 dr$$

The moment of the frictional resistance of the ring $= \frac{fV^2}{R^2} 2\pi r^4 dr$

The moment of the frictional resistance of the whole disc $= \frac{fV^2 2\pi}{R^2} \int_{r=R_1}^{r=R} r^4 dr$

$$= \frac{2\pi f V^2}{5}\left(\frac{R^5 - R_1^5}{R^2}\right)$$

and for the two discs—

The moment of the frictional resistance $= \frac{4\pi f V^2}{5}\left(\frac{R^5 - R_1^5}{R^2}\right)$

The horse-power wasted in overcoming the skin friction $= \frac{fV^3}{216}\left(\frac{R^5 - R_1^5}{R^3}\right)$

Taking the usual proportions for the runner, viz. $R_1 = \frac{R}{2}$, on substitution we get—

$$\text{The horse-power wasted in skin friction} = \frac{fV^3R^2}{223}$$

The practical outcome of an investigation of the disc friction is important; it shows that the work wasted in friction varies in the first place as the cube of the peripheral velocity of the runner. The head, however, against which the pump will raise water varies as the square of the velocity, hence the wasted power varies as $(\text{head})^{\frac{3}{2}}$. Thus, as the head increases, the work done in overcoming the skin friction also increases at a more rapid rate, and thereby imposes a limit on the head against which a centrifugal pump can be economically employed. Modern turbine-driven pumps are, however, made to work quite efficiently up to heads of 1000 feet. Further, the work wasted in disc skin friction varies as the square of the radius—other things being equal; hence a runner of small diameter running at a very great number of revolutions per minute, wastes far less in skin friction than does a large disc running at a smaller number of revolutions with the same peripheral velocity. An excellent example of this is found in the centrifugal pumps driven by the De Laval steam turbines. The author is indebted to Messrs. Greenwood and Batley, Leeds, for the following results:—

	Single stage.						Two-stage.		
Diameter of wheel ...	1·15 ft.			0·69			0·75 low, 0·24 high		
Width of waterway at circumference	0·15 ft.			0·17			—		
Diameter of pipes	0·67 ft.			0·67			0·50 low, 0·33 high		
Quantity in cub. ft. sec. ...	0	2·23	3·92	0	2·18	2·52	0·83	0·75	0·54
Lift in feet	142	126	100	68	53	46	136	420	781
Revolutions per minute ...	1565	1540	1547	2040	1997	2000	2104 (low) 20,500 high	2027	2029
Efficiency per cent. ...	—	67·6	75·6	—	72·0	73·0	not known		

In addition to the disc friction there is also a small amount of friction between the water and the interior of the runners, due to the outward flow of the water through the wheel passages. This, however, is negligible compared to the disc friction.

Capacity of Centrifugal Pumps.—When the velocity of flow V_r through the runner of a centrifugal pump is known, the quantity of water passing per second can be readily obtained by taking the product of this velocity and the circumferential area of the passages through the runner; or, if the velocity of whirl in the volute V_v be known, the quantity passing can be found from the product of this velocity and the sectional area of the volute close to the discharge pipe, which is usually made equal to the area of the discharge pipe itself. Whence from a knowledge of the dimensions of a pump both V_r and V_v can be readily obtained for any given discharge. The most important quantity to obtain is, however, the speed of the pump V for any given discharge. Expressions have already been given for V in terms of the head H, the angle θ, the velocities V_r and V_v; but on comparing the value of V calculated from these expressions with actual values, a discrepancy will be discovered which is largely due to the friction of the water in the suction and discharge pipes, and in the pump itself. The latter quantity can be most readily expressed in terms of the length of pipe, which produces an equal amount of friction. A few cases that the author has examined appear to show that the internal resistance of the pump and foot valve together is equivalent to the friction on a length of about 80 diameters of the discharge pipe. This head must be added to the actual lift of the pump in order to find the real head H, against which the pump is lifting.

When designing centrifugal pumps it is usual to make the velocity of flow V_r about $\frac{1}{4}\sqrt{2gH}$, and the diameter of the runner about twice the diameter of the "eye," the latter being usually equal to the diameter of the discharge pipe.

A concrete example of a pump in which the actual performances are known, will serve to show the extent to which our theoretical deductions may be trusted.

Diameter of runner	12 ins.
,, delivery pipe	6 ,,
Circumferential area of wheel passages	0·36 sq. ft.
Blade angle (θ)	45°
Lift	29 ft.
Length of delivery pipes, including four bends ...	42 ,,
,, suction pipe (straight)	8 ,,

The pump is fitted with a volute, but no whirlpool chamber or bell-mouth.

Quantity of water in cubic feet per second	V_r feet per	V_v second.	V in feet per second. Experiment.	Calculation.	Calculation allowing for friction in pipes and pump.	Efficiency. Total.	Calculated $\frac{gH}{VV_w}$
0·4	1·1	2·04	44·4	43·0	43·7	0·29	0·52
0·8	2·2	4·08	46·6	42·8	45·5	0·43	0·53
1·2	3·3	6·12	49·0	42·5	49·3	0·52	0·56
1·6	4·4	8·16	51·1	42·0	51·8	0·56	0·59
2·0	5·5	10·20	53·5	41·4	55·5	0·59	0·62

The curves in Fig. 732 show in more detail the actual results obtained from this pump.

Cavitation in Centrifugal Pumps.—In high speed centrifugal pumps cavitation, *i.e.* breaking up of the water in the runner, often gives rise to very serious troubles, not the least being extremely rapid corrosion and pitting of the runner.

The quantity of water Q delivered by a centrifugal pump can be very closely represented by an expression of the form

$$Q = \frac{N}{a} - b$$

where N is the number of revolutions per minute made by the runner, a and b are constants; also

$$N = a(Q + b)$$

when the pump just begins to deliver Q is very small; then

$$N = ab = \frac{60\sqrt{2gH}}{2\pi R} \text{ nearly}$$

The quantity entering the pump $\Big\} = Q = A\sqrt{2g\left(H_b - H_v - h\right)}$

Where A is the area in square feet of the section pipe, H_b the height of the water barometer in feet, *i.e.* 34 feet for cold water (and tight joints). H, the suction head in feet below the centre of the pump, h the head in feet corresponding to the friction of the suction pipe and foot valve. V_s the velocity in feet per second of the water in the suction pipe. The friction head can be expressed in terms of the quantity thus—

$$h = \frac{LV_s^2}{KD} = \frac{L\left(\frac{Q}{A}\right)^2}{KD}$$

$$\text{hence } Q = A\sqrt{2g\left(H_b - H_s - \frac{LQ^2}{KDA^2}\right)} = \frac{N}{a} - b$$

which reduces to

$$N = a\sqrt{\frac{2gA^2(H_b - H_s)KD}{KD + 2gL}} + ab$$

N is the speed at which cavitation occurs. In order to be on the safe side, H_b should not be taken over 30 feet for cold water, and even then all joints must be in good condition. In calculating the friction of the foot valve, one with ample passages may be taken as equivalent to 15 diameters of pipe, but in badly designed valves the loss may be much greater.

Multiple-lift Centrifugal Pumps.—In the case of a centrifugal pump which delivers against a very high lift, the peripheral speed of the runner becomes very great, and the skin friction of the discs assumes somewhat serious proportions. The very high speed of the runner is not a serious drawback for pumps driven direct from steam turbines or electric motors, and, in fact, such speeds are often found to be very convenient; the angle θ, however, becomes inconveniently small, and the skin friction is often so great as to materially reduce the efficiency. In order to avoid these drawbacks and yet obtain high lifts, two or more pumps may be arranged in series. The first pump delivers its water into the suction pipe of the second pump, which again delivers it to the third pump, and so on, according to the lift required; by this means the peripheral speed and the disc friction can be kept within moderate limits. One of the earliest and most successful pumps of this type is the Mather-Reynolds pump, made by Messrs. Mather and Platt, of Manchester, and by whose courtesy the section shown in Fig. 613*a* is reproduced.

The direction in which the water flows through the pump is indicated by arrows. It enters the runner axially from both sides, in order to eliminate axial thrust, and after traversing the runner the water is discharged into a chamber fitted with

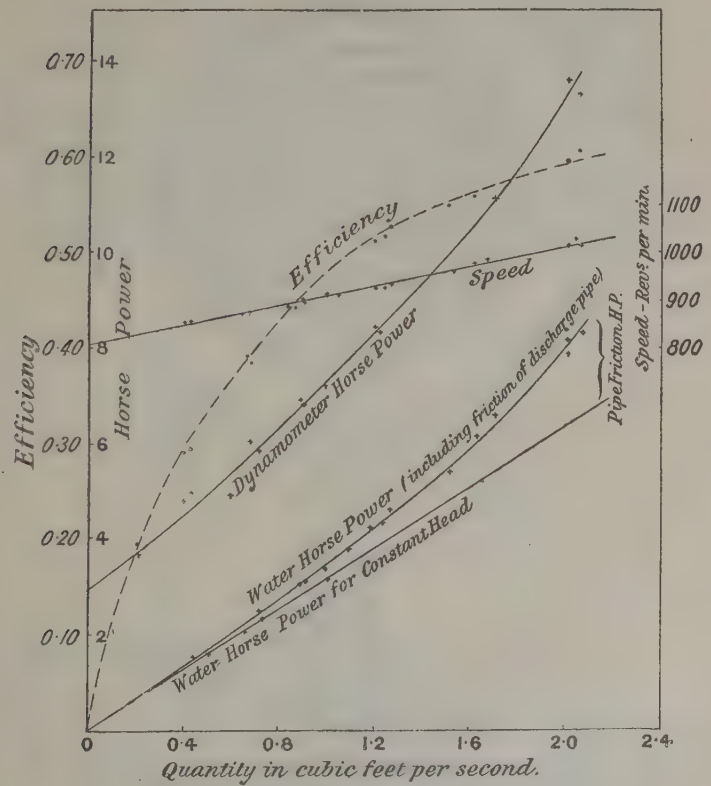


FIG. 732.

expanding guide-passages for the purpose of converting the kinetic energy of the water leaving the runner into pressure energy. This chamber is the special feature of the pump, and judging from the excellent results obtained, it much more

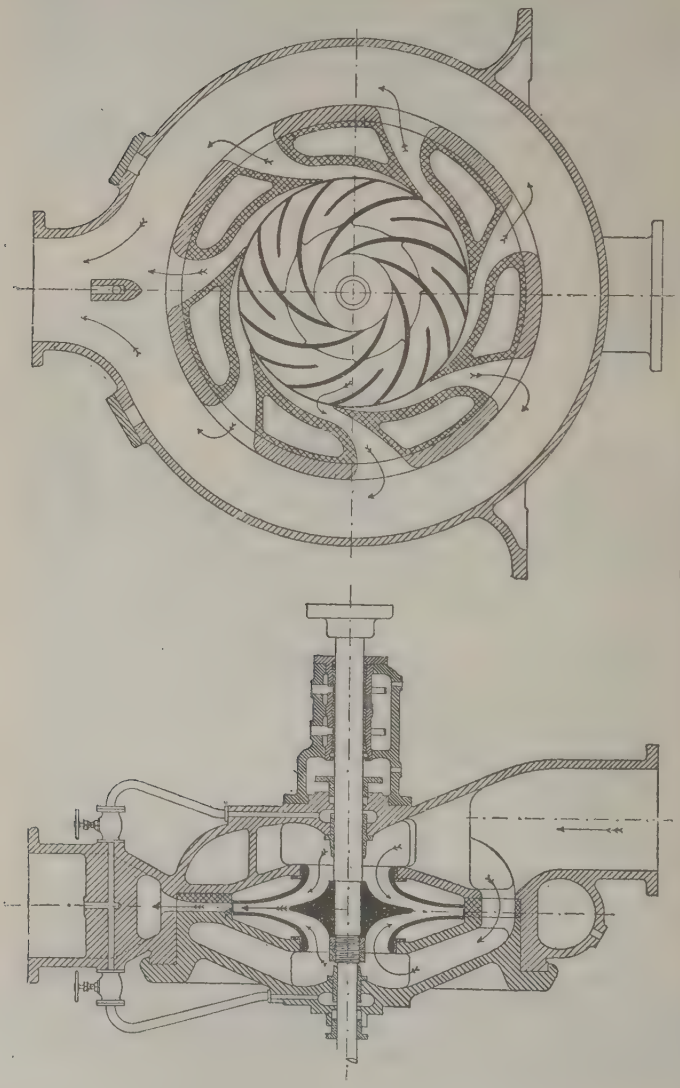


FIG. 733.

effectually produces the desired result than the ordinary whirlpool chamber mentioned in a previous paragraph. The reader should also notice the great care that has been bestowed on the design of the pump, in order to avoid sudden changes in the direction of flow, and in the cross-section of the passages.

In Fig. 734 is shown a multiple chamber pump of the same design. The water in this instance enters on one side only of the runner; then, after passing through expanding guide-passages, it is brought round to the other side of the runner, and after passing more guide-passages, it goes on to the next runner, and so on.

The following results of a test of one of these pumps will serve to show what excellent results are obtained; the figures are taken from curves published in the maker's catalogue:—

Diameter of suction pipe = 12 ins.
" delivery pipe = 10 "
Lift ... ... 100 ft.

Quantity of water in gallons per minute.	Revolutions per minute.	Efficiency per cent.
200	690	18
400	690	35
600	690	48
800	690	60
1000	700	67
1200	710	71
1400	720	72
1600	740	70
1800	770	65
2000	800	54

The following results of a test by the author on a multiple Parsons centrifugal pump, driven direct from one of their steam turbines, may be of interest:—

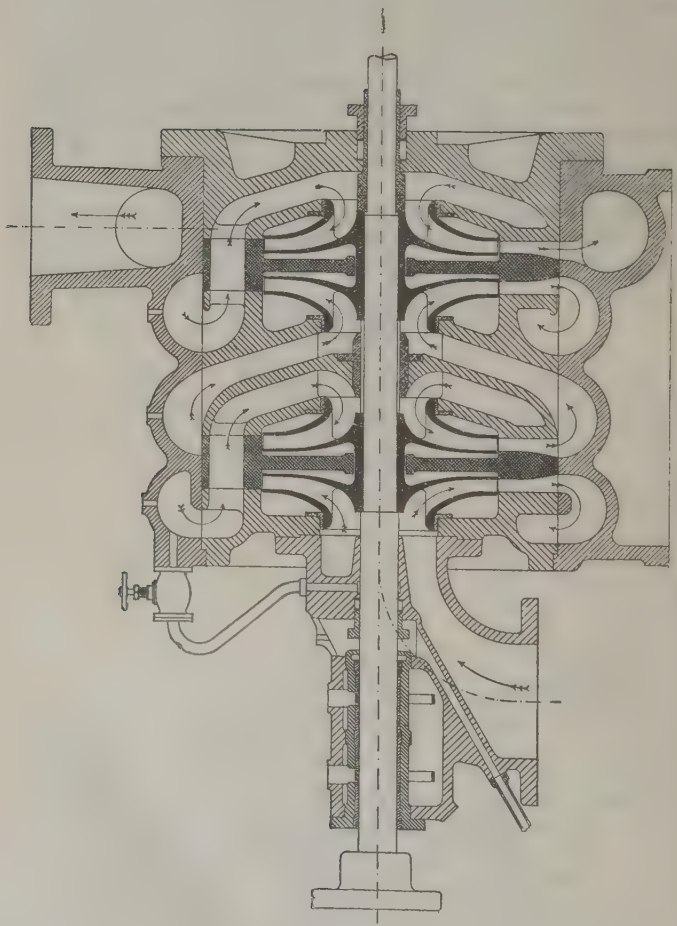


FIG. 734.

Destination of pump ... ... Sydney Waterworks
Suction-head in all tests ... ... 11 ft.

Lift in feet.			Millions of gallons pumped per day.	Water horse-power.	Speed in revolutions per minute.	Efficiency.
Pump A.	Pump B.	Pump C.				
223	475	751	1·573	252	3300	About 56 per cent.
235	479	733	1·499	235	3300	
237	484	745	1·503	239	3340	
300	606	906	1·689	326	3710	

In a short overload trial the pump delivered about two million gallons of water per day at a head of 1000 feet (see *Engineering*, vol. ii. p. 86 (1903).

The centrifugal pump in the near future will probably play a very much more important part amongst water-raising appliances than it has done in the past. Until quite recently it was universally considered that it was only applicable to the lifting of large quantities of water through low lifts, but now such pumps are employed for lifts up to 1000 feet, with an efficiency as high as that of many reciprocating pumps. The wear and tear is very small, and there are no sudden variations in the flow which produce such troublesome effects in the mains of plunger pumps.

APPENDIX

THE reader should get into the habit of checking the units in any expression he may arrive at, for the sake of preventing errors and for getting a better idea of the quantity he is dealing with.

A mere number or a constant may be struck out of an expression at once, as it in no way affects the units: thus, the length of the circumference of a circle is $2\pi r$ (p. 22); or—

The length = 2π (*a constant*) $\times$ r (*expressed in length units*)
= a certain number of length units

Similarly, the area of a circle = πr^2 (p. 28); or—

The area = π (*a constant*) $\times$ r (*in length units*) $\times$ r (*in length units*)
= a certain number of (length units)^2

Similarly, the volume of a sphere = $\frac{4}{3}\pi r^3$; or—

The volume = $\frac{4\pi}{3}$ (*a constant*) $\times$ r (*in length units*) $\times$ r (*in length units*) $\times$ r (*in length units*)
= a certain number of (length units)^3

The same relation holds in a more complex case, *e.g.* the slice of a sphere (p. 44).

The volume = $\frac{\pi}{3}[3R(Y_2^2 - Y_1^2) - Y_2^3 + Y_1^3]$

$= \frac{\pi}{3}$(*a constant*) {3 (*a constant*) $\times$ R (*in length units*) $[Y_2^2$ (in length units)$^2 - Y_1^2$ (*in length units*)$^2] - Y_2^3$ (*length units*)$^3 + Y_1^3$ (*in length units*)^3}
$=$ *a constant* {*a constant* [(*length units*)^3 $-$ (*length units*)^3 $-$ (*length units*)^3 $+$ (*length units*)^3]
$=$ *a constant* $\times$ (*length units*)^3
= a certain number of (length units)^3}

One more example may serve to make this question of units quite clear.

The weight of a flywheel rim is given by the following expression (p. 196) :—

$$\text{Weight of rim} = \frac{g \cdot E_n \cdot R^2}{KV^2R_w^2}$$

$$\text{The weight} = \frac{g\,(\textit{an acceleration}) \times E_n\,(\textit{force} \times \textit{space}) \times R^2\,(\textit{in length units})^2}{K\,(\textit{a constant}) \times V^2\,(\textit{velocity units})^2 \times R_w^2\,(\textit{in length units})^2}$$

$$= \frac{g\,(\textit{acceleration}) \times E_n\left(\textit{mass} \times \frac{\textit{space}}{\textit{time}^2} \times \textit{space}\right)}{V^2\left(\frac{\textit{space}}{\textit{time}^2}\right)^2}$$

[N.B.—The (length units)^2 cancel out, and the constant is omitted, as it does not affect the units.]

$$\text{The weight} = \frac{\text{acceleration} \times \text{mass} \times \text{space}^2 \times \text{time}^2}{\text{time}^2 \times \text{space}^2}$$
$$= \text{mass} \times \text{acceleration of gravity (see p. 9)}$$

Thus showing that the *form* or the *dimensions* of our flywheel equation is correct.

Operations of Differentiating and Integrating.—Not one engineer in a thousand ever requires to make use of any mathematics beyond an elementary knowledge of the calculus, but any one who wishes to get beyond the Kindergarten stage of the principles that underlie engineering work must have such an elementary knowledge. The labour spent in acquiring sufficient knowledge of the calculus to enable him to solve practically all the problems he is likely to come across, can be acquired in a fraction of the time that he would have to spend in dodging it by some roundabout process. There is no excuse for ignorance of the subject now that we have such excellent and simple books on the calculus for engineers as Perry's, Barker's, Smith's, Miller's, and others. For the benefit of those who know nothing whatever of the subject, the following treatment may be of some assistance; but it must be distinctly understood that it is only given here for the sake of helping beginners to get some idea of what such processes as differentiation and integration mean, not that they may stop when they can mechanically perform them, but rather to encourage them to read up the subject.

Suppose we have a square the length of whose sides is x units, then the area of the square is $a = x^2$. Now let the side be increased by a small amount Δx, and in consequence let the area be increased by a corresponding amount Δa; then we have—

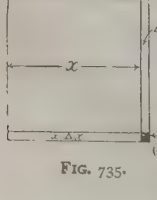


FIG. 735.

$$a + \Delta a = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$

Subtracting our original value of a, we have—

$$\Delta a = 2x\Delta x + (\Delta x)^2$$

Thus by increasing the side of the square by an amount Δx, we

have increased the area by the two narrow strips of length x and of width Δx, and by the small black square at the corner $(\Delta x)^2$.

For many problems we want to know how the area $\left\{\begin{matrix}\text{varies, } i.e. \\ \text{increases or} \\ \text{decreases}\end{matrix}\right\}$ with any given $\left\{\begin{matrix}\text{variation, } i.e. \\ \text{increase or} \\ \text{decrease}\end{matrix}\right\}$ in the length of the side; or, expressed in symbols, we want to know the value of $\frac{\Delta a}{\Delta x}$, which we can obtain from the expression above thus—

$$\frac{\Delta a}{\Delta x} = 2x + \Delta x$$

Now, if we make Δx smaller and smaller, the fraction $\frac{\Delta a}{\Delta x}$ will become more and more nearly equal to $2x$, and ultimately (which we shall term the limit) it will be $2x$ exactly. We then substitute da and dx for Δa and Δx, and write it thus—

$$\frac{da}{dx} = 2x$$

or, expressed in words, the area of the square varies $2x$ times as fast as the length of the side. This is actually and absolutely true, not a mere approximation, as we shall show later on by some examples.

The fraction $\frac{da}{dx}$ is termed the *differential coefficient* of a with regard to x. Then, having given $a = x^2$, we are said to differentiate a with regard to x when we write $\frac{da}{dx} =$

We will now go one step further, and consider a cube of side x; its volume $V = x^3$.

As before, let its side be increased by a very small amount Δx, and in consequence let the volume be increased by a corresponding amount ΔV. Then we get—

$$V + \Delta V = (x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

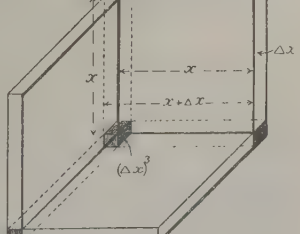


FIG. 736.

Subtracting our original value of V, we have—

$$\Delta V = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

The increase is shown in the figure, viz. three flat portions of

area x^2 and thickness Δx, three long prisms of length x and section $(\Delta x)^2$, and one small cube of volume $(\Delta x)^3$. From the above, we have—

$$\frac{\Delta V}{\Delta x} = 3x^2 + 3x\Delta x + \Delta x^2$$

which in the limit (*i.e.* when Δx becomes infinitely small) becomes—

$$\frac{dV}{dx} = 3x^2$$

By a similar process, we can show that if $s = t^4$—

$$\frac{ds}{dt} = 4t^3$$

Likewise, if we expand by the binomial theorem, or if we work out a lot of results and tabulate them, we shall find that the following relation holds:—

$$\text{when } x = y^n$$
$$\frac{dx}{dy} = ny^{n-1}$$

Suppose we had such a relation as—

$$x = Ay^2 + C$$

where A and C are constant quantities, and which of course do not vary; then, if x increases by an amount Δx, and in consequence y increases by a corresponding amount Δy, we have—

$$x + \Delta x = A(y + \Delta y)^2 + C$$
$$= A[y^2 + 2y\Delta y + (\Delta y)^2] + C$$

Subtracting the original value of x, we have—

$$\Delta x = 2Ay\Delta y + A(\Delta y)^2$$
$$\text{and } \frac{\Delta x}{\Delta y} = 2Ay + A\Delta y$$

which becomes in the limit—

$$\frac{dx}{dy} = 2Ay$$

It should be noticed that the constant C has disappeared, while the constant A is multiplied by the differential of y^2. Similarly for the constants in the following case.

Let $x = Ay^n \pm By^n \pm Cy \pm D$, where the capital letters are

constants. Then by a similar process to the one just given, we get—

$$\frac{dx}{dy} = mAy^{m-1} \pm nBy^{n-1} \pm C$$

It sometimes happens that x increases to a certain value, its maximum, and then decreases to a certain value, its minimum, and possibly increases again, and so on. If at any one instant it is found to be increasing, and the next to be decreasing, it is certain that there must be a point between the two when it neither increases nor decreases; this may occur at either the maximum or the minimum. At that instant we know that $\frac{dx}{dy} = 0$. Then, in order to find when a given quantity has a maximum or a minimum value, we have to find the value of $\frac{dx}{dy}$ and equate it to zero.

Thus, suppose we want to divide a given number N into two parts such that the product is a maximum. Let x be one part, and $N - x$ the other, and y be the product. Then—

$$y = (N - x)x = Nx - x^2$$
$$\text{and } \frac{dy}{dx} = N - 2x = 0 \text{ when } y \text{ is a maximum}$$
$$\text{or } 2x = N$$
$$\text{and } x = \frac{N}{2}$$

As an example, let N = 10:

N − x	x	y
9	1	9
8	2	16
7	3	21
6	4	24
5	5	25
4	6	24

and so on.

Thus we see that y has its maximum value when $x = \frac{N}{2} = 5$.

In some cases we may be in doubt as to whether the value we arrive at is a maximum or a minimum; in such cases the beginner had better assume one or two numerical values near the maximum or minimum, and see whether they increase or decrease, or what is very often a convenient method—to plot a diagram. This question is very clearly treated in either of the books mentioned above.

The process of integration follows quite readily from that of differentiation.

Let the line *ae* be formed by placing end to end a number of

short lines *ab* all of the same length. When the line gets to *c*, let its length be termed L_c, and when it gets to *e*, L_e. The length *ce* we have already said is equal to *ab*. Now, what is *ce*? It is simply the difference between the length of the line *ae* and the line *ac*, or $L_e - L_c$; then, instead of writing in full that *ce* is the difference in length of the two lines, we will write it Δl, *i.e.* the difference in the length after adding or subtracting one of the short lengths. Now, however long or however short the line may be, the difference in the length after adding or subtracting one of the lengths will be Δl. The whole length of the line L_e is the sum of all the short lengths of which it is composed; this we usually write briefly thus:

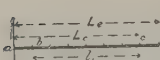


FIG. 737.

$$\Sigma_0^{L_e} \Delta l = \Big[l\Big]_0^{L_e} = L_e$$

or, the sum of (Σ) all the short lengths Δl between the limits when the line is of length L_e and of zero length is equal to L_e. Sometimes the sign of summation is written—

$$\sum_{l=0}^{l=L_e} \Delta l = L_e$$

Similarly, the length L_1 is the sum of all the short lengths between *b* and *c*, and may be written—

$$\Sigma_{L_b}^{L_c} \Delta l = \Big[l\Big]_{L_b}^{L_c} = L_c - L_b = L_1$$

the upper limit L_c being termed the superior, and the lower limit L_b the inferior limit: the superior limit is always the larger quantity. The lower limit of *l* is subtracted from the upper limit.

In the case of a line, the above statements are perfectly true, however large or however small the short lengths Δl are taken, but in some cases which we shall shortly consider it will be seen that if Δl be taken large, an error will be introduced, and that the error becomes smaller as Δl becomes smaller, and it disappears when Δl becomes infinitely small; then we substitute dl for Δl, but it still means the difference in length between the two lines L_e and L_c, although *ce* has become infinitely small. We shall now use a slightly different sign of summation, or, as we shall term it, integration. For the Greek letter Σ (sigma) we shall use the old English *s*, viz. $\int$, and—

$$\Sigma_0^{L_e} \Delta l \text{ becomes } \int_0^{L_e} dl = L_e$$

$$\text{and } \Sigma_{L^b}^{L_c} \Delta l \text{ becomes } \int_{L_b}^{L_c} dl = L_c - L_b = L_1$$

The expression still means that the sum of all the infinitely short lengths dl between the limits of length L_a and L_b is L_1, which is, of course, perfectly evident from the diagram.

Now that we have explained the meaning of the symbols, we will show a general connection between the processes of differentiation and integration. We have shown that when—

$$x = y^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

$$\frac{dx}{dy} = ny^{n-1}$$

$$\text{and } dx = ny^{n-1}dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

But the sum of all such quantities dx, viz. $\int dx$, $= x = y^n$ (iii.); hence, substituting the value of dx from (ii.), we have—

$$\int dx = \int ny^{n-1}dy$$

But from (iii.) we have—

$$\int dx = y^n$$

$$\text{hence } \int ny^{n-1}dy = y^n$$

This operation of integrating a function of a variable y may be expressed in words thus:

Add 1 *to the index of the power of* y, *and divide by the index so increased, and by the differential of the variable.*

Thus—

$$\int ny^{n-1}dy = \frac{ny^{n-1+1}dy}{(n-1+1)dy} = y^n$$

Similarly—

$$\int my^n dy = \frac{my^{n+1}}{n+1}$$

Let $m = 4$, $n = 2$. Then—

$$\int 4y^2 dy = \frac{4y^3}{3}$$

or when $m = 3{\cdot}2$, $n = \frac{1}{2}$—

$$\int 3{\cdot}2y^{\frac{1}{2}}dy = \frac{3{\cdot}2y^{1+\frac{1}{2}}}{1+\frac{1}{2}} = \frac{3{\cdot}2y^{1{\cdot}5}}{1{\cdot}5}$$

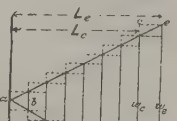


FIG. 738.

In order to illustrate this method of finding the sum of a large number of small quantities we will consider one or two simple cases.

Let the triangle be formed of a number of strips all of equal length, ab or Δl. The width w of each strip varies; not only is each one of different width from the next, but its own width is not

constant. If we take the greater width of, say, the strip *cefi*, viz. w_e, the area $w_e \Delta l$ is too great, and if we take the smaller width w_c, the area $w_c \Delta l$ will be too small. The narrower we take the strips the nearer will $w_e = w_c$, and when the strips are infinitely narrow, $w_e = w_c = w_0$, and the area will be $w_0 dl$ exactly, and the stepped figure gradually becomes a triangle.

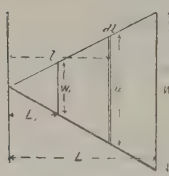


FIG. 739.

The area of the triangle is the sum of the areas of all the small strips $w \,.\, dl$ (Fig. 739).

But $\frac{w}{W} = \frac{l}{L}$, or $w = \frac{Wl}{L}$. Hence the area of the triangle is the sum of all the small strips of area $\frac{Wl}{L} dl$, which we write—

$$\int_{l=0}^{l=L} \frac{Wl}{L} dl, \text{ or } \frac{W}{L} \int_0^L l \,.\, dl = \frac{W}{L} \times \frac{L^2}{2}$$

$$= \frac{WL}{2}$$

The $\frac{W}{L}$ is placed outside the sign of integration, because neither W nor L varies. Now, *from other sources* we know that the area of the triangle is $\frac{WL}{2}$. Thus we have an independent proof of the accuracy of our reasoning.

Suppose we wanted the area of the trapezium bounded by W and W_1, distant L and L_1 from the apex. We have—

$$\frac{W}{L} \int_{L_1}^{L} l \,.\, dl = \frac{W}{L} \left(\frac{L^2 - L_1^2}{2} \right)$$

But here again we know, *from other sources*, that the area of the trapezium is—

$$\frac{WL}{2} - \frac{W_1 L_1}{2} = \frac{W}{L} \left(\frac{L^2 - L_1^2}{2} \right)$$

$$\text{since } W_1 = \frac{WL_1}{L}$$

which again corroborates our rule for integration.

By way of further illustrating the method, we will find the volume of a triangular plate of uniform thickness *t*. The volume of the plate is the sum of the volumes of the thin slices of thickness *dl*.

Volume of a thin slice (termed an elementary slice) $= w \,.\, t \,.\, dl$

$$= \frac{Wl}{L} t \,.\, dl$$

Vol. of whole plate $\frac{Wt}{L}\int l \,.\, dl = \frac{Wt}{L}\left(\frac{L^2}{2}\right) = \frac{WtL}{2}$

a result which we could have obtained by taking the product of the area $\frac{WL}{2}$ and the thickness t.

FIG. 740.

Similarly, the volume of the trapezoidal plate is—

$$\frac{Wt}{L}\int_{L_1}^{L} l \,.\, dl = \frac{Wt}{L}\left(\frac{L^2 - L_1^2}{2}\right)$$

One more case yet remains—that in which t varies, as in a pyramid.

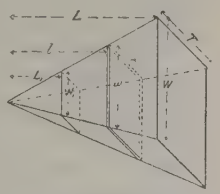


FIG. 741.

Volume of an elementary slice $= w \,.\, t \,.\, dl$

$$w = \frac{Wl}{L} \qquad t = \frac{Tl}{L}$$

hence the volume of slice $= \frac{WT}{L^2} l^2 \,.\, dl$

$$\text{Volume of pyramid} = \frac{WT}{L^2}\int l^2 \,.\, dl = \frac{WT}{L^2} \times \frac{L^3}{3}$$

$$= \frac{WTL}{3}$$

We know, however, from other sources, that the volume of a pyramid is one-third the volume of the circumscribing solid. This is easily shown by making a solid and the corresponding pyramid of the same material, and comparing the weights or the volumes of the two; or a cube can be so cut as to form three similar pyramids. In this case the volume of the circumscribing solid is WTL, and the volume of the pyramid $\frac{WTL}{3}$. Thus we have another proof of the accuracy of our method of integration. Similarly with the volume of the frustum of a pyramid.

$$\frac{WT}{L^2}\int_{L}^{L} l^2 \,.\, dl = \frac{WT}{L^2}\left(\frac{L^3 - L_1^3}{3}\right)$$

Until the reader has had an opportunity of following up the subject a little further, we must ask him to accept on faith the following results :—

$$\int \frac{1}{x} dx = \log_e x$$

$$\int_{x}^{x_1} \frac{1}{x} dx = \log_e x_1 - \log_e x_2 = \log_e \frac{x_1}{x_2}$$

and if $y = \log_e x$

$$\frac{dy}{dx} = \frac{1}{x}$$

Where $\log_e = 2{\cdot}3 \times$ the ordinary log. of the number, $\log_e$ is known as the hyperbolic logarithm. The reader should also read up the elements of conic sections, especially the parabola and hyperbola.

Checking Results.—The more experience one gets the more one realizes how very liable one is to make slips in calculations, and how essential it is to check results. "Cultivate the habit of checking every calculation" is, perhaps, the best advice one can give a young engineer.

First and foremost in importance is the habit of mentally reasoning out roughly the sort of result one would expect to get; this will prevent huge errors such as this. In an examination paper a question was asked as to the deflection of a beam of 10 feet span under a certain load. One student gave it as 34·2 inches, whereas it should have been 0·342 inch—an easy slip to make when working with a slide rule; but if he had thought for one moment, he would have seen that it is impossible to get nearly 3 feet elastic deflection on a 10-feet beam.

If an approximate method of arriving at a result is known, it should be used as a check on the more accurate method; or if there are two different ways of arriving at a result, it should be worked out by both to see if they agree. A graphical method is often an excellent check on an analytical method, or *vice versâ*. For example, suppose the deflection of an irregularly loaded beam has been arrived at by a graphical process; it can be roughly checked by calculating the deflection on the assumption that the load is evenly distributed. If the load be mostly placed in the middle, the graphical process should show a rather greater deflection; but if most of the load be near the two abutments, the graphical process should show a rather smaller deflection. In this way a very good check may be obtained. A little ingenuity will suggest ways of checking every result one obtains.

In solving an equation or in simplifying a vulgar fraction with several terms, rough cancelling can often be done which will enable one by inspection to see the sort of result that should be obtained, and calculations made on the slide rule can be checked by taking the terms in a different order.

Then, lastly, all very important work should be independently checked by another worker, or in some cases by two others.

Real and Imaginary Accuracy.—Some vainly imagine that the more significant figures they use in expressing the numerical

value of a quantity, the greater is the accuracy of their work. This is an exceedingly foolish procedure, for if the data upon which the calculations are based are not known to within 5 per cent., then all figures professing to give results nearer than 5 per cent. are false and misleading. For example, very few steam-engine indicators, with their attendant reducing gears, etc., are accurate to within 3 per cent., and yet one often sees the I.H.P. of an engine given as, say, 2345·67 ; the possible error here is $2345{\cdot}67 \times \frac{3}{100} = 70{\cdot}37$ I.H.P. Thus we are not even certain of the 45 in the 2345 ; hence the figures that follow are not only meaningless, but liable to mislead by causing others to think that we can measure the I.H.P. of an engine much more accurately than we really can. In the above case, it is only justifiable to state the I.H.P. as 2340, or rather nearer 2350 ; that is, to give one significant figure more than we are really certain of, and adding o's after the uncertain significant figure. In some cases a large number of significant figures is justifiable ; for example, the number of revolutions made by an engine in a given time, say a day. The counter is, generally speaking, absolutely reliable, and therefore the digits are as certain as the millions. Generally speaking, engineering calculations are not reliable to anything nearer than one per cent., and not unfrequently to within 10 or even 20 per cent. The number of significant figures used should therefore vary with the probable accuracy of the available data.

Experimental Proof of the Accuracy of the Beam Theory.—Each year the students in the author's laboratory make a tolerably complete series of experiments on the strength and elasticity of beams, in order to show the discrepancies (if any) between theory and experiment. The following results are those made during the past session, and are not selected for any special reason, but they serve well to show that there is no material error involved in the usual assumptions made in the beam theory. The deflection due to the shear has been neglected. The gear for measuring the deflection was attached at the neutral plane of the beam just over the end supports, in order to prevent any error due to external causes. If there had been any material error in the theory, the value of Young's Modulus E, found by bending, would not have agreed so closely with the values found by the tension and compression experiments. The material was mild steel ; all the specimens were cut from one bar, and all annealed together. The section was approximately 2 inches square, but the exact dimensions are given in each case.

Bending.				Tension.		Compression	
Central load. $E_b = \frac{WL^3}{48\delta I}$		Two loads dividing span into three equal parts. $E_b = \frac{23WL^3}{648\delta I}$		$E_t = \frac{WL}{Ax}$		$E_c = \frac{WL}{Ax}$	
Depth of sect. (h) 2·026″ Breadth „ (b) 2·035″ I . . . . . 1·41 Span (L) . . 36·25″		Depth of sect. (h) 2·020″ Breadth „ (b) 2·026″ I . . . . . 1·40 Span (L) . . 36·25″		Sect. 1·066″ × 2·038″ Area (A) 2·172 sq. ins. Length between datum points (L) 20″		Sect. 2·041″ × 2·0 Area (A) 4·157 sq Length between datum points (L)	
Load in tons (W).	Deflection in inches (δ).	Load in tons (W).	Deflection in inches (δ).	Load in tons (W).	Strain in inches (x).	Load in tons (W).	Strain inches
0·1	0·007	0·1	0·004	1	0·0003	2	0·00
0·2	0·012	0·2	0·009	2	0·0006	4	0·00
0·3	0·017	0·3	0·014	3	0·0010	6	0·00
0·4	0·023	0·4	0·019	4	0·0013	8	0·00
0·5	0·028	0·5	0·023	5	0·0017	10	0·00
0·6	0·033	0·6	0·028	6	0·0020	12	0·00
0·7	0·037	0·7	0·032	7	0·0023	14	0·00
0·8	0·043	0·8	0·037	8	0·0026	16	0·00
0·9	0·047	0·9	0·042	9	0·0029	18	0·00
1·0	0·053	1·0	0·047	10	0·0033	20	0·00
1·1	0·058	1·1	0·051	11	0·0037	22	0·00
1·2	0·064	1·2	0·056	12	0·0040	24	0·00
1·3	0·069	1·3	0·060	13	0·0044	26	0·00
1·4	0·075	1·4	0·065	14	0·0048	28	0·00
1·5	0·080	1·5	0·070	15	0·0051	30	0·00
1·6	0·086	1·6	0·075	16	0·0054	32	0·00
1·7	0·091	1·7	0·079	17	0·0057	34	0·00
1·8	0·097	1·8	0·083	18	0·0060	36	0·00
1·9	0·103	1·9	0·088	19	0·0063	38	0·00
2·0	0·109	2·0	0·093	20	0·0068	40	0·00
2·1	0·114	2·1	0·098	21	0·0072	42	0·00
2·2	0·120	2·2	0·103	22	0·0075	44	0·00
2·3	0·126	2·3	0·108	23	0·0079	46	0·00
2·4	0·131	2·4	0·112	24	0·0083	48	0·00
2·5	0·136	2·5	0·117	25	0·0087	50	0·00
2·6	0·142	2·6	0·122	26	0·0098	52	0·0
2·7	0·148	2·7	0·126	27	0·0130	53	0·0
2·8	0·154	2·8	0·133	28	0·0200	54	0·0
2·9	0·160	2·9	0·138	29	0·0300	56	0·1
3·0	0·168	3·0	0·142	30	0·0400	58	0·1
3·1	0·176	3·1	0·147	31	0·14	60	0·1
3·2	0·183	3·2	0·153	32	0·16	62	0·1
3·3	0·194	3·3	0·157	33	0·18	64	0·1
3·4	0·208	3·4	0·162	34	0·20	68	0·2
3·5	0·236	3·5	0·168	36	0·22	72	0·2
3·6	0·57	3·6	0·173	38	0·30	76	0·3

Load in tons (W).	Deflection in inches (δ).	Load in tons (W).	Deflection in inches (δ).	Load in tons (W).	Strain in inches (ϵ).	Load in tons (W).	Strain in inches (ϵ).
3·7	0·82	3·7	0·177	40	0·35	80	0·3[illegible]
3·8	1·05	3·8	0·180	42	0·42	84	0·4[illegible]
3·9	1·18	3·9	0·184	44	0·52		
4·0	1·59	4·0	0·189	46	0·62		
4·1	1·91	4·1	0·196	48	0·75		
4·2	2·19	4·2	0·203	50	0·95		
4·3	2·60	4·3	0·210	52	1·30		
4·4	2·77	4·4	0·218	53	1·74		
4·5	3·01	4·5	0·228	53·8	3·01		
4·6	3·40	4·6	0·250	48·0	Broke		
4·7	4·12	4·7	0·300				
4·8	4·97	4·8	0·35				
4·9	6·41	4·9	0·45				
5·0	8·88	5·0	0·50				
		5·1	0·60				
		5·2	0·80				
		5·3	1·10				
		5·4	1·45				
		5·5	2·50				
		6·0	3·65				
		6·5	5·25				
E_δ = 13,000 tons sq. inch.		E^δ = 12,940 tons sq. inch.		E_ϵ = 13,420 tons sq. inch.		E_ϵ = 13,180 tons sq. inch.	

It will be seen that the E obtained by the bending experiments agrees closely with that found by tension and compression; the mean of the former is 2·5 per cent. lower than the latter. If we had taken into account the deflection due to the shear, the discrepancy would have been even smaller. Further, the E_ϵ in this case is 1·8 per cent. less than the E_ϵ, hence we should expect the E_δ to be 0·5 per cent. less than the E_ϵ (see p. 466).

The elastic limit of the tension bar occurs at about 27 tons, corresponding to a tensile stress of 12·43 tons per square inch; in compression it occurs at about 53 tons, corresponding to a compressive stress of 12·75 tons per square inch. For reasons stated on p. 467, we cannot arrive at the true elastic limit in bending.

EXAMPLES

Chapter I.

Introductory.

1. Express 25 tons per square inch in kilos. per square millimetre.
Ans. 39·4.

2. A train is running at 30 miles per hour. What is its speed in feet per second?
Ans. 44.

3. What is the angular speed of the wheels in the last question? Diameter = 6 feet.
Ans. 14⅔ radians per second.

4. A train running at 30 miles per hour is pulled up gradually and evenly in 100 yards by brakes. What is the negative acceleration?
Ans. 3·23 feet per second per second.

5. What is the negative acceleration, if the train in Question 4 be pulled up from 40 miles to 5 miles per hour in 80 yards?
Ans. 7·06 feet per second per second.

6. If the train in Question 5 weighed 300 tons, what was the total retarding force on the brakes?
Ans. 65·8 tons.

(Check your result by seeing if the work done in pulling up the train is equal to the change of kinetic energy in the train.)

7. Water at 60° Fahr. falls over a cliff 1000 feet high. What would be the temperature in the stream below if there were no disturbing causes?
Ans. 61·29° Fahr.

8. A body weighing 10 lbs. is attached to the rim of a rotating pulley of 8 feet diameter. If a force of 50 lbs. were required to detach the body, calculate the speed at which the wheel must rotate in order to make it fly off.
Ans. 61 revolutions per minute.

9. (I.C.E., October, 1897.) Taking the diameter of the earth as 8000 miles, calculate the work in foot-pounds required to remove to an infinite distance from the earth's surface a stone weighing 1 lb.
Ans. 21,120,000 foot-lbs.

10. (I.C.E., October, 1897.) If a man coasting on a bicycle down a uniform slope of 1 in 50 attains a limiting speed of 8 miles per hour, what horse-power must he exert to drive his machine up the hill at the same speed, there being no wind in either case? The weight of man and bicycle together is 200 lbs.
Ans. 0·17.

11. Find the maximum and minimum speeds with which the body represented in Fig. 2 is moving. Express the result in feet per second and metres per minute.

Ans. Maximum, 13 feet per second, or 238 metres per minute at about the fifth second; minimum, 0·5 foot per second, or 9·15 metres per minute at about the third second.

12. A flywheel is running at 500 revolutions per minute and is brought to rest by the friction of the bearings, which may be assumed to be constant, in 4 minutes. How many revolutions will it make before stopping? What was the angular acceleration in radians per second per second?

Ans. 1000 revolutions ; 0·218 radians per second per second.

13. (I.C.E., February, 1910.) At a height of 4 feet above the level floor of a gate-chamber, a jet of water issues horizontally through a small orifice in the gate, and strikes the floor at a distance of 12 feet from the gate. What is the velocity of the issuing jet? *Ans.* 24·1 feet per second.

14. A pit cage weighing 10 tons is being hauled up the shaft by a rope. What is the tension in the rope—(i) when the cage is being lifted with an acceleration of 2 feet per second per second ; (ii.) when the cage is being lifted at uniform speed ; (iii.) when the cage is being retarded 2 feet per second per second? *Ans.* 10·62 ; 10 ; 9·38 tons.

15. (I.C.E., February, 1903.) With what speed must a locomotive be running on level railway lines, forming a curve of 968 feet radius, if it produces a horizontal thrust on the outer rail equal to $\frac{1}{61}$ of its weight?

Ans. 15·05 miles per hour.

16. (I.C.E., February, 1912.) A 2-ton motor-car is working at constant horse-power and travels a distance of 20 miles in 30 minutes, rising 500 feet in the time. The frictional resistance to motion is equivalent to an average force of 50 lbs., and its initial and final speeds are the same. Find the horse-power of the car. *Ans.* 7·6.

17. (I.C.E., October, 1911.) Find in direction and magnitude the force required to compel a body weighing 10 lbs. to move in a curved path, the radius of curvature at the point considered being 20 feet, the velocity of the body 40 feet per second, and the acceleration in its path 48 feet per second per second. *Ans.* 29 lbs. ; 59° with path.

Chapter II.

MENSURATION.

1. A locomotive wheel 8 feet 6 inches diameter slips 17 revolutions per mile. How many revolutions does it actually make per mile?

Ans. 214·7.

2. In Fig. 13, C = 200 feet, C_1 = 120 feet. Find the length of the arc by both methods. *Ans.* 256 feet, 253·3 feet.

3. Find the length of a semicircular arc of 2 inches radius by means of the method shown in Fig. 15, and see whether it agrees with the actual length, viz. 6·28 inches.

4. Find the weight of a piece of ¾-inch boiler plate 12′ 3″ × 4′ 6″, having an elliptical manhole 15″ × 10″ in it. (One-inch plate weighs 40 lbs. per square foot.) *Ans.* 1629 lbs., or 0·73 ton.

5. Find the area of a regular hexagon inscribed in a circle of 50 feet radius. *Ans.* 6495 square feet.

6. Find the area in square yards of a quadrilateral ABCD. Length of AB = 200 yards, BC = 650 yards, CD = 905 yards, DA = 570 yards, AC = 800 yards.

(N.B.—The length BD can be calculated by working backwards when the area is known, but the work is very long. The tie-lines AC and DB in a survey are often checked by this method.)

Ans. 272,560 square yards.

7. A piece of sheet iron 5 feet wide is about to be corrugated. Assuming the corrugations to be semicircular of 3-inch pitch, what will be the width when corrugated? *Ans.* 3·18 feet.

8. Find the error per cent. involved in using the first formula of Fig. 25, when $H = \frac{r}{2}$ for a circle 5 feet diameter, assuming the second formula to be exact. *Ans.* 5·8.

9. Find the area of the shaded portion of Fig. 426. S = 4 inches; the outlines are parabolic. *Ans.* 2·67 square inches.

10. Cut an irregular-shaped figure out of a piece of thin cardboard or metal with smooth edges. Find its area by—(1) the method shown in Fig. 32; (2) the mean ordinate method; (3) Simpson's method; (4) the planimeter; (5) by weighing.

11. Find the area of Figs. 29, 31, 32, 33, 34 expressed in square inches. *Ans.* (29) 0·29, (31) 0·16, (32) 1·14, (33) 0·83, (34) 0·86.

12. Find the sectional area of a hollow circular column, 12 inches diameter outside, 9 inches diameter inside. *Ans.* 49·48 square inches.

13. Bend a piece of thin wire to form the quarter of an arc of a circle, balance it to find the c. of g., and then find the surface of a hemisphere by the method shown in Fig. 35. Check the result by calculation by the method given in Fig. 36.

14. Find the heating surface of a taper boiler-tube—length, 3 feet; diameter at one end, 5 inches; at the other, 8 inches. *Ans.* 5·1 square feet.

15. Find the mean height of an indicator diagram in which the initial height Y is 2 inches, and cut-off occurs at $\frac{1}{4}$ stroke, log 4 = 0·602—(i.) with hyperbolic expansion; (ii). with adiabatic expansion. ($n = 1·4$.) *Ans.* (i.) 1·19 inch; (ii.) 1·02 inch.

16. Find the weight of a cylindrical tank 4 feet diameter inside, and 7 feet deep, when full of water. Thickness of sides $\frac{3}{8}$ inch; the bottom, which is $\frac{1}{4}$ inch thick, is attached by an internal angle $3'' \times 3'' \times \frac{1}{2}''$. There are two vertical seams in the sides, having an overlap of 3 inches; pitch of all rivets, $2\frac{1}{2}$ inches; diameter, $\frac{3}{4}$ inch. Water weighs 62·5 lbs. per cubic foot, and the metal weighs 480 lbs. per cubic foot. *Ans.* 7270 lbs.

17. (Victoria, 1893.) A series of contours of a reservoir bed have the following areas:—

At water-level	4,800,000 sq. yards.
2 feet down	3,700,000 ,,
4 ,, ...	2,000,000 ,,
6 ,, ...	700,000 ,,
8·4 ,, ...	Zero

Find the volume of the reservoir in cubic yards. *Ans.* 5,900,000 (approx.).

18. The area of a half-section of a concrete building consisting of cylindrical walls and a dome is 12,000 square feet. The c. of g. of the section is situated 70 feet from the axis of the building. Find the number of cubic yards of concrete in the structure. *Ans.* 195,400.

19. Find the weight of water in a spherical vessel 6 feet diameter, the depth of water being 5 feet. *Ans.* 6545 lbs.

20. In the last question, how much water must be taken out in order to lower the level by 6 inches. *Ans.* 581 lbs.

(Roughly check the result by making a drawing of the slice, measure the mean diameter of it, then calculate the volume by multiplying the mean area by the thickness of the slice.)

21. Calculate the number of cubic feet of water in an egg-ended boiler, diameter, 6 feet; total length, 30 feet; depth of water, 5·3 feet. *Ans.* 746.

22. Find the number of cubic feet of solid stone in a heap having a rectangular base 60′ × 18′, standing on level ground; slope of sides, 1½ to 1 foot; flat-topped, height 4 feet; the voids being 35 per cent. *Ans.* 1881.

23. Find the weight of a steel projectile with solid body and tapered point, assumed to be a paraboloid. Material weighs 0·28 lb. per cubic inch; diameter, 6 inches; length over all, 24 inches; length of tapered part, 8 inches. *Ans.* 158 lbs.

24. Find the weight of a wrought-iron anchor ring, 6 inches internal, and 10 inches external diameter, section circular. *Ans.* 22 lbs.

Chapter III.

MOMENTS.

1. A uniform bar of iron, ⅜ inch square section and 6 feet long, rests on a support 18 inches from one end. Find the weight required on the short arm at a distance of 15 inches from the support in order to balance the long arm. *Ans.* 13·5 lbs.

2. In the case of a lever such as that shown in Fig. 65, $W_1 = 1\frac{1}{4}$ ton, $l_1 = 28$ inches, $l = 42{\cdot}5$ inches, $l_2 = 50$ inches, $l_3 = 4$ inches, $w_2 = 1$ ton. Find W when $w_3 = 0$, and find w_3 when $w_2 l_2$ is clockwise and $l_2 = 12{\cdot}5$ feet. *Ans.* W = 2 tons; $w_3 = 50$ tons.

(N.B.—In the Buckton testing-machine, w_3 is the load on the bar under test.)

3. A lever safety-valve is required to blow off at 70 lbs. per square inch. Diameter of valve, 3 inches; weight of valve, 3 lbs.; short arm of lever, 2½ inches; weight of lever, 11 lbs.; distance of c. of g. of lever from fulcrum, 15 inches. Find the distance at which a cast-iron ball 6 inches diameter must be placed from the fulcrum. The weight of the ball-hook = 0·6 lb. *Ans.* 35·5 inches.

4. In a system of compound levers the arms of the first lever are 3 inches and 25 inches, of the second 2·5 inches and 25·5 inches, of the third 2·1 inches and 25·9 inches, a force of 84 lbs. is exerted on the end of the long arm of the third lever. Find the force that must be exerted on the short arm of the first lever when the system is arranged to give the greatest possible leverage. *Ans.* 88,060 lbs.

5. Find the position of the c. of g. of a beam section such as that shown in Fig. 430.

Top flange	3 inches wide,	1½ inch thick.
Bottom flange	15 ,, wide,	1¾ ,, thick.
Web	1½ ,,	thick.
Total height	18 ,,	

Ans. 5·72 inches from the bottom edge.

6. Find the height of the c. of g. of a T-shaped section from the foot, the top cross-piece being 12 inches wide, 4 inches deep, the stem 3 feet deep, 3 inches wide. *Ans.* 24·16 inches.

(Check by seeing if the moments are equal about a line passing through the section at the height found.)

7. Cut out of a piece of thin card or metal such a figure as is shown in Figs. 78 and 87. Find the c. of g. by calculation or by the graphical process, and check the result by balancing as shown on p. 75.

8. In such a figure as 77, the width of the base is 4 inches, and at the base of the top triangle 1·5 inch. The height of the trapezium = 2 inches, and the total height = 5 inches. Find the height of the c. of g. from the apex. *Ans.* 3·54 inches.

9. A trapezoidal wall has a vertical back and a sloping front face: width of base, 10 feet; width of top, 7 feet; height, 30 feet. What horizontal force must be applied at a point 20 feet from the top in order to overturn it, *i.e.* to make it pivot about the toe? Width of wall, 1 foot; weight of masonry in wall, 130 lbs. per cubic foot. *Ans.* 18,900 lbs.

10. Find the height of the c. of g. of a column 4 feet square and 40 feet high, resting on a tapered base forming a frustum of a square-based pyramid 10 feet high and 8 feet square at the base.

Ans. 20·4 feet from base.

11. Find the position of the c. of g. of a piece of wire bent to form three-fourths of an arc of a circle of radius R.

Ans. On a line drawn from the centre of the circle to a point bisecting the arc, and at a distance 0·3R from the centre.

12. Find the position of the c. of g. of a balance weight having the form of a circular sector of radius R, subtending an angle of 90°.

Ans. 0·6R from centre of circle.

13. Find the second moment (I_0) of a thin door about its hinges: width, 7 feet; height, 4 feet. *Ans.* 457 in feet^4 units.

14. Find the second moment (I) of a rectangular section 9 inches deep, 3 inches wide, about an axis passing through the c. of g., and parallel to the short side. *Ans.* 183 inch^4 units.

15. Find the second moment (I_0) of a square section of 4-inch side about an axis parallel to one side and 5 inches from the nearest edge.

Ans. 805·3 inch^4 units.

16. Find the second moment (I_0) of a triangle 0·9 inch high, base 0·6 inch wide as in Fig. 100. *Ans.* 0·109 inch^4 units.

17. Find the second moment (I) of a triangle 4 inches high and 3 inches base, about an axis passing through the c. of g. of the section and parallel with the base. *Ans.* $5\frac{1}{3}$ inch^4 units.

Ditto (I_0) about the base of the triangle. *Ans.* 16 inch^4 units.

Ditto (I_0) of a trapezium (Fig. 103); B = 3 inches, B_1 = 2 inches, h = 2 inches. *Ans.* 7·3 inch^4 units.

Ditto (I_0) ditto, as shown in Fig. 104. *Ans.* 6 inch^4 units.

Ditto (I) ditto, as shown in Fig. 105. *Ans.* 1·64 inch^4 units.

Ditto ditto, by the approximate method in Fig. 106.

Ans. 1·67 inch^4 units.

18. Find the second moment (I) of a square of 6 inches edge about its diagonal. *Ans.* 108 inch^4 units.

19. Find the second moment (I) of a circle 6 inches diameter about a diameter. *Ans.* 63·6 inch^4 units.

20. Find the second moment (I) of a hollow circle 6 inches external and 4 inches internal diameter about a diameter. *Ans.* 51 inch^4 units.

21. Find the second moment (I) of a hollow eccentric circle, as shown in Fig. 110. $D_e = 6$ inches, $D_i = 4$ inches. The metal is $\frac{1}{2}$ inch thick on the one side, and $1\frac{1}{2}$ inch on the other side. *Ans.* 45·4 inch^4 units.

22. Find the second moment (I) of an ellipse about its minor axis. $D_2 = 6$ inches, $D_1 = 4$ inches. *Ans.* 42·4 inch^4 units.
Ditto, ditto about the major axis. *Ans.* 18·84 inch^4 units.

23. Find the second moment (I) of a parabola about its axis. H = 6 inches, B = 4 inches. *Ans.* 102·4 inch^4 units.
Ditto, ditto (I_o) about its base, as in Fig. 114.
Ans. 263·3 inch^4 units.

24 Take an irregular figure, such as that shown in Fig. 115, and find its second moment (I_o) by calculation. Check the result by the graphical method given on p. 96.

25. Find the second polar moment (I_p) of a rectangular surface as shown in Fig. 117. B = 4 inches, H = 6 inches. *Ans.* 104 inch^4 units.
Ditto, ditto, ditto for Fig. 118. D = 6 inches.
Ans. 127·2 inch^4 units.
Ditto, ditto, ditto for Fig. 119. $D_e = 6$ inches, $D_i = 4$ inches.
Ans. 102 inch^4 units.

26. Find the second polar moment (I_p) of such a bar as that shown in Fig. 120. B = $\frac{1}{2}$ inches, H = 2 inches, L = 16 inches.
Ans. 2120 inch^5 units.

27. Find the second polar moment (I_p) for a cylinder as shown in Fig. 121. D = 16 inches, H = 2 inches. *Ans* 12,870 inch^5 units.

28. Find the second polar moment (I_p) for a hollow cylinder as shown in Fig. 122. $R_e = 8$ inches, $R_i = 5$ inches, H = 2 inches.
Ans. 10,903 inch^5 units.

29. Find the second polar moment (I_p) of a disc flywheel about its axis.

External radius of rim = 8 inches
Internal radius of rim = 6 ,,
Internal radius of web = 1·5 ,,
Thickness of web = 0·7 ,,
Internal radius of boss = 0·75 ,,
Thickness of boss = 3 ,,
Width of rim = 3 ,,

Ans. 14,640 inch^5 units.

30. Find the second polar moment (I_p) of a sphere 6 inches diameter about its diameter. *Ans.* 407 inch^5 units.

Find the second polar moment (I_{op}) about a line situated 12 inches from the centre of the sphere. *Ans.* 16,690 inch^5 units.

Find the second polar moment (I_p) of a cone about its axis. H = 12 inches, R = 2 inches. *Ans.* 60·3 inch^5 units.

In examples 26, 27, 28, 29, 30, $\frac{W}{g}$ is taken as unity.

Chapter IV.

Resolution of Forces.

1. Set off on a piece of drawing-paper six lines representing forces acting on a point in various directions. Find the resultant in direction and magnitude by the two methods shown in Fig. 128.

2. Set off on a piece of drawing-paper six lines as shown in Fig. 130. Find the magnitude of the forces required to keep it in equilibrium in the position shown.

3. In the case of a suspension bridge loaded with eight equal loads of 1000 lbs. each placed at equal distances apart. Find the forces acting on each link by means of the method shown in Fig. 131.

4. (I.C.E., October, 1897.) A pair of shear legs make an angle of 20° with one another, and their plane is inclined at 60° to the horizontal. The back stay is inclined at 30° to the plane of the legs. Find the force on each leg, and on the stay per ton of load carried.

Ans. Each leg, 0·88 ; stay, 1·0 ton.

5. A telegraph wire $\frac{1}{10}$ inch diameter is supported on poles 170 feet apart and dips 2 feet in the middle. Find the pull on the wire.

Ans. 47·4 lbs.

6. A crane has a vertical post DC as in Fig. 135, 8 feet high. The tie AC is 10 feet long, and the jib AB 14 feet long. Find the forces acting along the jib and tie when loaded with 5 tons simply suspended from the extremity of the jib. *Ans.* Tie, 6·3 tons ; jib, 8·8 tons.

7. In the case of the crane in Question 6, find the radius, viz. l, by means of a diagram. If the distance x be 4 feet, find the pressures p_1 and p_2. *Ans.* $p_1 = p_2 = 12{\cdot}2$ tons.

8. Again referring to Question 6. Taking the weight of the tie as 50 lbs., and that of the jib as 350 lbs., and the pulley, etc., at the end of the jib as 50 lbs. find the forces acting on the jib and tie.

Ans. Jib, 8·99 ; tie, 6·39 tons.

9. In the case of the crane in Question 6, find the forces when the crane chain passes down to a barrel as shown in Fig. 137. Let the sloping chain bisect the angle between the tie and the jib. Find W_1 if $l_1 = 5$ feet ; also find the force acting along the back stay.

Ans. Jib, 11·4 ; tie = 3·6 ; $W_1 = 4{\cdot}9$; back stay, 5·8 tons.

10. Each leg of a pair of shear legs is 50 feet long. They are spread out 20 feet at the foot. The back stay is 75 feet long. Find the forces acting on each member when lifting a load of 20 tons at a distance of 20 feet from the foot of the shear legs, neglecting the weight of the structure. *Ans.* Legs, 16·7 ; back stay, 16·5 tons.

11. In the last question, find the horizontal pull on the screw and the total upward pull on the guides, also the force tending to thrust the feet of the shear legs apart.

Ans. Horizontal pull, 16·7 ; upward pull, 8·75; thrust, 4·1 tons.

12. The three legs of a tripod AO,BO,CO, are respectively 28, 29, and 27 feet long. The horizontal distances apart of the feet are AB, 15 feet ; BC, 17 feet ; CA, 13 feet. A load of 12 tons is supported from the apex, find the thrust on each leg.

Ans. A = 2·94 tons ; B = 2·22 tons ; C = 7·35 tons.

13. A simple triangular truss of 30 feet span and 5 feet deep supports a load of 4 tons at the apex. Find the force acting on each member.

Ans. 6 tons on the tie; 6·32 tons on the rafters.

14. (I.C.E., October, 1897.) Give a reciprocal diagram of the stress in the bars of such a roof as that shown in Fig. 139, loaded with 2 tons at each joint of the rafters; span, 40 feet; total height, 10 feet; depth of truss in the middle, 8 feet 7½ inches.

15. The platform of a suspension foot bridge 100 feet span is 10 feet wide, and supports a load of 150 lbs. per square foot, including its own weight. The two suspension chains have a dip of 20 feet. Find the force acting on each chain close to the tower and in the middle, assuming the chain to hang in a parabolic curve.

Ans. 60,000 lbs. close to tower; 47,000 lbs. in middle.

16. (London U., 1914.) A footbridge 8 feet wide has to be supported across a river 75 feet wide by means of two cables of uniform cross section. The dip of the cables at the centre is 9 feet, and the maximum load on the platform is to be taken as 140 lbs. per square foot of platform area. The working stress in the cables is not to exceed 5 tons per square inch, and the steel from which the cables are made weighs 0·28 lb. per cubic inch. Determine a suitable cross sectional area for these cables.

Ans. 4·4 square inches.

17. (Leeds U., 1914.) Three spheres which are kept in contact with one another rest on a plane horizontal surface. Another similar sphere is placed on the top. Find the pressure the top one exerts on the three lower spheres. Weight of each sphere, 100 lbs. *Ans.* 40·8 lbs. on each lower sphere.

18. A ladle is lifted by means of three sling chains each 3 feet in length, the upper ends are attached to a ring and the lower ends to three hooks placed 4 feet apart, *i.e.* at the angular points of an equilateral triangle. Find the load on each chain. Weight of ladle 5 tons. *Ans.* 2·6 tons.

19. (London U., 1913.) In a tripod the lengths of the legs are AO, 18 feet, BO 16 feet, CO 17 feet, and the distances apart of the feet are AB 15 feet, BC 14 feet, CA 14 feet 6 inches.

A load of 40 tons is suspended from the apex O. Find the force acting down each leg. Assuming that the feet rest on a perfectly smooth, level plane, and that they are prevented from spreading by means of ties attached to the feet. Find the tension in each tie.

Ans. OA, 10 tons; OB, 21·6 tons; OC, 14 tons; AB, 4·2 tons; BC, 5·7 tons; CA, 2·5 tons.

20. (I.C.E., February, 1903). A Warren girder, length 100 feet, is divided into five bays on the lower flange, the length of the inclined braces being 20 feet; if loads of 30 tons are carried by the girder at the joints, 20 feet and 40 from one end. Find the stresses in members.

Ans. Middle bottom bay, 52 tons; middle top bay remote from loads, 41·6 tons; middle diagonal remote from loads, 20·8 tons.

Chapter V.

MECHANISMS.

1. Construct the centrodes of $O_{b,d}$ and $O_{a,c}$ for the mechanism shown in Fig. 148, when the link *d* is fixed.

2. In Fig. 149, if the link *b* be fixed, the mechanism is that of an oscillating cylinder engine, the link *c* representing the cylinder, *a* the crank, and *d* the piston-rod. Construct a diagram to show the relative angular velocity of the piston and the rod for one stroke when the crank rotates uniformly.

3. Construct a curve to show the velocity of the cross-head at all parts of the stroke for a uniformly revolving crank of 1 foot radius—length of connecting rod = 3 feet—and from this curve construct an acceleration curve, scale 2 inches = 1 foot. State the scale of the acceleration curve when the crank makes 120 revolutions per minute.

4. The bars in a four-bar mechanism are of the following lengths: *a* 1·2, *b* 2, *c* 1·9, *d* 1·4. Find the angular velocity of *c* when normal to *d*, having given the angular velocity of *a* as 2·3 radians per second.

5. In Fig. 153, find the weight that must be suspended from the point 6 in order to keep the mechanism in equilibrium, when a weight of 30 lbs. is suspended from the point 5, neglecting the friction and the weight of the mechanism itself.

6. In Fig. 158, take the length of $a = 2$, $b = 10$, $c = 4$, $d = 10{\cdot}75$. The interior angle at $3 = 156°$. Find the angular velocity of *c* when that of *a* is 5. *Ans.* 2·19.

7. Find the angular velocity of the connecting rod in the case of an engine. Radius of crank = 0·5 foot; length of connecting rod = 5 cranks. Revolutions per min. 200; crank angle, 45°.

Ans. 2·5 radians per second.

8. In Fig. 161*a*, the length of $a = 1{\cdot}5$, $b = 5$, find the angular velocity of *c* when the crank is in its lowest position, and when making 100 revolutions per minute. *Ans.* 0·45 radians per second.

9. Construct velocity and acceleration curves, such as those shown in Fig. 163, for the mechanism given in Question 4.

10. In a Stephenson's link motion, the throw of the eccentrics is $3\frac{1}{2}$ inches. The angle of advance 16°, *i.e.* 106° from the crank. The length of the eccentric rods is $57\frac{3}{8}$ inches. The rods are attached to the link in the manner shown in Fig. 164*c*. The distance ST = $17\frac{3}{8}$ inches. The suspension link is attached at T, and is $25\frac{1}{2}$ inches long. The point U is $10\frac{7}{8}$ inches above the centre line of the engine, and is 5 feet from the centre of the crank shaft. Find the velocity of the top and bottom pins of the link when the engine makes 100 revolutions per minute, and the link is in its lowest position, and when the crank has turned through 30° from its inner dead centre.

Ans. Top pin, 2·8 feet per second; bottom pin, 4·4 feet per second.

11. In a Humpage gear, the numbers of teeth in the wheels are:— A = 42, B = 28, B_1 = 15, C = 24, E = 31. Find the number of revolutions made by C for one revolution of the shaft. *Ans.* 10·03.

12. Construct (i.) an epicycloidal, (ii.) a hypocycloidal, tooth for a spur

wheel ; width of tooth on pitch line, 1 inch ; depth below pitch line, $\frac{1}{4}$ inch ; do. above, $\frac{3}{4}$ inch ; diameter of pitch circle, 18 inches ; do. of rolling circle, 6 inches. Also construct an involute and a cycloidal tooth for a straight rack.

13. The cam, shown in Fig. 175, rotates at 84 revolutions per minute ; find the speed and acceleration of the follower when the cam is in the position shown.

Ans. Speed 1·3 feet per second ; acceleration 45 feet per second per second.

14. In Fig. 200 the wheel *a* rotates at 1000 revolutions per minute, the arm D rotates in the same sense at 975 revolutions per minute ; find the speed of *c* and the sense of rotation.

Ans. 950 revolutions per minute in the same sense as *a*.

15. Referring to the data given in Question 14, find the speed of D when *c* rotates at 950 revolutions per minute in the opposite sense to *a*.

Ans. 25 revolutions per minute.

16. Find the angular velocity of a connecting-rod when the crank makes 140 revolutions per minute and has moved through 30° from its inner dead centre. The connecting-rod is 4 cranks long.

Ans. 186 radians per minute.

17. Three wheels are arranged as shown in Fig. 192. The numbers of teeth are (*a*) 60, (*b*) 120, (*c*) 150. Find the revolutions of the unfixed wheels when

(i) The frame *d* is fixed and *a* is given + 1 revolution.
(ii) *a* is fixed and *d* is given − 1 revolution.
(iii) *b* is fixed and *d* is given + 0·5 revolution.
(iv) *c* is fixed and *d* is given − 0·4 revolution.

Ans.

	d	*a*	*b*	*c*
i	0	+ 1	− 0·5	+ 0·4
ii	− 1	0	− 1·5	− 0·6
iii	+ 0·5	+ 1·5	0	+ 0·9
iv	− 0·4	+ 0·6	− 0·9	0

Chapter VI.

Dynamics of the Steam Engine.

1. Find the acceleration pressure at each end of the stroke of a vertical inverted high-speed steam-engine when running at 500 revolutions per minute ; stroke, 9 inches ; weight of reciprocating parts, 110 lbs. ; diameter of cylinder, 8 inches ; length of connecting-rod, 1·5 foot.

Ans. 54·8 lbs. sq. inch at bottom ; 85·5 lbs. sq. inch at top.

2. Find the acceleration pressure at the end of the stroke of a pump having a slotted cross-head as shown in Fig. 160 ; speed, 100 revolutions per minute ; stroke, 8 inches ; diameter of cylinder, 5 inches ; weight of reciprocating parts, 95 lbs. *Ans.* 5·49 lbs. sq. inch.

3. To what pressure should compression be carried in the case of a horizontal engine running at 60 revolutions per minute ; stroke, 4 feet ; weight of reciprocating parts per square inch of piston, 3·2 lbs. ; length of connecting-rod, 9 feet.

Ans. 9·57 lbs. sq. inch at "in" end, 6·09 lbs. sq. inch at "out" end.

4. In a horizontal engine, such as that shown in Fig. 205, the weights of the parts were as follows :—

Piston	54 lbs.
Piston and tail rods	40 ,,
Both cross-heads	100 ,,
Small end of connecting-rod	18 ,,
Plain part of ,, ...	24 ,,
Air-pump plunger	18 ,,
l_1	3 feet
l_2	1 foot
w_1	20 lbs.
w_2	7 ,,
Diameter of cylinder	8 inches
Revolutions per minute	140
Length of connecting-rod	40 inches
Length of stroke	18 ,,

Find the acceleration pressure at each end of the stroke.

Ans. 27·8, 17·6 lbs. sq. inch.

5. (Victoria, 1902.) In an inverted vertical engine the radius of the crank is 1 foot ; the length of the connecting-rod, 4 feet ; diameter of the piston, 16 inches ; revolutions per minute, 200 ; weight of the reciprocating parts, 500 lbs. Find the twisting moment on the crank shaft when the piston is descending and the crank has turned through 30° from the top centre. The effective pressure on the piston is 50 lbs. sq. inch. *Ans.* 2400 lbs.-feet.

6. (Victoria, 1903.) A weight of 50 lbs. is attached to a long projecting pin on the ram of a slotting machine. Find the downward force acting on the pin when the ram is at the top and the bottom of its downward stroke. The speed of the machine is 60 revolutions per minute, the stroke is 12 inches, and the connecting-rod which is below the crank-pin is 24 inches long. *Ans.* bottom, 87·5 lbs. ; top, 27·5 lbs.

7. In an engine of the Willans type the annular air cushion cylinder is 12 inches and 9 inches diameter, the weight of the reciprocating parts is 1000 lbs. The radius of the crank is 3 inches, the length of the connecting-rod is 12 inches. Revolutions per minute 450, the air pressure at the bottom of the stroke is atmospheric. Find the required clearance at the top of the stroke, assuming (i.) hyperbolic, (ii.) adiabatic compression of the air. *Ans.* (i.) 0·20 inch, (ii.) 0·58 inch.

8. Take the indicator diagrams from a compound steam-engine (if the reader does not happen to possess any, he can frequently find some in the Engineering papers), and construct diagrams similar to those shown in Figs. 208 to 213. Also find the value of m, p. 196, and the weight of flywheel required, taking the diameter of the wheel as five times the stroke.

9. Taking the average piston speed of an engine as 400 feet per minute, and the value of m as $\frac{1}{3}$; the fluctuation of velocity as 1 % on either side of the mean, namely K = 0·02 ; the radius of the flywheel as twice the stroke ; show that the weight of the rim may be taken as 50 lbs. per I.H.P. for a single-cylinder engine, running at 100 revolutions per minute.

10. A two-cylinder engine with cranks at right angles indicates 120 H.P. at 40 revolutions per minute. The fluctuation of energy per stroke is 12 % (*i.e.* $m = 0{\cdot}12$) ; the percentage fluctuation of velocity is 2 % on either side of the mean (*i.e.* K = 0·04) ; the diameter of the flywheel is 10 feet. Find the weight of the rim. *Ans.* 4·87 tons.

11. Taking the average value of the piston speed of a gas-engine as 600 feet per minute, and the value of m as 8·5, the fluctuation of velocity as 2 % (*i.e.* K = 0·04); on either side of the mean, the radius of the fly-wheel as twice the stroke, show that the weight of the rim may be taken at about 150 lbs. per I.H.P. for a single cylinder engine running at 200 revolutions per minute.

12. Find the weight of rim required for the flywheel of a punching machine, intended to punch holes $1\frac{1}{2}$-inch diameter through $1\frac{1}{4}$-inch plate, speed of rim 30 feet per second. *Ans.* 1180 lbs.

13. (I.C.E., October, 1897.) A flywheel supported on a horizontal axle 2 inches in diameter is pulled round by a cord wound round the axle carrying a weight. It is found that a weight of 4 lbs. is just sufficient to overcome the friction. A further weight of 16 lbs., making 20 in all, is applied, and after two seconds starting from rest it is found that the weight has gone down 12 feet. Find the moment of inertia of the wheel.

Ans. 0·014 mass feet^2 units.

(N.B.—For the purposes of this question, you may assume that the speed of the wheel when the weight is released is twice the mean.)

14. (I.C.E., October, 1897.) In a gas-engine, using the Otto cycle, the I.H.P. is 8, and the speed is 264 revolutions per minute. Treating each fourth single stroke as effective and the resistance as uniform, find how many foot-lbs. of energy must be stored in the flywheel in order that the speed shall not vary by more than one-fortieth above or below its mean value. *Ans.* 15,000 foot-lbs.

15. Find the stress due to centrifugal force in the rim of a cast-iron wheel 8 feet diameter, running at 160 revolutions per minute.

Ans. 431 lbs. sq. inch.

16. (S. & A., 1896.) A flywheel weighing 5 tons has a mean radius of gyration of 10 feet. The wheel is carried on a shaft of 12 inches diameter, and is running at 65 revolutions per minute. How many revolutions will the wheel make before stopping, if the coefficient of friction of the shaft in its bearing is 0·065? (Other resistances may be neglected.)

Ans. 352.

17. Find the bending stress in the middle section of a coupling-rod of rectangular section when running thus: Radius of coupling-crank, 11 inches; length of coupling-rod, 8 feet; depth of coupling-rod, 4·5 inches; width, 2 inches; revolutions per minute, 200. *Ans.* 2·4 tons per sq. inch.

18. Find the bending stress in the rod given in the last question if the sides were fluted to make an I section; the depth of fluting, 3 inches; thickness of web, 1 inch. *Ans.* 1·87 ton sq. inch.

19. Assuming that one-half of the force exerted by the steam on one piston of a locomotive is transmitted through a coupling-rod, find the maximum stress in the rod mentioned in the two foregoing questions. Diameter of cylinder, 16 inches; steam-pressure, 140 lbs. per sq. inch.

Ans. 3·09 tons sq. inch for rectangular rod; 2·92 tons sq. inch for fluted rod.

20. If the rod in Question 17 had been tapered off to each end instead of being parallel in side elevation, find the bending stress at the middle section of the rod. The rod has a straight taper of $\frac{1}{8}$ inch per foot.

Ans. 2·22 tons sq. inch.

21. Find the skin stress due to bending in a connecting-rod when running thus: Radius of crank, 10 inches; length of rod, 4 feet; diameter of rod, 3 inches; number of revolutions per minute, 120. *Ans.* 450 lbs. sq. inch.

22. A railway carriage wheel is found to be out of balance to the extent of 3 lbs. at a radius of 18 inches. What will be the amount of "hammer blow" on the rails when running at 60 miles per hour? Diameter of wheels, 3 feet 6 inches. *Ans.* 354 lbs.

23. In Mr. Hill's paper (I.C.E., vol. civ. p. 265) on locomotive balancing, the following problem is set :—

The radius of the crank = 12 inches ; radius of balance-weight, 33 inches ; weight of the reciprocating parts, 500 lbs. ; weight of the rotating parts, 680 lbs. ; distance from centre to centre of cylinders = 24 inches ; ditto of balance-weights, 60 inches. Find the weight of the balance-weight if both the reciprocating and rotating weights are fully balanced. *Ans.* 325 lbs.

24. Find the weight and position of the balance-weights of an inside cylinder locomotive working under the following conditions :—

Weight of connecting rod, big end ...	...	150 lbs.
,, ,, ,, small end	...	70 ,,
,, ,, ,, plain part	...	180 ,,
,, cross-head and slide blocks	...	170 ,,
,, piston and rod	...	200 ,,
,, crank-web and pin... ...	...	330 ,,
Radius of c. of g. of crank-web and pin	...	10 inches
,, ,, crank (R) ...	...	12 ,,
,, ,, balance-weight (R_b)	...	30 ,,
Cylinder centres (C)	...	24 ,,
Wheel centres (y)	...	72 ,,

Find the requisite balance-weight and its position for balancing the rotating and two-thirds of the reciprocating parts.

Ans. 259 lbs.; $\theta = 26\frac{1}{4}°$.

25. An English railway company, instead of taking two-thirds of the reciprocating, and the whole of the rotating parts, take as a convenient compromise the whole of the reciprocating parts. Find the amount of each balance-weight for such a condition, taking the values given in Question 23. *Ans.* 138 lbs.

26. Find the amount of balance-weight required for the conditions given in Question 24 for an uncoupled outside-cylinder engine. Weight of coupling crank and pin, 80 lbs. ; radius of c. of g. of ditto, 11 inches ; R_c = 12 inches. *Ans.* 266 lbs.

27. Find the amount and position of the balance-weight required for a six-wheel coupled inside-cylinder engine, where w from a to b and c to d = 140 lbs., and w from b to c = 300 lbs. ; weight of coupling-crank and pin, 80 lbs. ; radius of c. of g. of ditto, 11 inches ; R_c = 12 inches. The other weights may be taken from Question 24.

Ans. 85 lbs. on trailing and leading wheels ; 140 lbs. on driving wheel ; $\theta = 55°$.

28. Find the balance-weights for such an engine as that shown in Fig. 234. C_2 = 100 inches, C_1 = 108 inches. For other details see Question 24.

Ans. 200 lbs. on trailing wheel ; 480 lbs. on driving wheel.

29. Find the speed at which a simple Watt governor runs when the arm makes an angle of 30° with the vertical. Length of arm from centre of pin to centre of ball = 18 inches. *Ans.* 47·5 revs. per minute

30. (Victoria, 1896.) A loaded Porter's governor geared to an engine with a velocity ratio 5 has rods and links 1 foot long, balls weighing 2 lbs. each, and a load of 100 lbs.; the valve is full open when the arms are at 30° to the vertical, and shut when at 45°. Find the extreme working speeds of the engine. *Ans.* 92 and 83·2 revs. per minute.

31. (Victoria, 1898.) The balls of a Porter governor weigh 4 lbs. each, and the load on it is 44 lbs. The arms of the links are so arranged that the load is raised twice the distance that the balls rise for any given increase of speed. Calculate the height of this governor for a speed of 180 revolutions per minute. Calculate also the force required to hold the sleeve for an increase of speed of 3%.

Ans. Height, 13·05 inches; force, 2·92 lbs.

32. Find the amount the sleeve rises in the case of a simple Watt governor when the speed is increased from 40 to 41 revolutions per minute. The sleeve rises twice as fast as the balls. Find the weight of each ball required to overcome a resistance on the sleeve of 1 lb. so that the increase in speed shall not exceed the above-mentioned amount.

Ans. 2·14 inches; 20 lbs.

33. Find the speed at which a crossed-arm governor runs when the arms make an angle of 30° with the vertical. The length of the arms from centre of pin to centre of ball = 29 inches; the points of suspension are 7 inches apart. *Ans.* 43 revs. per minute.

34. In a Wilson-Hartnell governor (Fig. 247) r_0 and $r = 6$ inches when the lower arm is horizontal $K = 12$, $W = 2$ lbs., $n = 1$. Find the speed at which the governor will float. *Ans.* 133 revs. per minute.

35. In a McLaren governor (Fig. 251), the weight W weighs 60 lbs., the radius of W about the centre of the shaft is 8 inches, the load on the spring $s = 1000$ lbs., the radius of the c. of g. of W about the point J is 9 inches, and the radius at which the spring is attached is 8 inches. Find the speed at which the governor will begin to act.

Ans. 256 revs. per minute.

36. A simple Watt governor 11 inches high lags behind to the extent of 10% of its speed when just about to lift. The weight of each ball is 15 lbs. The sleeve moves up twice as fast as the balls. Calculate the equivalent frictional resistance on the sleeve. *Ans.* 3·2 lbs.

37. In the governor mentioned in Question 30, find the total amount of energy stored. For what size of engine would such a governor be suitable if controlled by a throttle valve?

Ans. 32·44 foot-lbs. About 35 I.H.P.

38. Find the energy stored in the McLaren governor mentioned in Question 35. There are two sets of weights and springs; the weight moves out 3 inches, and the final tension on the spring is 1380 lbs.

Ans. 529 foot-lbs.

39. (London U., 1914.) A loaded governor has arms and links 13 inches long, the two rotating balls each weigh 3 lbs., and the central weight is 108 lbs. When the arms are inclined at an angle of 45° the steam valve is full open, and when the arms are inclined at 30° to the vertical the steam valve is shut.

Assuming that the frictional resistances to the motion of the sleeve are equivalent to a load of 2½ lbs., determine the extreme range of speed.

Ans. 381 and 336 revs. per minute; range 45 revs. per minute.

40. (London U., 1914.) In a spring controlled governor the radial force acting on the balls was 900 lbs. when the centre of the balls was

8 inches from the axis, and 1500 lbs. when at 12 inches, assuming that the force varies directly as the radius, find the radius of the ball-path when the governor runs at 270 revolutions per minute ; also find what alteration in the spring load is required in order to make the governor isochronous, and at what speed would it then run?

Weight of each ball = 60 lbs.

Ans. 11·5 inches. Initial load on spring, 300 lbs. Speed 297 revs. per minute when the governor is isochronous.

Chapter VII.

VIBRATION.

1. A particle moving with simple harmonic motion makes one complete oscillation in 4 seconds and has an amplitude of 2 feet. Find its velocity and acceleration after 0·2, 0·5, and 1·4 seconds respectively from one end of the swing.

Ans. 0·97, 2·22, 2·54 ft./sec. 4·69, 3·48, 2·90 ft./sec./sec.

2. (I.C.E., October, 1913.) An engine piston weighing 400 lbs. has a stroke of 2 feet, and makes 160 strokes per minute. If its motion is regarded as simple harmonic, find its velocity, kinetic energy, and acceleration when it has travelled a quarter stroke from one end position.

Ans. 7·25 ft./sec. ; 327 ft. lbs. ; 35·1 ft./sec./sec.

3. (I.C.E., October, 1911.) The table of a machine weighs 160 lbs. and moves horizontally with simple harmonic motion. The travel is 1 foot and the mean speed 80 feet per minute. Find the force required to overcome the inertia at the ends of the stroke and the kinetic energy at quarter stroke. *Ans.* 43·5 lbs., 8·2 feet lbs.

4. (I.C.E., February, 1912.) A vertical helical spring, whose weight is negligible, extended 1 inch by an axial pull of 100 lbs. A weight of 250 lbs. is attached to it and set vibrating axially. Find the time of a complete vibration. If the amplitude of the oscillation is 2 inches, find the kinetic energy when the weight is $\frac{3}{4}$ inch below the central position.

Ans. 0·505 second ; 14·82 feet lbs.

5. (I.C.E., February, 1911.) When a carriage underframe and body is mounted on its springs these are observed to deflect $1\frac{1}{2}$ inch. Calculate the time of a vertical oscillation. *Ans.* 0·392 second.

6. A simple pendulum is 5 feet long. How many complete swings will it make per minute? *Ans.* 24·4.

7. (I.C.E., February, 1914.) A connecting-rod weighing 200 lbs., when set vibrating round an axis through the centre of the small end, makes one "to and fro" vibration in 2·08 secs. Its centre of gravity is distant 33 inches from the centre of the small end. Find its moment of inertia about an axis parallel to the axis of vibration, and passing through the centre of gravity. *Ans.* 13·3 lbs. feet units.

8. A fan shaft 3·5 inches diameter, length between centres of swivelled bearings 7 feet, weight of fan disc 1170 lbs. Find the speed at which whirling will commence.

Ans. 725 revolutions per minute (an actual test gave 770).

9. (Victoria, 1900.) A cylindrical disc 4 inches diameter, 0·2 inches thick (weight 0·3 lb. per cubic inch) vibrates about its centre and is

controlled by the elastic twist of a vertical wire held rigidly at its upper end. The wire twists 20 degrees for a twisting moment of 0·01 pounds feet. Find the time of a complete oscillation of the disc, neglecting the mass of the wire. *Ans.* 0·67 seconds.

10. (Victoria, 1900.) Find the period of vibration of a uniform heavy bar, hinged at one end, so that it is free to turn in a vertical plane, and supported horizontally in equilibrium by a light vertical spring of such stiffness that 200 lbs. deflects it one foot. The spring is 4 feet from the hinge and 8 feet from the other end. The bar may be considered as rigid, its weight being 100 lbs. *Ans.* 1·33 seconds.

11. Calculate the approximate speed at which a plain cylindrical unloaded shaft will whirl when supported freely at each end.

Ans. 300 revolutions per minute.

12. Find the approximate whirling speed for a shaft which is parallel for a length of 14 inches in the middle and has a straight taper to each end. Diameter in middle, 10 inches; diameter at ends, 7 inches. Length, 9 feet. Central load, 8 tons. E = 12,000 tons per square inch.

Ans. 840 revolutions per minute.

13. A wheel weighing 3500 lbs. is mounted in the middle of a tapered shaft which is supported in swivelled bearings at each end. The taper is such that the moment of inertia of the shaft at every section varies directly as the bending moment at that section.

The diameter in the middle is 4 inches.

Length between bearings, 53 inches.

The weight of the shaft itself may be neglected.

Find the speed at which whirling will occur.

Ans. 903 revolutions per minute.

14. Find the frequency of natural torsional vibration in the case of an engine crank shaft which has (*a*) one fly-wheel weighing 4000 lbs. situated 15 inches from the middle of the crank-pin; (*b*) two fly-wheels weighing 2000 lbs. each, 30 inches apart symmetrically placed as regards the crank-pin. The radius of gyration in all cases is 28 inches, the diameter of the shaft is 4 inches. *Ans.* (*a*) 476; (*b*) 673 rev. per minute.

In case (*a*) if the engine ran at 238, 159, 119 revolutions per minute dangerous oscillations might be set up which might lead to disaster. In case (*b*) the corresponding speeds are 336, 224, 168 revolutions per minute.

15. A circular bar 1·125 inch diameter and 2 feet in length and weighing 0·28 lb. per cubic inch, is suspended horizontally by a vertical wire 10 feet long and 0·04 inch diameter. G = 10,000,000 lbs. square inch. Find the time of one vibration. *Ans.* 28·3 seconds.

16. Find the time of one complete oscillation of a helical spring, mean diameter of coils 5·37 inches, diameter of wire 0·627 inch, number of free coils 7·5, oscillating weight 311·7 lbs., weight of spring 13·8 lbs. G = 4400 tons per square inch. *Ans.* 0·44 second.

17. A thin ring 6 feet in diameter hangs on a horizontal peg. Find the time of a complete oscillation.

Ans. The equivalent length of a simple pendulum is 6 feet, hence the time is 2·71 seconds.

18. (London U.) A steel shaft $4\frac{1}{2}$ inches in diameter has two wheels weighing 3000 and 4000 lbs. respectively, fixed upon it at a distance of 42 inches apart. The radii of gyration of the wheels are 30 inches and 33 inches respectively.

Calculate the frequency of the natural torsional vibrations, and prove

the formula employed. The effect of the inertia of the shaft may be neglected.

Modulus of transverse elasticity or rigidity = 12,000,000 lbs. per square inch.

Ans. 494 per minute.

Chapter VIII.

GYROSCOPIC ACTION.

1. A rotor of a marine steam turbine revolves at 950 revolutions per minute and propels the vessel at a speed of 32 knots (1 knot = 6080 feet per hour). Find the gyroscopic moment exerted on the stator of the turbine when the vessel is moving in a circle of 1140 feet radius.

Weight of rotor . . . . . 10,000 lbs.
Radius of gyration of rotor . . . . . 1·45 ft.

Ans. 3077 pounds-feet.

2. (Lanchester in paper before the Incorporated Institute of Automobile Engineers.) A car of 10 feet wheel base travelling at 50 feet per second along a curve of 250 feet radius. Radius of gyration, 0·75 feet; speed of motor, 20 revolutions per second. Find the gyroscopic twisting moment.

Ans. 47 pounds-feet, or 4·7 pounds increase on one axle and decrease on the other.

3. If the same car experiences a side slip on a greasy road and turns through 180° in one second, find the gyroscopic twisting moment.

Ans. 690 pounds-feet.

4. The wheels of a bicycle weigh 7 lbs., the radius of gyration is 12·9 inches, and the diameter of the wheels 28 inches. Find the gyroscopic moment when turning a corner of 44 feet radius at a speed of 20 miles per hour. *Ans.* 4·2 pounds-feet.

5. (London U.) The revolving parts of a motor-car engine rotate clockwise when looked at from the front of the car, and have a moment of inertia of 400 in pound and foot units. The car is being steered in a circular path of 400 feet radius at 12 miles per hour, and the engine runs at 800 revolutions per minute. (*a*) What are the effects on the steering and driving axles due to gyroscopic action? The distance between these axles is 8 feet. (*b*) Suppose the car to be turned and driven in the reverse direction over the same curve at the same speed, what will be the effects on the axles?

Ans. The load on the steering axle will be reduced (increased), and that on the back axle increased (reduced), when the car turns to the right (left) to the extent of 5·7 lbs.

6. The wheels of a motor bicycle weigh 16 lbs., the radius of gyration is 12·9 inches, and the diameter of the wheels 28 inches. There is a plain disc flywheel 12 inches diameter on the engine, weighing 30 lbs., which makes 2000 revolutions per minute, in a plane parallel to the road wheels. Find the gyroscopic moment when rounding a corner of 70 feet radius at a speed of 40 miles per hour, when the flywheel rotates in (*a*) the same direction; (*b*) the opposite direction to the road-wheels.

Ans. (*a*) 40·8; (*b*) 3·4 pounds-feet. Thus, if the machine and rider together weighed 350 pounds, the centre of gravity will have to be $\frac{40{\cdot}8 \times 12}{350} = 1{\cdot}4$ inch further from the vertical than if centrifugal force only were acting.

Chapter IX.

Friction.

1. (S. and A., 1896.) A weight of 5 cwt., resting on a horizontal plane, requires a horizontal force of 100 lbs. to move it against friction. What is the coefficient of friction? *Ans.* 0·179.

2. (S. and A., 1891.) The saddle of a lathe weighs 5 cwt.; it is moved along the bed of the lathe by a rack-and-pinion arrangement. What force, applied at the end of a handle 10 inches in length, will be just capable of moving the saddle, supposing the pinion to have twelve teeth of 1½ inch pitch, and the coefficient of friction between the saddle and lathe bed to be 0·1, other friction being neglected?

Ans. 13·37 lbs.

3. A block rests on a plane which is tilted till the block commences to slide. The inclination is found to be 8·4 inches at starting, and afterwards 6·3 inches on a horizontal length of 2 feet. Find the coefficient of friction when the block starts to slide, and after it has started.

Ans. 0·35; 0·26.

4. In the case of the block in the last question, what would be the acceleration if the plane were kept in the first-mentioned position?

Ans. 2·7 feet per second per second.

5. A block of wood weighing 300 lbs. is dragged over a horizontal metal plate. The frictional resistance is 126 lbs. What would be the probable frictional resistance if it be dragged along when a weight of 600 lbs. is placed on the wooden block, (i.) on the assumption that the value of μ remains constant; (ii.) that it decreases with the intensity of pressure as given on p. 230. *Ans.* (i.) 378 lbs.; (ii.) 308 lbs.

6. In an experiment, the coefficient of friction of metal on metal was found to be 0·2 at 3 feet per second. What will it probably be at 10 feet per second? Find the value of K as given on p. 288.

Ans. 0·11; $K = 0·83$.

7. (S. and A., 1897.) A locomotive with three pairs of wheels coupled weighs 35 tons; the coefficient of friction between wheels and rails is 0·18. Find the greatest pull which the engine can exert in pulling itself and a train. What is the total weight of itself and train which it can draw up an incline of 1 in 100, if the resistance to motion is 12 lbs. per ton on the level? *Ans.* 6·3 tons; 410 tons.

8. (Victoria, 1896.) Taking the coefficient of friction to be μ, find the angle θ at which the normal to a plane must be inclined to the vertical so that the work done by a horizontal force in sliding a weight w up the plane to a height h may be $2wh$. *Ans.* 10° 35′.

9. A body weighing 1000 lbs. is pulled along a horizontal plane, the coefficient of friction being 0·3; the line of action being (i.) horizontal; (ii.) at 45°; (iii.) such as to give the least pull. Find the magnitude of the pull, and the normal pressure between the surfaces.

Ans. Pull, (i.) 300 lbs., (ii.) 326 lbs., (iii.) 287 lbs. Normal pressure, (i.) 1000 lbs., (ii.) 770 lbs., (iii.) 917 lbs.

10. A horse drags a load weighing 35 cwt. up an incline of 1 in 20. The resistance on the level is 100 lbs. per ton. Find the pull on the traces when they are (i.) horizontal, (ii.) parallel with the incline, (iii.) in the position of least pull.

Ans. (i.) 371·7 lbs., (ii.) 370·67 lbs., (iii.) 370·04.

11. A cotter, or wedge, having a taper of 1 in 8, is driven into a

cottered joint with an estimated pressure of 600 lbs. Taking the coefficient of friction between the surfaces as 0·2, find the force with which the two parts of the joint are drawn together, and the force required to withdraw the wedge. *Ans.* 1128 lbs.; 307 lbs.

12. Find the mechanical efficiency of a screw-jack in which the load rotates with the head of the jack in order to eliminate collar friction. Threads per inch, 3; mean diameter of threads, $1\frac{3}{4}$ inch; coefficient of friction, 0·14. Also find the efficiency when the load does not rotate.
Ans. 30 per cent.; 17·1 per cent.

13. (Victoria, 1897.) Find the turning moment necessary to raise a weight of W lbs. by a vertical square-threaded screw having a pitch of 6 inches, the mean diameter of the thread being 4 inches, and the coefficient of friction $\frac{1}{8}$. *Ans.* 1·93 W lbs.-inches.

14. (Victoria, 1898.) Find the tension in the shank of a bolt in terms of the twisting moment T on the shank when screwing up, (i.) when the thread is square, (ii.) when the angle of the thread is 55°; neglecting the friction between the head of the bolt and the washer; taking r = the outside radius of the thread, a = the depth of thread, n = the number of threads to an inch, f = the coefficient of friction.

$$\textit{Ans.}\ \text{(i.)}\ \frac{2\pi \mathrm{T}n}{1+2\pi nf\left(r-\frac{a}{2}\right)}\ \text{(ii.)}\ \frac{2\pi \mathrm{T}n}{1+2{\cdot}26\pi nf\left(r-\frac{a}{2}\right)}$$

In the answers given it was assumed as a close approximation that $\tan(\alpha+\phi)=\tan\alpha+\tan\phi$. An exact solution for (i.) gives—

$$\frac{2\pi n\mathrm{T}\left(r-\frac{a}{2}\right)-f\mathrm{T}}{\left(r-\frac{a}{2}\right)\left[2\pi nf\left(r-\frac{a}{2}\right)+1\right]}$$

The approximate method in a given instance gave 1675 lbs., and the exact method 1670 lbs. The simpler approximate solution is therefore quite accurate enough in practice.

15. The mean diameter of the threads of a $\frac{1}{2}$-inch bolt is 0·45 inch, the slope of the thread 0·07, and the coefficient of friction 0·16. Find the tension on the bolt when pulled up by a force of 20 lbs. on the end of a spanner 12 inches long. *Ans.* 1920 lbs.

16. The rolling resistance of a waggon is found to be 73 lbs. per ton on the level; the wheels are 4 feet 6 inches diameter. Find the value of K.
Ans. 0·88.

17. A 4-inch axle makes 400 revolutions per minute on anti-friction wheels 30 inches diameter, which are mounted on 3-inch axles. The load on the axle is 5 tons; $\mu = 0{\cdot}1$; $\theta = 30°$; $\mathrm{K} = 0{\cdot}01$. Find the horse-power absorbed. *Ans.* 1·7.

18. A horizontal axle 10 inches diameter has a vertical load upon it of 20 tons, and a horizontal pull of 4 tons. The coefficient of friction is 0·02. Find the heat generated per minute, and the horse-power wasted in friction, when making 50 revolutions per minute.
Ans. 155 thermal units; 3·63 H.P.

19. Calculate the length required for the two necks in the case of the axle given in the last question, if placed in a tolerably cool place ($t = 0{\cdot}3$). *Ans.* 26 inches.

20. Calculate the horse-power of the bearing mentioned in Question 18 by the rough method given on p. 329, taking the resistance as 2 lbs. per square inch. *Ans.* 4·13 H.P.

21. Calculate the horse-power absorbed by a footstep bearing 8 inches diameter when supporting a load of 4000 lbs., and making 100 revolutions per minute. $\mu = 0{\cdot}03$, with (i.) a flat end; (ii.) a conical pivot; $\alpha = 30°$; (iii.) a Schiele pivot. $t = R_1$, $R_2 = \frac{R_1}{2}$.

Ans. (i.) 0·51; (ii.) 1·02; (iii.) 0·76.

22. The efficiency of a single-rope pulley is found to be 94%. Over how many of such pulleys must the rope pass in order to make it self-sustaining, *i.e.* to have an efficiency of under 50%. *Ans.* 12 pulleys.

23. In a three-sheaved pulley-block, the pull W on the rope was 110 lbs., and the weight lifted, W_u, was 369 lbs. What was the mechanical efficiency? and if the friction were 80 % of its former value when reversed, what would be the reversed efficiency, and what resistance would have to be applied to the rope in order to allow the weight to gently lower? *Ans.* 55·9 per cent.; 36·9 per cent., 22·7 lbs.

(N.B.—The probable friction when reversed may be roughly arrived at thus: The total load on the blocks was 369 + 110 = 479 lbs., when raising the load. Then, calculating the resistance when lowering, assuming the friction to be the same, we get 369 + 22·7 = 391·7 lbs. Assuming the friction to vary as the load, we get the friction when lowering to be $\frac{391{\cdot}7}{479}$ = 0·82. This is only a rough approximation, but experiments on pulleys show that it holds fairly well. In the question above, the experiment gave 23·4 lbs. resistance against 22·7 lbs., and the efficiency as 38 % against 36·9 %. Many other experiments agree equally well.)

24. (S. and A., 1896.) A lifting tackle is formed of two blocks, each weighing 15 lbs.; the lower block is a single movable pulley, and the upper or fixed block has two sheaves. The cords are vertical, and the fast end is attached to the movable block. Sketch the arrangement, and determine what pull on the cord will support 200 lbs. hung from the movable block, and also what will then be the pressure on the point of support of the upper block. *Ans.* 71⅔ lbs. pull; 301⅔ lbs. on support.

25. (S. and A., 1896.) If in a Weston pulley block only 40 % of the energy expended is utilized in lifting the load, what would require to be the diameter of the smaller part of the compound pulley when the largest diameter is 8 inches in order that a pull of 50 lbs. on the chain may raise a load of 550 lbs.? *Ans.* 7·42 inches.

26. Find the horse-power that may be transmitted through a conical friction clutch, the slope of the cone being 1 in 6, and the mean diameter of the bearing surfaces 18 inches. The two portions are pressed together with a force of 170 lbs. The coefficient of friction between the surfaces is 0·15. Revolutions per minute, 200. *Ans.* 4·4.

27. An engine is required to drive an overhead travelling crane for lifting a load of 30 tons at 4 feet per minute. The power is transmitted by means of 2¼-inch shafting, making 160 revolutions per minute. (For the purposes of this question use the expression at foot of p. 344.) The length of the shafting is 250 feet; the power is transmitted from the shaft

through two pairs of bevel wheels (efficiency 90 % each including bearings) and one worm and wheel having an efficiency of 85 % including its bearings. Taking the mechanical efficiency of the steam-engine at 80 %, calculate the required I.H.P. of the engine. *Ans.* 22.

28. (I.C.E., October, 1897.) When a band is slipping over a pulley, show how the ratio of the tensions on the tight and slack sides depends on the friction and on the angle in contact. Apply your result to explain why a rope exerting a great pull may be readily held by giving it two or three turns round a post.

29. (S. and A., 1896.) What is the greatest load that can be supported by a rope which passes round a drum 12 inches in diameter of a crab or winch which is fitted with a strap friction brake worked by a lever, to the long arm of which a pressure of 60 lbs. is applied? The diameter of the brake pulley is 30 inches, and the brake-handle is 3 feet in length from its fulcrum; one end of the brake strap is immovable, being attached to the pin forming the fulcrum of the brake-handle, while the other end of the strap is attached to the shorter arm, 3 inches in length, of the brake-lever. The angle $\alpha = \frac{3\pi}{2}$. The gearing of the crab is as follows: On the shaft which carries the brake-wheel is a pinion of 15 teeth, and this gears into a wheel of 50 teeth on the second shaft; a pinion of 20 teeth on this latter shaft gears into a wheel with 60 teeth carried upon the drum or barrel shaft. $\mu = 0{\cdot}1$. *Ans.* 4·83 tons.

30. (S. and A., 1896.) The table of a small planing-machine, which weighs 1 cwt., makes six double strokes of 4½ feet each per minute. The coefficient of friction between the sliding surfaces is 0·07. What is the work performed in foot-pounds per minute in moving the table?

Ans. 423·3.

31. (S. and A., 1897.) A belt laps 150° round a pulley of 3 feet diameter, making 130 revolutions per minute; the coefficient of friction is 0·35. What is the maximum pull on the belt when 20 H.P. is being transmitted and the belt is just on the point of slipping? *Ans.* 898 lbs.

32. (Victoria, 1897.) Find the width of belt necessary to transmit 10 H.P. to a pulley 12 inches in diameter, so that the greatest tension may not exceed 40 lbs. per inch of the width when the pulley makes 1500 revolutions per minute, the weight of the belt per square foot being 1·5 lbs., taking the coefficient of friction as 0·25. *Ans.* 8 inches.

33. A strap is hung over a fixed pulley, and is in contact over an arc of length equal to two-thirds of the total circumference. Under these circumstances a pull of 475 lbs. is found to be necessary in order to raise a load of 150 lbs. Determine the coefficient of friction between the strap and the pulley-rim. *Ans.* 0·275.

34. Power is transmitted from a pulley 5 feet in diameter, running at 110 revolutions per minute, to a pulley 8 inches in diameter. Thickness of belt = 0·24 inch; modulus of elasticity of belt, 9000 lbs. per square inch; tension on tight side per inch of width = 60 lbs.; ratio of tensions, 2·3 to 1. Find the revolutions per minute of the small pulley. *Ans.* 792.

35. How many ropes, 4 inches in circumference, are required to transmit 200 H.P. from a pulley 16 feet in diameter making 90 revolutions per minute? *Ans.* 10.

36. Find the horse-power that can be transmitted through a friction clutch having conical surfaces under the following conditions:—

Mean diameter of coned surfaces	18 inches
Apex angle of cone	30 degrees
Coefficient of friction	0·24
Axial pressure on cone	550 lbs.
Revolutions per minute	200

Ans. 14·5.

37. Find the maximum horse-power transmitted by a rope running in grooved pulleys under the following conditions :—

Speed of rope in feet per minute	5000
Coefficient of friction	0·25
Angle of groove	45 degrees
Angle embraced by rope	200 degrees
Weight of rope per foot run	0·28 lb.
Maximum tension in rope	200 lbs.

Ans. 19·5.

38. A load of 10 tons has to be gently lowered from a pier into a boat by one man who has no appliances beyond a rope. The vertical loaded portion of the rope hangs over the rounded edge of the pier and from thence passes horizontally to a vertical cylindrical post around which it is wound a sufficient number of times to reduce the tension at the slack end to 30 lbs. The coefficient of friction on the pier edge and on the post is 0·3. Find the number of turns the rope must be wound round the post.

Ans. 3·28.

39. Find the maximum working load for a thrust ball bearing with grooved races which contains 12 balls $1\frac{1}{4}$ inch diameter, the diameter of the ball race being 6 inches. Speed (*a*) 10 ; (*b*) 100 ; (*c*) 1000; (*d*) 10,000 revolutions per minute.

Ans. (*a*) 75,600 lbs. ; (*b*) 27,500 lbs. ; (*c*) 3750 lbs ; (*d*) 390 lbs.

40. Referring to the data given in the last question, find the corresponding loads for a cylindrical bearing.

Ans. (*a*) 18,300 lbs. ; (*b*) 15,100 lbs. ; (*c*) 5510 lbs. ; (*d*) 750 lbs.

41. A steel ball $1\frac{1}{2}$ inch diameter is compressed between two flat plates, find the amount they approach one another under the following loads. (*a*) 50 lbs. ; (*b*) 100 lbs. ; (*c*) 200 lbs. ; (*d*) 500 lbs.

Ans. (*a*) 0·00037 inch ; (*b*) 0·00057 ; (*c*) 0·00093 ; (*d*) 0·00172.

42. Find the maximum working load for a roller bearing of cheap design, *i.e.* cast-iron casing with no liner, soft steel rollers, soft shaft. (See *Proc. Inst. C. E.*, vol. clxxxix.) Speed, 150 r.p.m. Length of rollers, 4 inches. Diameter of rollers, 0·75 inch. Number of rollers, 14. Diameter of shaft, 4 inches. *Ans.* 6000 lbs.

Chapter X.

Stress, Strain, and Elasticity.

1. Plot a stress-strain and a real stress diagram for the following test : Scales—elastic strains, 2000 times full size ; permanent strains, twice full size ; loads, 10,000 lbs. to an inch.

Calculate the stress at the elastic limit, the maximum stress; the percentage of extension on 10 inches; the reduction in area; the work done in fracturing the bar. Compare the calculated work with that obtained from the diagram; the modulus of elasticity (mean up to 28,000 lbs.). Original dimensions: Length, 10 inches; width, 1·753 inch; thickness, 0·611 inch. Final dimensions: Length, 12·9 inches; width, 1·472 inch; thickness, 0·482 inch.

Loads in pounds ...	4000	8000	12,000	16,000	20,000
Extension in inches	0·0009	0·0021	0·0033	0·0045	0·0057
24,000	28,000	32,000	34,000	36,000	40,000
0·0069	0·0082	0·0103	0·016	0·07	0·19
44,000	48,000	52,000	56,000	59,780	54,900
0·30	0·47	0·75	1·36	2·5	2·9

Ans. Elastic limit, 15 tons square inch. Maximum stress, 24·92. Extension, 29 per cent. on 10 inches. Reduction, 34 per cent. Work (by calculation), 6·27 inch-tons per cubic inch. Modulus of elasticity, 31,000,000 lbs. per square inch; 13,840 tons per square inch.

2. (I.C.E., October, 1897.) Distinguish between stress and strain. An iron bar 20 feet long and 2 inches in diameter is stretched $\frac{1}{20}$ of an inch by a load of 7 tons applied along the axis. Find the intensity of stress on a cross-section, and the coefficient of elasticity of the material (E).

Ans. Stress, 2·23 tons per sq. inch; E = 10,700 tons per sq. inch.

3. (I.C.E., October, 1897.) A bar 4" × 2" in cross-section is subjected to a longitudinal tension of 40 tons. Find the normal and shearing stresses on a section inclined at 30° to the axis of the bar.

Ans. Normal, 1·25 tons per square inch; tangential, 2·16 tons per square inch.

4. A bar of steel 4" × 1" is rigidly attached at each end to a bar of brass 4" × $\frac{3}{8}$"; the combined bar is then subjected to a load of 20 tons. Find the load taken by each bar. E for steel = 13,000 tons per square inch; brass, 4000 tons per square inch.

Ans. Load on steel bars, 17·93 tons; load on brass bar, 2·07 tons.

5. The nominal tensile stress (reckoned on the original area) of a bar of steel was 32·4 tons per square inch, the reduction in area at the point of fracture was 54%. What would be the approximate tensile strength of hard drawn wire made from such steel? *Ans.* 70 tons per square inch.

6. Plot a stress-strain and a real stress diagram for the following compression test of a specimen of copper. Scale of loads, 5 tons per inch; strain twice full size.

Loads in tons	13	14	15	16	18	20
Length of specimen in inches	2·50	2·47	2·29	2·19	2·02	2·86
22	24	26	28	30	32	34	36	38
1·73	1·61	1·52	1·43	1·35	1·30	1·23	1·17	1·11
40	42	44	46	48	50			
1·08	1·02	0·99	0·96	0·92	0·90			

Original length, 2·52 inches; diameter, 2·968 inches.

7. (S. and A., 1896.) A bar of iron is at the same time under a direct

tensile stress of 5000 lbs. per square inch, and to a shearing stress of 3500 lbs. per square inch. What would be the resultant equivalent tensile stress in the material? *Ans.* 6800 lbs. per square inch.

8. (I.C.E., October, 1897.) Explain what is meant by Poisson's Ratio. A cube of unit length of side has two simple normal stresses p_1 and p_2 on pairs of opposite faces. Find the length of the sides of the cube when deformed by the stresses (tensile).

$$Ans.\ 1 + \frac{1}{E}\left(p_1 - \frac{p_2}{n}\right);\ 1 + \frac{1}{E}\left(p_2 - \frac{p_1}{n}\right);\ 1 - \frac{1}{En}\left(p_1 + p_2\right)$$

9. Find the pitch of the rivets for a double row lap joint. Plates $\frac{1}{2}$ inch thick; rivets, 1 inch diameter; clearance, $\frac{1}{16}$ inch; thickness of ring damaged by punching, $\frac{1}{16}$ inch; $f_t = 23$ tons per square inch; $f_r = 25$ tons per square inch. *Ans.* 4·27 inches.

10. Calculate the bearing pressure on the rivets when the above-mentioned joint is just about to fracture. *Ans.* 33·3 tons per square inch.

11. Calculate the pitch of rivets for a double cover plate riveted joint with diamond riveting as in Fig. 396, the one cover plate being wide enough to take three rows of rivets, but the other only two rows; thickness of plates, $\frac{7}{8}$ inch; drilled holes; material steel; the pitch of the outer row of the narrow cover plate must not exceed six times the diameter of the rivet. What is the efficiency of the joint and the bearing pressure?

Ans. 10·84 inches outer row, 5·42 inches inner row; 88·7 per cent., 50·5 tons per square inch.

12. Two lengths of a flat tie-bar are connected by a lap riveted joint. The load to be transmitted is 50 tons. Taking the tensile stress in the plates at 5 tons per square inch, the shear stress in the rivets at 4 tons per square inch, and the thickness of the plates as $\frac{3}{4}$ inch; find the diameter and the number of rivets required, also the necessary width of bar for both the types of joint as shown in Figs. 399 and 400. What is the efficiency of each, and the working bearing pressure?

Ans. 0·94 inch. 18 are sufficient, but 20 must be used for convenience; 17 inches and 14¼ inches; 78 and 93·4 per cent.; 3·6 tons per square inch.

13. Find the thickness of plates required for a boiler shell to work at a pressure of 160 lbs. per square inch: diameter of shell, 8 feet; efficiency of riveted joint, 89%; stress in plates, 5 tons per square inch.

Ans. 0·77 inch, or say $\frac{25}{32}$ inch.

14. Find the maximum and minimum stress in the walls of a thick cylinder; internal diameter, 8 inches; external diameter, 14 inches; internal fluid pressure, 2000 lbs. per square inch. (Barlow's Theory.)

Ans. 4670 lbs. per square inch; 1520 lbs. per square inch.

15. A thick cylinder is built up in such a manner that the initial tensile stress on the outer skin and the compressive stress on the inner skin are both 3 tons per square inch. Calculate the resultant stress on both the outer and the inner skin when under pressure. Internal fluid pressure, 4·5 tons per square inch. (Barlow's Theory.) Internal diameter, 6 inches; external diameter, 15 inches. *Ans.* 4·2 and 4·5 tons per square inch.

16. Find the strain energy stored per square foot of plate in the cylindrical shell of a boiler 6 feet 6 inches diameter, $\frac{5}{8}$ inch thick, steam-pressure 150 lbs. per square inch.

Young's modulus, 13,000 tons per square inch.
Poisson's ratio = $\frac{1}{4}$. *Ans.* 135 inch lbs.

17. Find the normal and tangential components of stress on a section inclined at 45° to the axis of a boiler shell. Diameter, 8 feet; thickness, $\frac{3}{4}$ inch; steam pressure, 160 lbs. per square inch.

Ans. 7680 lbs. square inch, normal; 2560, tangential.

18. (London U., 1913.) A symmetrically shaped body weighing 20 tons rests on three vertical cylinders placed in line at equal intervals and of approximately 1·5 inch in height. The block and the base on which the cylinders rest may be regarded as rigid. The centre of gravity of the block is immediately over the central cylinder. The diameter of each cylinder is 1 inch.

Find the load supported by each cylinder when the central cylinder is 0·001 inch (*a*) shorter, (*b*) longer than the others.

Young's Modulus of Elasticity: 12,000 tons per square inch.

Ans. Two outer cylinders (*a*) 8·75 tons each. (*b*) 4·6 tons.
Central cylinder (*a*) 2·5 tons. (*b*) 10·8 tons.

19. The covers are held in position on the ends of a piece of pipe by means of a central bolt. Find by how much the nut must be screwed up in order to subject the pipe to a compressive stress of 3 tons per square inch.

Length of pipe to outside of covers	32 in.
Internal diameter of pipe	2·4 in.
External ,, ,, ...	3·2 in.
Diameter of bolt	1·75 in.
Modulus of elasticity of bolt .	13,000 tons square inch
Modulus of elasticity of pipe .	6,000 ,, ,,

The strain on the covers may be neglected, also any effect of the reduced sectional area at the bottom of the thread. *Ans.* 0·0268 inch.

20. Find the pressure required to produce permanent deformation in a steel cylinder cover, thickness $\frac{7}{8}$ inch; diameter 14 inches; elastic limit of steel, 16 tons per square inch. *Ans.* 560 lbs. per square inch.

21. Find the thickness of a flat cast-iron rectangular cover 18″ × 12″ to withstand a water pressure of 80 lbs. per square inch. Stress 1·5 ton per square inch. *Ans.* 1·1 inch.

22. What intensity of wind pressure would be required to blow in a window pane 50″ × 20″, thickness 0·1 inch. Tensile strength of glass 2000 lbs per square inch. *Ans.* 16·7 pounds per square foot.

23. An empirical formula used by boiler designers for finding the largest flat area that may remain unstayed is

$$S = \frac{75{,}000t^2}{p},$$

where S is the area in square inches of the largest circle that can be drawn between the stays on a flat surface, t is the thickness of the plates in inches, p is the working pressure in pounds per square inch.

What intensity of stress in the plates does this represent?

Ans. 23,900 lbs. square inch.

Chapter XI.

Beams.

1. (Victoria, 1897.) Find the greatest stress which occurs in the section of a beam resting on two supports, the beam being of rectangular section, 12 inches deep, 6 inches wide, carrying a uniform load of 5 cwt. per foot run, span 35 feet. *Ans.* 3·12 tons square inch.

2. Rolled joists are used to support a floor which is loaded with 150 lbs. per square foot including its own weight. The pitch of the joists is 3 feet; span, 20 feet; skin stress, 5 tons per square inch. Find the required Z and a suitable section for the joists, taking the depth at not less than $\frac{1}{25}$ of the span. *Ans.* Z = 24·1; say 10″ × 5″ × ½″.

3. A rectangular beam, 9 inches deep, 3 inches wide, supports a load of ½ ton, concentrated at the middle of an 8-foot span. Find the maximum skin stress. *Ans.* 0·3 ton square inch.

4. A beam of circular section is loaded with an evenly distributed load of 200 lbs. per foot run; span, 14 feet; skin stress, 5 tons per square inch. Find the diameter. *Ans.* 3·77 inches.

5. Calculate approximately the safe central load for a simple web riveted girder, 6 feet deep; flanges, 18 inches wide, 2½ inches thick. The flange is attached to the web by two 4″ × ½″ angles. Neglecting the strength of the web, and assuming that the section of each flange is reduced by two rivet-holes ⅞ inch diameter passing through the flange and angles. Span of girder, 50 feet; stress in flanges, 5 tons per square inch. *Ans.* 110 tons.

6. Find the relative weights of beams of equal strength having the following sections: Rectangular, $h = 3b$; Square; circular; rolled joist, $h_1 = 2b_1 = 12t$.

Ans. Rectangular, 1·00; square, 1·44; circular, 1·61; joist, 0·54.

7. Find the safe distributed load for a cast-iron beam of the following dimensions: Top flange, 3″ × 1″; bottom flange, 8″ × 1·5″; web, 1·25 inch thick; total depth, 10 inches; with (i.) the bottom flange in tension, (ii.) when inverted; span, 12 feet; skin stress, 3000 lbs. per square inch. *Ans.* (i.) 5·7; (ii.) 3·2 tons.

8. In the last question, find the safe central load for a stress of 3000 lbs. per square inch, including the stress due to the weight of the beam. The beam is of constant cross-section. *Ans.* (i.) 2·65; (ii.) 1·4 ton.

9. (S. and A., 1896.) If a bar of cast iron, 1 inch square and 1 inch long, when secured at one end, breaks transversely with a load of 6000 lbs. suspended at the free end, what would be the safe working pressure, employing a factor of safety of 10, between the two teeth which are in contact in a pair of spur-wheels whose width of tooth is 6 inches, the depth of the tooth, measured from the point to the root, being 2 inches, and the thickness at the root of the tooth 1⅛ inch? (Assume that one tooth takes the whole load.) *Ans.* 4050 lbs.

10. (S. and A., 1897.) Compare the resistance to bending of a wrought-iron I section when the beam is placed like this, I, and like this, ⊢. The flanges of the beam are each 6 inches wide and 1 inch thick, and the web is ¾ inch thick and measures 8 inches between the flanges.

Ans. 4·56 to 1.

11. A trough section, such as that shown in Fig. 425, is used for the flooring of a bridge; each section has to support a uniformly distributed load of 150 lbs. per square foot, and a concentrated central load of 4 tons. Find the span for which such a section may be safely used. Skin stress = 5 tons per square inch; pitch of corrugation, 2 feet; depth, 1 foot; width of flange (B, Fig. 425) = 8 inches; thickness = $\frac{5}{8}$ inch.

Ans. 19 feet.

12. A 4″ × 4″ × $\frac{1}{2}$″ ⊥ section is used for a roof purlin, the load being applied on the flanges; the span is 12 feet; the evenly distributed load is 100 lbs. per foot run. Find the skin stress at the top and the bottom of the section.

Ans. Top, 4·9 tons square inch compression; bottom, 2·1 tons square inch tension.

13. A triangular knife-edge of a weighing-machine overhangs $1\frac{1}{2}$ inch, and supports a load of 2 tons (assume evenly distributed). Taking the triangle to be equilateral, find the requisite size for a tensile stress at the apex of 10 tons per square inch. *Ans.* S = 1·84 inch.

14. A cast-iron water main, 30 inches inside diameter and 32 inches outside, is unsupported for a length of 12 feet. Find the stress in the metal due to bending. *Ans.* 180 lbs. square inch.

15. In the case of a tram-rail, the area A of one part of the modulus figure is 4·12 square inches, and the distance D between the two centres of gravity is 5·55 inches; the neutral axis is situated at a distance of 3·1 inches from the skin of the bottom flange. Find the I and Z.

Ans. 70·9; 22·86.

16. Find the Z for the sections given on pp. 454, 455, and 456, which are drawn to the following scales: Figs. 439, 440, and 441, 4 inches = 1 foot; Fig. 442, 3 inches = 1 foot; Fig. 443, 1 inch = 1 foot; Fig. 444, $\frac{1}{4}$ full size; Fig. 445, $\frac{1}{3{\cdot}5}$ full size; Fig. 446, $\frac{1}{2}$ inch = 1 foot. (You may assume that the crosses correctly indicate the c. of g. of each figure.) The areas must be measured by a planimeter or by one of the methods given in Chapter II. *Ans.* 4·86; 6·03; 17·3; 8·78; 36·3; 15·5; 11·4; 460.

17. When testing a 9″ × 9″ timber beam, the beam split along the grain by shearing along the neutral axis under a central load of 15 tons. Calculate the shear stress. *Ans.* 310 lbs. square inch.

18. Calculate the ratio of the maximum shear stress to the mean in the case of a square beam loaded with one diagonal vertical.

Ans. 1; *i.e.* they are equal.

19. Calculate the central deflection of a tram rail due to (i.) bending, (ii.) shear, when centrally loaded on a span of 3 feet 6 inches with a load of 10 tons. E = 12,300 tons square inch. I = 80·5 inch units; A = 10·5 square inches. G = 4900 tons square inch; K = 4·03.

Ans. (i.) 0·016 inch; (ii.) 0·008 inch.

20. Find the position of the neutral axis, also the moments of resistance of a reinforced concrete beam of rectangular section 14 inches deep, 6 inches wide; area of rods, 2·8 square inches; centre of rods, 1 inch from the tension edge of the concrete. Ratio of moduli of elasticity, 12. Working stress in rods, 6 tons per square inch, and 0·25 tons per square inch in concrete. *Ans.* The N.A. is 7·7 inches from the compression skin. Concrete moment = 60·2 inch tons. Rod moment = 175·2.

21. An open-topped rectangular reinforced concrete conduit is used for conveying water over a roadway; it has a clear span of 48 feet. Find the stress in the concrete and in the reinforcements when the depth of water in the conduit is 4 feet 6 inches.

External depth	8 feet
Internal depth	5 ,,
External width	6 ,,
Internal width	4 ,,

The reinforcements consist of 20 rods $\frac{3}{4}$ inch diameter, the centres of which are 2 inches from the bottom of the section. The rods take the whole of the tension.

Weight of concrete	140 lbs. per cubic foot
Ratio of moduli of elasticity	12

Ans. 0·31 tons per square inch in concrete; 10·3 in rods.

22. The evenly distributed load on an overhung crank-pin is 75,000 lbs. The intensity of pressure on the projected area is 700 lbs. per square inch. The bending stress in the pin is 8000 lbs. per square inch, the shearing stress may be neglected. Find the diameter and length of the pin.
Ans. Diameter 8·46", length 12·7".

23. Find the tension and compression Z's for an unequal I section. Flanges, 3" × 1·5" and 12" × 2". Web, 1·75" thick; overall depth, 10".
Ans. 120; 55.

24. Find the "modulus of the section" of a solid circular beam section 12 inches diameter, having a 4-inch hole through the middle, *i.e.* across the beam and coinciding with the neutral axis. *Ans.* 158·7.

25. Find the Z_c and Z_t for a rectangular section 10 inches deep, 3 inches wide, having a horizontal hole $1\frac{1}{2}$ inch diameter through the section parallel to the neutral axis and having its centre 3 inches from the top edge.
Ans. 42·6 and 49 respectively.

26. Find the Z for a corrugated floor, having semicircular corrugations of 6 in. pitch. Thickness 0·5 inch. *Ans.* 3·1 per corrugation.

Chapter XII.

Bending Moments and Shear Forces.

1. (Victoria, 1897.) The total load on the axle of a truck is 6 tons. The wheels are 6 feet apart, and the two axle-boxes 5 feet apart. Draw the curve of bending moment on the axle, and state what it is in the centre. *Ans.* 18 tons-inches.

2. (Victoria, 1896.) A beam 20 feet long is loaded at four points equi-distant from each other and the ends, with equal weights of 3 tons. Find the bending moment at each of these points, and draw the curve of shearing force. *Ans.* 24, 36, 36, 24 tons-feet.

3. (I.C.E., October, 1898.) In a beam ABCDE, the length (AE) of 24 feet is divided into four equal panels of 6 feet each by the points B, C, D. Draw the diagram of moments for the following conditions of

loading, writing their values at each panel-point : (i.) Beam supported at A and E, loaded at D with a weight of 10 tons ; (ii.) beam supported at B and D, loaded with 10 tons at C, and with a weight of 2 tons at each end A and E ; (iii.) beam *encastré* from A to B, loaded with a weight of 2 tons at each of the points C, D, and E.

Ans. (i.) $M_B = 15$, $M_C = 30$, $M_D = 45$ tons-feet.
(ii.) M_B and $M_D = 12$, $M_C = 18$,,
(iii.) $M_B = 72$, $M_C = 36$, $M_D = 12$,,

4. Find the bending moment and shear at the abutment, also at a section situated 4 feet from the free end in the case of a cantilever loaded thus : length, 12 feet ; loads, 3 tons at extreme end, $1\frac{1}{2}$ ton 2 feet from end, 4 tons 3 feet 6 inches from end, 8 tons 7 feet from end.

Ans. M at abutment, 125 tons-feet ; ditto 4 feet from end, 17 tons-feet.
Shear ,, 16·5 tons ,, ,, 8·5 ,,

5. Construct bending moment and shear diagrams for a beam 20 feet long resting on supports 12 feet apart. The left-hand support is 3 feet from the end. The beam is loaded thus : 2 tons at the extreme left-hand end, 1 ton 2·5 feet from it, 3 tons 5 feet from it, 8 tons 12 feet from it, and 6 tons on the extreme right-hand end. Write the values of the bending moment under each load.

Ans. 30 tons-feet at the right abutment ; 4·62 tons-feet under the 8-ton load ; 1·42 under the 3-ton load ; 6·5 over the left abutment.

6. A beam arranged with symmetrical overhanging ends, as in Fig. 485, is loaded with three equal loads—one at each end and one in the middle. What is the distance apart of the supports, in terms of the total length l, when the bending moment is equal over the supports and in the middle, and at what sections is the bending moment zero?

Ans. $\frac{3}{5}l$. At sections situated between the supports distant $\frac{l}{5}$ from them.

7. A square timber beam of 12 inches side and 20 feet long supports a load of 2 tons at the middle of its span. Calculate the skin stress at the middle section, allowing for its own weight. The timber weighs 50 lbs. per cubic foot. *Ans.* 1037 lbs. per square inch.

8. The two halves of a flanged coupling on a line of shafting are accidentally separated so that there is a space of 2 inches between the faces. Assuming the bolts to be a driving fit in each flange, calculate the bending stress in the bolts when transmitting a twisting moment of 10,000 lbs.-inches. Diameter of bolt circle, 6 inches ; diameter of bolts, $\frac{3}{4}$ inch ; number of bolts, four. *Ans.* 9 tons square inch.

9. A uniformly tapered cantilever is loaded at its extreme outer end, where the diameter is 12 inches, and it increases $\frac{1}{8}$ inch per inch distance from the end. Find the position of the most highly stressed section.

Ans. 40 inches from the end.

10. A beam 28 feet long rests on supports 20 feet apart ; the left-hand end overhangs 8 feet. A load of (i.) 4 tons is placed at the overhanging end ; (ii.) 2·5 tons, 10 feet from it ; (iii.) 11 tons, 17 feet from it. In addition the beam supports a uniformly distributed load of 0·8 tons per foot run throughout its length. Construct bending moment and shear diagrams, and state their values at all the principal points.

Ans. Bending moments ; 8 feet from end, 57·6 tons feet ; 10 feet, 20·84 tons feet ; 17 feet, – 65·12 tons feet. Shear at corresponding points, 10·4 tons, – 17·58 tons, 1·52 tons.

11. A beam 30 feet long rests on two supports 18 feet apart, the left-hand end overhangs 8 feet. Find the bending moment on the beam due to its own weight, which is 0·4 tons per foot run, at the following places: (i.) over each support; (ii.) at the section where it has its maximum value between the supports. Also find the position of the points of contrary flexure.

Ans. At the left-hand support $-$ 12·8 tons feet, at the right-hand support $-$ 3·2 tons feet. The maximum occurs at a section 11·7 feet from the right-hand end, and there amounts to 8·6 tons feet. The points of contrary flexure are 5·13 feet and 18·27 feet from the right-hand end.

12. (London U.) A thin plate bulkhead in a ship is 14 feet in height, and is strengthened by vertical rolled joists riveted to the plating at 2 feet centres. If the compartment on the one side of the bulkhead is filled with water, draw the diagram of shearing force on one of the beams and calculate the maximum bending moment if the ends are simply supported and not rigidly held as regards direction.

Ans. The maximum bending moment occurs where the shear changes sign, *i.e.* 8·08 feet from the top. Its value is 9·8 tons-feet.

13. Find the bending moment at the foot of a circular chimney on the assumption that the intensity of the wind pressure varies as the square root of the height above the ground. At 100 feet it is 24 lbs. per square foot on a flat surface. The coefficient of wind resistance for a cylinder is 0·6. The diameter of the chimney is 18 feet at the base and tapers at the rate of $\frac{1}{30}$th of a foot per foot in height. Height of chimney 230 feet.

Ans. 3045 tons feet.

Chapter XIII.

Deflection of Beams.

1. (Victoria, 1897.) A round steel rod $\frac{1}{2}$ inch in diameter, resting upon supports A and B, 4 feet apart, projects 1 foot beyond A, and 9 inches beyond B. The extremity beyond A is loaded with a weight of 12 lbs., and that beyond B with a weight of 16 lbs. Neglecting the weight of the rod, investigate the curvature of the rod between the supports, and calculate the greatest deflection between A and B. Find also the greatest intensity of stress in the rod due to the two applied forces. ($E = 30{,}000{,}000$.)

Ans. The rod bends to the arc of a circle between A and B. $\delta = 0{\cdot}45$ inch; $f = 11{,}750$ lbs. square inch.

2. A cantilever of length l is built into a wall and loaded evenly. Find the position at which it must be propped in order that the bending moment may be the least possible, and find the position of the virtual joints or points of inflection.

(N.B.—The solution of this is very long.)

Ans. Prop must be placed $0{\cdot}265l$ from free end. Virtual joints $0{\cdot}55l$, $0{\cdot}58l$ from wall.

3. Find an expression for the maximum deflection of a beam supported at the ends, and loaded in such a manner that the bending-moment diagram is (i.) a semicircle, the diameter coinciding with the beam; (ii.) a rectangle; height $= \frac{1}{2}$ the span (L).

Ans. (i.) $\frac{L^3}{17{\cdot}7EI}$; (ii.) $\frac{L^3}{16EI}$.

4. What is the height of the prop relative to the supports in a centrally propped beam with an evenly distributed load, when the load on the prop is equal to that on the supports?

Ans. $\frac{7}{1152}\frac{WL^3}{EI}$ below the end supports.

5. (S. and A., 1897.) A uniform beam is fixed at its ends, which are 20 feet apart. A load of 5 tons in the middle; loads of 2 tons each at 5 feet from the ends. Construct the diagram of bending moment. State what the maximum bending moment is, and where are the points of inflection.

Ans. $M_{max} = 20$ tons-feet close to built-in ends. Points of inflection 4·4 feet from ends.

6. Find an expression in the usual terms for the end deflection of a cantilever of uniform depth, but of variable width, the plan of the beam being a triangle with the apex loaded and the base built in.

Ans. $\delta = \frac{WL^3}{2EI}$.

(NOTE.—In this case and in that given in Question 12, the I varies directly as the bending moment, hence the curvature is constant along the whole length, and the beam bends to the arc of a circle.)

7. Find an expression for the deflection at the free end of a cantilever of length L under a uniformly distributed load W and an *upward* force W_0 acting at the free end. What proportion must W_0 bear to the whole of the evenly distributed load in order that the end deflection may be zero? What is the bending moment close to the built-in end?

Ans. $\frac{L^3}{EI}\left(\frac{W}{8} - \frac{W_0}{3}\right)$. W_0 must be $\frac{3}{8}W$; M at wall, $\frac{WL}{8}$.

8. Find graphically the maximum deflection of a beam loaded thus: Span, 15 feet; load of 2 tons, 2 feet from the left-hand end, 4 tons 3 feet from the latter, 1 ton 2 feet ditto, 3 tons 4 feet ditto, *i.e.* at 4 feet from the right-hand end. Take I = 242; E = 12,000 tons square inch.

Ans. 0·32 inch.

9. (I.C.E., October, 1898.) In a rolled steel beam (symmetrical about the neutral axis), the moment of inertia of the section is 72 inch units. The beam is 8 inches deep, and is laid across an opening of 10 feet, and carries a distributed load of 9 tons. Find the maximum fibre stress, also the central deflection, taking E at 13,000 tons square inch.

Ans. 7·5 tons square inch; 0·215 inch.

10. (I.C.E., February, 1898.) A rolled steel joist, 40 feet in length, depth 10 inches, breadth 5 inches, thickness throughout $\frac{1}{2}$ inch, is continuous over three supports, forming two spans of 20 feet each. What uniformly distributed load would produce a maximum stress of $5\frac{1}{2}$ tons per square inch? Sketch the diagrams of bending moments and shear force.

Ans. 0·23 ton foot run.

11. (I.C.E., October, 1898.) A horizontal beam of uniform section, whose moment of inertia is I, and whose total length is 2L, is supported at the centre, one end being anchored down to a fixed abutment. Neglecting the weight of the beam, suppose it to be loaded at the other end with a single weight W. Find an expression for the vertical deflection at that end below its unstrained position.

Ans. $\delta = \frac{2WL^3}{3EI}$.

12. A laminated spring of 3 feet span has 20 plates, each 0·375 inch

thick and 2·95 inches wide. Calculate the deflection when centrally loaded with 5 tons. E = 11,600 tons square inch. *Ans.* 2·45 inches.

Actual deflection, 2·2 inches when loading.
" " 2·9 " " unloading.

13. A laminated spring of 40 inches span has 12 plates each 0·375 inch thick, and 3·40 inches wide. Calculate the deflection when centrally loaded with 4 tons. E = 11,600 tons square inch. *Ans.* 3·9 inches.

Actual deflection 3·35 inches when loading.
" " 4·37 " " unloading.

14. A laminated spring of 75 inches span has 13 plates, each 0·39 inch thick, and 3·5 inches wide. Calculate the deflection when centrally loaded with 1 ton. E = 11,600 tons square inch. *Ans.* 5·0 inches.

Actual deflection 4·21 inches when loading.
" " 4·97 " " unloading.

The deflection due to shear in this long spring is evidently less than in the short stumpy springs in Examples 12 and 13. The E by experiment comes out 12,600. See page 461 for the effects of shear on short beams.

15. One of the set of beams quoted in the Appendix was tested with a load applied at a distance of 12 inches from one end, total length 36 inches, $h = 1·984$ inch; $b = 1·973$ inch; taking E = 13,000 tons square inch. Find the maximum deflection. *Ans.* 0·51 inch.

Actual deflection = 0·53 inch.

16. A cantilever is to be made of several round rods of steel placed side by side. It is to project 2 feet and is to support a load of 1000 lbs. at the end. The deflection is to be $\frac{1}{10}$ inch and the maximum stress in the steel 4 tons per square inch. Find the diameter of the rods and the number needed. E = 30,000,000 lbs. per square inch. *Ans.* 1·14"; 18 rods.

17. Calculate the deflection at the free end of a cantilever of uniform section when loaded with 4 tons at the extreme end and six tons in the middle. Length = 14 feet. Depth = 18 inches. Stress = 5 tons per square inch. E = 13,000 tons per square inch. *Ans.* 0·34.

18. A horizontal cantilever is uniformly loaded throughout its length. Find the position at which a prop must be placed in order that it may support one half of the load.

Ans. At a distance = 0·838 × length from the wall.

19. A cantilever loaded with an evenly distributed load is propped at the outer end in such a manner that the bending moment at the wall is one half of that due to an evenly distributed load on the cantilever when not propped. Find (i.) the load on the prop, (ii.) the amount the top of the prop is above or below the unloaded position of the beam. Length = 80 inches. Load per foot run = 0·2 tons. Stress in beam = 5 tons per square inch. Depth of beam section = 8 inches. E = 12,000 tons per square inch. *Ans.* (i.) 0·33 ton, (ii.) 0·11 inch down.

20. A beam 20 feet long is loaded at a point 4 feet from the left-hand end. Find the deflection at the middle of the beam and at a point 14 feet from the left-hand abutment. Load 8 tons. I = 300. E = 12,000 tons per square inch. *Ans.* 0·36 inch in middle, 0·27 inch at 14 feet.

21. Find graphically the deflection of a beam at points *a*, *d*, *f* when loaded thus: Load at *a* = 3 tons (end of beam), distance *ab* = 2 feet, load at *b* = 3 tons, *bc* = 2 feet, *c* is a support; load at *d* = 12 tons, *cd* = 5 feet, *e* is a support; *de* = 7 feet, load at *f* = 5 tons, *ef* = 5 feet, *f* is the end of beam.

Ans. Deflection at *a* = 0·05 inch, at *d* = 0·015 inch, at *f* = 0·16 inch.

22. A beam of length L is freely supported at each end and supports two loads W at distances $\frac{L}{4}$ from each end. Find an expression for the central deflection.

Ans. $\frac{11}{384}\frac{WL^3}{EI}$.

23. A beam 30 feet long, freely supported at each end, is loaded thus: Distances from left-hand end, 3 feet, 9, 15, 18; respective loads 2·0 tons, 3·2, 1·5, 2·7. Find the deflection at a point 12 feet from the same end in terms of E and I.

Ans. $\frac{6{,}700{,}000}{EI}$.

24. A beam 36 feet long is built at one end into a wall and is held by a sloping tie at the free end; it is attached to the wall at a point 18 feet above the beam. Find the tension in the tie when the beam is loaded with 10 tons evenly distributed, also the bending moment close to the wall.

Ans. 8·4 tons, 45 tons-feet.

25. The vertical sides of an earthwork trench are timbered with horizontal planks which are freely supported at the ends by vertical posts placed 8 feet apart; the thickness of the planks is 3 inches, and the depth (vertical) is 9 inches. The central deflection of the planks due to the earth pressure is 5 inches. Find (i.) the pressure per square foot exerted by the earthwork, (ii.) the stress in the planks. E = 900,000 lbs. per square inch. *Ans.* (i.) 1320 lbs.; (ii.) 7030 lbs. square inch.

26. Three beams of 100 inches span support a central weight of 600 lbs. in such a way that the weight just touches the tops of the beams when the weight is lowered and the beams support no load. The two outside beams are 3 inches deep and 6 inches wide, the middle beam is 6 inches square. Find the load supported by each beam and the stress in each.

Ans. Outside beams 60 lbs. each, middle beam 480 lbs. Stress in outer beams 166 lbs. per square inch, and in the middle beam 333 lbs. per square inch.

27. A steel shaft 16 feet in length is supported by three bearings. Diameter 8 inches. E = 12,000 tons per square inch. Find the amount the central bearing must be set low in order that the pressure on each bearing may be the same. *Ans.* 0·022 inch.

28. A shaft 8 inches diameter, and 20 feet in length, is supported on three bearings. It has two flywheels, each weighing 2 tons, placed one on each side of the central bearing, and 3 feet from its middle. Find by how much the central bearing must be set low in order to secure that each bearing shall support one-third of the total load. *Ans.* 0·32 inch

Chapter XIV.

Combined Bending and Direct Stresses.

1. A brickwork pier, 18 inches square, supports a load of 4 tons; the resultant pressure acts at a distance of 4 inches from the centre of the pier. Calculate the maximum and minimum stresses in the brickwork.

Ans. 4·15 tons square foot compression.
0·59 „ „ tension.

2. (I.C.E., 1897.) The total vertical pressure on a horizontal section of a wall of masonry is 100 tons per foot length of wall. The thickness of

the wall is 4 feet, and the centre of stress is 6 inches from the centre of thickness of the wall. Determine the intensity of stress at the opposite edges of the horizontal joint. *Ans.* Outer edge, 43·75 tons square foot.
Inner ,, 6·25 ,, ,,

3. (I.C.E., October, 1898.) A hollow cylindrical tower of steel plate, having an external diameter of 3 feet, a thickness of ½ inch, and a height of 60 feet, carries a central load of 50 tons, and is subjected to a horizontal wind-pressure of 56 lbs. per foot of its height. Calculate the vertical stresses at the fixed base of the tower on the windward and on the leeward side. (Allow for the weight of the tower itself.)
Ans. Windward side, 0·11 tons square inch.
Leeward ,, 2·09 ,, ,,

4. (Victoria, 1897.) A tension bar, 8 inches wide, 1¼ inch thick, is slightly curved in the plane of its width, so that the mean line of the stress passes 2 inches from the axis at the middle of the bar. Calculate the maximum and minimum stress in the material. Total load on bar, 25 tons.
Ans. Maximum, 6·25 tons square inch tension.
Minimum, 1·25 ,, ,, compression.

5. (Victoria, 1896.) If the pin-holes for a bridge eye-bar were drilled out of truth sideways, and the main body of the bar were 5 inches wide and 2 inches thick, what proportion would the maximum stress bear to the mean over any cross-section of the bar at which the mean-line of force was ⅛ inch from the middle of the section. *Ans.* 1·15.

6. The cast-iron column of a 100-ton testing-machine has a sectional area of 133 square inches. $Z_t = 712$; $Z_c = 750$. The distance from the line of loading to the c. of g. of the section is 17·5 inches. Find the maximum tensile and compressive skin stresses.
Ans. 3·08 tons square inch compression.
1·71 ,, ,, tension.

7. A tube in a bicycle frame is cranked ½ inch, the external diameter being 0·70 inch, and the internal 0·65 inch. Find how much the stress is increased in the cranked tube over that in a similar straight tube under the same load. *Ans.* Four times (nearly).

8. The section through the back of a hook is a trapezium with the wide side inwards. The narrow side is 1 inch, and the wide side 2 inches; the depth of the section is 2½ inches; the line of pull is 1½ inch from the wide side of the section. Calculate the load on the hook that will produce a tensile skin stress of 7 tons per square inch. *Ans.* 3·87 tons.

9. In the case of a punching-machine, the load on the punch is estimated to be 160 tons. It has a gap of 3 feet from the centre of the punch. The tension flange is 40″ × 3″, and the compression flange 20″ × 2″. There are two 2-inch webs; the distance from centre to centre of flanges is 4 feet. Calculate the stress in the tension flange. *Ans.* 2·05 tons square inch.

10. A cylindrical steel plate chimney 160 feet in height and 12 feet external diameter is exposed to wind pressure, the intensity of which varies as the square root of the height above the ground, and is 30 lb. per square foot at 100 feet high taken on the projected area. The coefficient of wind resistance is 0·6. Find the stress in the plates, which are 0·5 inch in thickness at the foot of the chimney. *Ans.* 3980 lbs. square inch.

11. A tower 150 feet high is 3 feet out of perpendicular towards the east. A wind of 10 lbs. per square foot blows from the south. The external diameter is 20 feet, the internal 10 feet. The masonry weighs 130 lbs. per cubic foot. Find (*a*) the bending moment due to the overhang,

(*b*) the bending moment due to the wind, (*c*) the maximum compressive stress on the masonry due to wind and overhang.

Ans. (*a*) 3082 tons-feet, (*b*) 1003 tons-feet, (*c*) 13·1 tons per square foot.

12. (See Morley's *Strength of Materials*, p. 330). The central horizontal section of a hook is a symmetrical trapezium $2\frac{1}{4}$ inches deep, the inner width being 2 inches and the outer width 1 inch. Estimate the extreme intensities of stress when the hook carries a load of 1·25 ton, the load line passing 2 inches from the inside edge of the section, and the centre of curvature being in the load line.

Ans. Maximum tension 3·83 tons per square inch.
,, compression 2·25 ,, ,,

Chapter XV.

STRUTS.

1. Find the buckling load of a steel strut of 3 inches solid square section with rounded ends, by both Euler's and Gordon's formula, for lengths of 2 feet 6 inches, 9 feet, 15 feet. $E = 30{,}000{,}000$, $f = 49{,}000$, $S = 60{,}000$.

Ans. Euler, 2,250,000, 173,600, 62,500 lbs.
Gordon, 465,500, 176,000, 79,900 lbs.

2. Calculate the buckling load of a 2″ × 1·03″ rectangular section strut of hard cast iron, 16 inches long, rounded ends. Take the tabular values for a and s. *Ans.* 40·3 tons. (Experiment gave 41·6 tons.)

3. Calculate the buckling load of a piece of cast-iron pipe, length, 24 inches; external diameter, 4·4 inches; internal diameter, 3·9 inches; rounded ends. Take the tabular values for a and S.

Ans. 159 tons (hard), 98 tons (soft), 128 tons (mean). (Experiment, 113·2 tons.)

4. Calculate the buckling load of a cast-iron column, 9 feet long, external diameter, 3·56 inches; internal diameter, 2·91 inches; ends flat, but not fixed; flanges 8 inches diameter.

Ans. 78·7 tons (hard), 48·4 (soft), 63·6 (mean). (Experiment, 61 tons.)

5. Ditto, ditto, but with pivoted ends.

Ans. 28·4 tons (hard). The iron was very hard, with fine close-grained fracture. (24·9 tons by experiment.)

6. Calculate the safe load for a cast-iron column, external diameter, 3·54 inches; internal, 2·86 inches; pivoted ends; loaded 2 inches out of the centre; safe tensile stress, 1 ton square inch. *Ans.* 2 tons (nearly).

(N.B.—Cast-iron columns loaded thus nearly always fail by tension.)

7. Calculate the buckling load of T iron struts as follows:—

	Area.	a
(1) $r = 8{\cdot}2$	1·67 sq. ins.	$\frac{1}{467}$
(2) $r = 16{\cdot}0$	1·67 ,,	$\frac{1}{467}$
(3) $r = 18{\cdot}0$	1·95 ,,	$\frac{1}{824}$
(4) $r = 14{\cdot}3$	1·20 ,,	$\frac{1}{461}$
(5) $r = 19{\cdot}8$	2·04 ,,	$\frac{1}{806}$
(6) $r = 26{\cdot}8$	1·20 ,,	$\frac{1}{461}$

Ans. Calculated, (1) 26·1 tons; (2) 19·2; (3) 21·5; (4) 14·8; (5); 20·3; (6) 8·4.
By experiment, 29·0 ,, 18·5 19·7 15·9 19·3 12·7.

8. A hollow circular mild-steel column is required to support a load of 50 tons, length, 28 feet; ends rigidly held; external diameter, 6 inches; factor of safety, 4. Find the thickness of the metal. *Ans.* 0·9 inch.

9. A solid circular-section cast-iron strut is required for a load of 20 tons, length, 15 feet; rounded ends; factor of safety, 6. Find the diameter. *Ans.* 6·25 inches.

(This is most easily arrived at by trial and error, by assuming a section, calculating the buckling load, and altering the diameter until a suitable size is arrived at.)

10. Find the buckling load of a rolled joist when used as a strut. The lower end is rigidly built in, and the upper is simply guided.

Total length	25 feet.
Depth of section	7 inches.
Breadth of flanges	12 ,,
Thickness of flanges	1·1 ,,
Thickness of web	0·6 ,,
Crushing strength of a short strut	30 tons square inch.
Elastic limit of material . . .	14 ,, ,,
E	13,000 ,, ,,

Check your result by Euler's expression.
Ans. 696 tons (Gordon), 913 tons (Euler).

11. Find the safe load (factor of safety 5) for a strut of channel section 12″ × 7″ × 1″. Total length 16 feet, both ends rigidly held. Check by Euler. *Ans.* Gordon 110 tons, Euler 300 tons.

Chapter XVI.

Torsion.

1. (S. and A., 1897.) Find the diameter of a wrought-iron shaft to transmit 90 H.P. at 130 revolutions per minute. If there is a bending moment equal to the twisting moment, what ought to be the diameter? Stress = 5000 lbs. per square inch. *Ans.* 3·54 inches; 4·75 inches.

2. A winding drum 20 feet diameter is used to raise a load of 5 tons. If the driving shaft were in pure torsion, find the diameter for a stress of 3 tons per square inch. *Ans.* 10·1 inches.

3. Find the diameter for the above case if the load is accelerated at the rate of 40 feet per second per second when the winding engine is first started. *Ans.* 13·2 inches.

4. Find the diameter of shaft for a steam-engine having an overhung crank. Diameter of cylinder, 18 inches; steam pressure, 130 lbs. per square inch; stroke, 2 feet 6 inches; overhang of crank, *i.e.* centre of crank-pin to centre of bearing, 18 inches; stress, 7000 lbs. per square inch. *Ans.* 10 inches.

5. Find the diameter of a hollow shaft required to transmit 5000 H.P. at 70 revolutions per minute; stress, 7500 lbs. per square inch; the external diameter being twice the inner; maximum twisting moment = $1\frac{1}{2}$ times the mean. *Ans.* 16·9 inches.

6. A 4-inch diameter shaft, 30 feet long, is found to spring 6·2° when transmitting power; revolutions per minute, 130. Find the horse-power transmitted. G = 12,000,000. *Ans.* 187.

7. (Victoria, 1898.) If a round bar, 1 inch in diameter and 40 inches between supports, deflects 0·0936 inch under a load of 100 lbs. in the middle, and twists through an angle of 0·037 radian when subjected to a twisting moment of 1000 lbs.-inches throughout its length of 40 inches, find E, G, and K; Young's modulus, the moduli of distortion, and volume.

Ans. E = 29,020,000, G = 11,020,000, K = 26,250,000, all in pounds per square inch.

8. A square steel shaft is required for transmitting power to a 30-ton overhead travelling crane. The load is lifted at the rate of 4 feet per minute. Taking the mechanical efficiency of the crane gearing as 35 %, calculate the necessary size of shaft to run at 160 revolutions per minute. The twist must not exceed 1° in a length equal to 30 times the side of the square. G = 13,000,000. *Ans.* 2 inches square.

9. A horse tram-car, weighing 5 tons, when travelling at 8 miles an hour, is pulled up by brakes. Find what weight of helical springs of solid circular section would be required to store this energy. Stress in springs, 60,000 lbs. per square inch. *Ans.* 0·48 ton.

10. Calculate the amount a helical spring having the following dimensions will compress under a load of one ton. Number of coils, 20·5; mean diameter of spring, 2·5 inches; coils of square section steel, 0·51 inch side. *Ans.* 4·61 inches. (Experiment gave 4·57 inches.)

11. Calculate the stretch of a helical spring under a load of 112 lbs. Number of coils, 22; mean diameter, 0·75 inch. Diameter of wire, 0·14 inch. *Ans.* 1·80 inch. (Experiment gave 1·82 inch.)

12. A square shaft ($Z^p = 0{\cdot}208\ S^3$) is required to transmit 60 H.P. at 130 revolutions per minute. Find the size of shaft required, also the amount it will twist on a length of 80 feet.

Stress . . . . . 6000 lbs. per square inch.
Modulus of rigidity . . . . 12,000,000 ,, ,,
Ans. 2·86 inches; 14·3°.

13. A close coiled helical spring is required to stretch to four times its original length. Find the mean diameter of the coils and the load.

Diameter of wire . . . . . 0·3 inches.
Stress . . . . . 60,000 lbs. per square inch.
Modulus of rigidity . . . . 12,000,000 ,, ,,
Ans. 4·16 inches diameter. Load 152 lbs.

14. (London U., 1913). A vessel having a single propeller shaft 12 inches in diameter, and running at 160 revolutions per minute, is re-engined with turbines driving two equal propeller shafts at 750 revolutions per minute and developing 60 per cent. more horse-power. If the working stresses of the new shafts are 10 per cent. greater than that of the old shaft, find their diameters. *Ans.* 6·5 inches.

15. Find the diameter of shaft required to transmit 400 horse-power at 160 revolutions per minute. The twist must not exceed one degree in a length of fourteen diameters.

G = 5000 tons square inch. *Ans.* 4·86 inches.

16. Find the relative strength of two shafts of the same material, one of which contains no keyway, and the other a sunk keyway whose depth is $\frac{1}{8}$th the diameter of the shaft. *Ans.* 1 to 0·824.

17. A crank shaft having the crank midway between the bearings, is required for an engine. Find the diameter to satisfy the maximum shear

and maximum tension conditions. Diameter of cylinder, 18 inches. Stroke, 2 feet. Bearing centres, 4·4 feet. Pressure, 90 lbs. square inch. Maximum tensile stress, 3·5 tons square inch. Maximum shear stress, 2 tons square inch.

Ans. The diameter comes to practically the same in both cases, viz. 7·74 inches.

18. A shaft overhangs 4 feet from the nearest bearing, and has a belt pulley 5 feet in diameter at the free end. The pulley weighs 180 lbs. The tension on the tight side of a vertical belt is 580 lbs., and on the slack side 110 lbs. Find the size of the shaft to satisfy the conditions that the maximum tensile stress shall not exceed 3 tons per square inch, and the maximum shear stress shall not exceed 2 tons per square inch.

Ans. For tension conditions, 4·03 inches.
For shear conditions, 3·69 „

19. A helical spring is made of steel of circular section 1·4 inch diameter, the mean diameter of the coils is 11 inches. Number of coils 28. Find the load required to elongate it 4 inches. What is the load that may be applied to the spring if the stress is not to exceed 80,000 lbs. per square inch? $G = 12{,}000{,}000$. *Ans.* 618 lbs. and 7826 lbs.

20. A weight W falls on to a helical buffer-spring consisting of 18 coils of 1 inch circular section; the mean diameter of the coils is 9 inches. Find the height the weight must fall before it strikes the spring in order to compress it 6 inches.

$G = 11{,}000{,}000$ lbs. per square inch. $W = 50$ lbs. *Ans.* 31·7 inches.

Chapter XVII.

STRUCTURES.

1. Find the forces, by means of a reciprocal diagram, acting on the members of such a roof as that shown in Fig. 585, with an evenly distributed load of 16 lbs. per square foot of covered area. Span, 50 feet; distance apart of principals, 12 feet, which are fixed at both ends. Calculate the force *pa* by taking moments about the apex of the roof. Find the force *tu* by the method of sections. See how they check with the values found from the reciprocal diagram.

2. In the case given above, find the forces when one of the ends is mounted on rollers, both when the wind is acting on the roller side and on the fixed side of the structure, taking a horizontal wind-pressure of 30 lbs. per square foot on a vertical surface.

3. Construct similar diagrams for Fig. 586.

4. Construct a polygon and a reciprocal diagram for the Island Station roof shown in Fig. 587. Check the accuracy of the work by seeing whether the force F is equal to *li*.

5. (Victoria, 1896.) A train of length T, weighing W tons per foot run, passes over a bridge of length l greater than T, which weighs w tons per foot run. Find an expression for the maximum shear at any part of the structure, and sketch the shear diagram.

Let y = the distance of the part from the middle of the structure.

$$\text{Shear} = \frac{W T}{2l}(l - T + 2y) + wy$$

N.B.—As far as the structure is concerned, the length of the train T is only that portion of it actually upon the structure at the time. This expression becomes that on p. 604, when the length of the train is not less than the length of the structure.

6. Find an expression for the focal length X of a bridge which may be entirely covered with a rolling load of W tons per foot run. The dead load on the bridge = w tons per foot run.

Let $M = \frac{w}{W}$.

$$x = l(1 - 2\sqrt{M + M^2} + 2M)$$

7. A plate girder of 50 feet span, 4 feet deep in the web, supports a uniformly distributed load of 4 tons per foot run. Find the thickness of web required at the ends. Shear stress in web and rivets, 4 tons square inch. Also find the pitch of rivets, $\frac{7}{8}$ inch diameter, for shearing and bearing stress. Bearing pressure, 9 tons square inch.

Ans. Thickness, $\frac{1}{2}$ inch. Pitch for single row in shear, 2·4 inches; better put two rows of say 4·8-inch pitch. Ditto for bearing, 1·97 inch and say 4 inches.

8. Find the skin stress due to change of curvature in a two-hinged arch rib, on account of its own dead weight, which produces a mean compressive stress f_c of 6 tons per square inch. E = 12,000 tons square inch. Span, 550 feet; rise, 114 feet; depth of rib, 15 feet. Also find the deflection due to a stationary test load which produces a further mean compressive stress f_c of 1 ton per square inch.

Ans. Stress, 0·8 ton square inch; $\delta = 0{\cdot}79$ inch.

(NOTE.—The above arch is that over the Niagara River. Some of the details have been assumed, but are believed to be nearly accurate. When tested, the arch deflected 0·81 inch.)

9. Bridge members are subjected to the following loads. State for what static loads you would design the members: (i.) Due to a dead load on the structure of 10 tons in tension, and a live load of 30 tons in tension. (ii.) Dead, 100 tons tension; live, 30 tons compression. (iii.) Dead, 80 tons tension; live, 60 tons compression. (iv.) Dead, 50 tons tension; live, 70 tons compression.

Ans. (i.) 70 tons tension. (ii.) 100 tons tension; when under the action of the live load, the stress is diminished. (iii.) Either for 80 tons tension or 40 tons compression, whichever gave the greatest section. (iv.) 90 tons compression.

10. Find the number of $4\frac{1}{2}$-inch rings of brickwork required for an arch of 40 feet clear span; radius of arch, 30 feet.

Ans. By approximate formula, 34·4 inches = 8 rings; by Trautwine, 31·9 inches = 7 rings.

11. (Victoria, 1902). A load w, if applied slowly, induces an intensity of stress f_o in a bar of length l. Show that (provided the limit of elasticity be not exceeded) if the same load fall a height h before extending the bar the stress induced is given by—

$$f_o\left\{1 + \sqrt{1 + \frac{2EH}{f_o l}}\right\}$$

in which E represents Young's modulus of elasticity for the material of the bar.

12. A sliding weight of 4000 lbs. drops down a vertical rod 12 feet

long and 2 inches diameter on to a collar at the bottom on which a buffer-spring is placed. The spring deflects 1 inch for each 1000 lb. load. Find the height, reckoned from the free position of the spring, from which the weight must be dropped in order to produce a momentary stress of 10,000 lbs. per square inch in the bar. E = 30,000,000 lbs. per square inch.

Ans. 92 inches with spring, 0·14 inch without.

13. A weight of $\frac{1}{2}$ ton is dropped on to the middle of a rolled joist $12'' \times 6'' \times \frac{3}{8}''$ and 15 feet span. Find the height the weight must be dropped in order to produce a momentary stress of 7 tons per square inch. E = 12,000 tons per square inch. *Ans.* 2·32 inches.

14. Calculate the equivalent momentary loads on members of structures when subjected to the following loads :—

Dead load due to the weight of the structure.	Load when a train stands on the structure.
(*a*) 18 tons tension	32 tons tension
(*b*) 18 ,, ,,	13 ,, ,,
(*c*) 18 ,, ,,	4 tons compression
(*d*) 18 ,, ,,	18 ,, ,,

Ans. (*a*) 46 tons ; (*b*) 8 tons ; (*c*) − 26 tons ; (*d*) 54 tons.

Chapter XVIII.

HYDRAULICS.

1. Taking the weight of 1 cubic foot of water at 62·5 lbs. at atmospheric pressure and at 60° Fahr., calculate the weight of one cubic foot when under a pressure of 3 tons per square inch and at a temperature of 100° Fahr. K = 140 tons square inch. *Ans.* 63·48 lbs.

2. Find the depth of the centre of pressure of an inclined rectangular surface making an angle of 30° with the surface, length, 10 feet ; bottom edge 15 feet below surface and horizontal. *Ans.* 12·6 feet.

3. (Victoria, 1897.) Find the horizontal pull on a chain fixed to the top of a dock gate to keep it from overturning, there being no resistance at the sides of the gate, which is hinged horizontally at the bottom. Height of gate, 40 feet ; depth of water on one side, 35 feet. Depth of water on the other side, 23 feet ; width of gate, 70 feet ; weight of salt water, 64 lbs. per cubic foot. *Ans.* 256 tons.

4. The height of a dam such as that shown in Fig. 621 is 8 feet, and the width 3 feet 6 inches ; the back stay is inclined at an angle of 45°. Calculate the pressure on the stay. *Ans.* 9950 lbs.

5. A channel of triangular (equilateral) cross-section of 8 feet sides is closed by a gate supported at each corner. Find the pressure on each support when the channel is full of water.

Ans. 1000 lbs. at each top corner ; 2000 lbs. at bottom corner.

6. Calculate the number of cubic feet per hour that will pass through a plain orifice in the bottom of a tank ; diameter, 1 inch ; head, 8·2 inches ; K = 0·62. *Ans.* 80·6.

7. Find the size of a circular orifice required in the bottom of a tank to pass 10,000 gallons per hour. h = 3 feet ; K = 0·62.

Ans. 3·07 inches diameter.

8. Calculate the coefficient of discharge for a pipe orifice having a slightly rounded mouth, the coefficient of contraction for which is 0·75

when running a clear stream without touching the sides of the pipe. The discharge is reduced 5 per cent. by friction. *Ans.* 0·90.

9. Water issues from an orifice in a vertical plate, and passes through a ring whose centre is 4 feet away from the plate and 3 feet below the orifice. The head of water over the orifice is 2 feet. The area of the orifice is 0·01 square foot. The discharge is 25 gallons per minute. Find K_v, K_c, K_d, and the coefficient of resistance.

Ans. $K_v = 0·818$, $K_c = 0·72$, $K_d = 0·59$, coefficient of resistance = 0·391.

10. How many gallons per minute will be discharged through a short pipe 2 inches diameter leading from and flush with the bottom of a tank? Depth of water, 25 feet. *Ans.* 268.

11. In the last question, what would be the discharge if the pipe were made to project 6 inches into the tank, the depth of water being the same as before? *Ans.* 172.

12. In an experiment with a diverging mouthpiece, the vacuum at the throat was 18·3 inches of mercury, and the head of water over the mouthpiece was 30 feet, the diameter of the throat 0·6 inch. Calculate the discharge through the mouthpiece in cubic feet per second. *Ans.* 0·112.

13. (I.C.E., October, 1897.) A weir is 30 feet long, and has 18 inches of head above the crest. Taking the coefficient at 0·6, find the discharge in cubic feet per second. *Ans.* 176.

14. A rectangular weir, for discharging daily 10 million gallons of compensation water, is arranged for a normal head over the crest of 15 inches. Find the length of the weir. Take a coefficient of 0·7.

Ans. 3·56 feet.

15. Find the number of cubic feet of water that will flow over a right-angled V notch per second. Head of water over bottom of notch, 4 inches; $K = 0·6$. *Ans.* 0·165.

16. (I.C.E., February, 1898.) The miner's inch is defined as the flow through an orifice in a vertical plane of 1 square inch in area under an average head of 6½ inches. Find the water-supply per hour which this represents. *Ans.* 90 cubic feet per hour when $K = 0·61$.

17. (Victoria, 1898.) Find the discharge in gallons per hour from a circular orifice 1 inch in diameter under a head of 2 feet, the pipe leading to the orifice being 6 inches in diameter. *Ans.* 885.

18. A swimming bath 60 feet long, 30 feet wide, 6 feet 6 inches deep at one end, and 3 feet 6 inches at the other, has two 6-inch outlets, for which $K_d = 0·9$. Find the time required to empty the bath, assuming that all the conditions hold to the last. *Ans.* 29·7 minutes.

19. Find the time required to lower the water in a hemispherical bowl of 5 feet radius from a depth of 5 feet down to 2·5 feet, through a hole 2 inches diameter in the bottom of the bowl. *Ans.* 14·3 minutes.

20. A horizontal pipe 4 inches diameter is reduced very gradually to half an inch. The pressure in the 4-inch pipe is 50 lbs. square inch above the atmosphere. Calculate the maximum velocity of flow in the small portion without any breaking up occurring in the stream.

Ans. 87 feet per second.

21. A slanting pipe 2 inches diameter gradually enlarges to 4 inches; the pressure at a given section in the 2-inch pipe is 25 lbs. square inch absolute, and the velocity 8 feet per second. Calculate the pressure in the 4-inch portion at a section 14 feet below. *Ans.* 31·2 lbs. square inch.

22. In the last question, calculate the pressure for a sudden enlargement. *Ans.* 30·96 lbs. square inch.

23. (I.C.E., October, 1898.) A pipe tapers to one-tenth its original area, and then widens out again to its former size. Calculate the reduction of pressure at the neck, of the water flowing through it, in terms of the area of the pipe and the velocity of the water.

$$Ans.\ p_1 - p_2 = \frac{w V_1^2}{2g}(100 - 1).$$

N.B.—The 100 is the ratio $\left(\frac{A_1}{A_2}\right)^2$.

24. Find the loss of head due to water passing through a socket in a pipe, the sectional area of the waterway through the socket being twice that of the pipe. Velocity in pipe, 10 feet per second. *Ans.* 0·85 foot.

25. A Venturi water-meter is 13·90 inches diameter in the main, and 5·68 inches diameter in the waist, the difference of head in inches of mercury is 20·75 (both limbs of the mercury gauge being full of water above the mercury). Taking the coefficient of velocity K_v as 0·98, find the quantity of water passing per minute. *Ans.* 2451 gallons.

26. Find the loss of head due to water passing through a diaphragm in a pipe, the area being one-third of that of the pipe. Velocity, 10 feet per second. *Ans.* 21·3 feet.

27. A locomotive scoops up water from a trough 1 mile in length. Calculate the number of gallons of water that will be delivered into the tender when the train travels at 40 miles per hour. The water is lifted 8 feet, and is delivered through a 4-inch pipe. The loss of head by friction and other resistances is 30 %. *Ans.* 2130.

28. The head of water in a tank above an orifice is 4 feet. Calculate the time required to lower the head to 1 foot. The area of the surface of the water is 1000 times as great as the sectional area of the jet.

Ans. 250 seconds.

29. (I.C.E., February, 1898.) A canal lock with vertical sides is emptied through a sluice in the tail gates. Putting A for area of lock basin, a for area of sluice, H for the lift, find an expression for the time of emptying the lock. (K_c = coefficient of contraction.)

$$Ans.\ \frac{2A\sqrt{H}}{K_c a\sqrt{2g}}.$$

30. (I.C.E., October, 1898.) A pipe 2 feet diameter draws water from a reservoir at a level of 550 feet above the datum; it falls for a certain distance, and again rises to a level of 500 feet 5 miles from its starting-point; it then falls to a reservoir at a level of 400 feet 1 mile away. Calculate the rate of delivery into the lower reservoir.

Ans. 12·2 cubic feet per second.

31. (I.C.E., October, 1898.) Taking skin friction to be 0·4 lb. per square foot at 10 feet per second, find the skin resistance in pounds of a ship of 12,000 square feet immersed surface at 15 knots (a knot = 6086 feet per hour). Also the horse-power to overcome skin friction.

Ans. 30,860 lbs. resistance; 1423 H.P.

32. (I.C.E., October, 1897.) A pipe 12 inches diameter connects two reservoirs 5 miles apart, and having a difference of level of 20 feet. Find the velocity of flow and discharge of the pipe, taking the coefficient of friction at 0·01.

Ans. 11 feet per second; 0·85 cubic feet per second.

33. (Victoria, 1897.) The jet condenser of a marine engine is 15 feet below the surface of the water. Suppose a barometer connected with the

condenser to stand at 5 inches (say 30 inches of vacuum); find the quantity of water discharged per second into the condenser through a pipe 1 inch diameter and 20 feet long. The contraction on entering the pipe may be taken at 0·5. *Ans.* 0·1 cubic foot.

34. (Victoria, 1896.) Water is admitted by a contracting mouthpiece from the bottom of a tank into a 4-inch pipe 600 feet long, and then allowed to escape vertically as a fountain through a 2-inch nozzle into the air, the nozzle being 100 feet below the level of the surface in the tank. Find the height above the nozzle to which the water will rise if the coefficient of resistance of the pipe is 0·005. *Ans.* 30 feet.

35. (Victoria, 1898.) Find the amount of water in gallons per day which will be delivered by a 24-inch cast-iron pipe, 3 miles in total length, when the surface of the water under which it discharges is 175 feet below the surface of the reservoir from which it is drawn, (i.) when the mouth is full open; (ii.) when restricted by a conical nozzle to one-fourth the area.

(NOTE.—When a pipe is provided with a nozzle, the total energy of the water is equal to the frictional resistance due to V, and the kinetic energy of the escaping water at the velocity V_1, together with the resistance at entry into pipe, and resistances due to sudden enlargements, etc., if any.)

Ans. (i.) 12,580,000; (ii.) 12,120,000.

36. Calculate the velocity of discharge by Thrupp's formula in the case of a new cast-iron pipe, 0·2687 foot diameter, having a fall of 28 feet per mile. $n = 1{\cdot}85$, $C = 0{\cdot}005347$, $x = 0{\cdot}67$. *Ans.* 1·81 foot per second.

37. Water is conveyed from a reservoir by a pipe 1 foot diameter and 2 miles long, which contains, (i.) a short sudden enlargement to three times its normal diameter; (ii.) a half-closed sluice-valve; (iii.) a conical nozzle on the bottom end of one-fifth the area of the pipe. The nozzle is 80 feet below the surface of the water in the reservoir. Find the discharge in cubic feet per second. *Ans.* 3·3.

38. A tank in the form of a frustum of a cone is 8 feet diameter at the top, 2 inches at the bottom, and 12 feet high. Find the time required to empty the tank through the bottom hole. $K = 0{\cdot}9$. *Ans.* 430 seconds.

39. A sluice ABC, in which AB is at right angles to BC, is pivoted about a horizontal axis passing through B. The sluice is subjected to water pressure on the inside. Find the force acting at C and making an angle of 45° with BC, required to keep it in equilibrium. AB = 1′ 2″, BC = 3′ 4″; AB makes an angle of 60° to the horizontal, and BC 30°, both sides being below the pivot. The width of the sluice is 6 feet, and the head of water above the hinge 42 feet. *Ans.* 14·8 tons.

40. A tank in the form of an inverted pyramid is provided with a square door in one of the sides, the sides of the tank make an angle of 60° to the horizontal. Find the force required to open the door by means of a vertical chain which is attached to the middle of the bottom edge. The door is hinged along the top horizontal edge.

Dimensions of door . . . . . 4 feet square.
Depth of submersion of hinge . . 15 feet.

Ans. 7·7 tons.

41. A reservoir having vertical sides is provided with an overflow rectangular notch or weir 3 feet wide; the area of the water surface is 284,000 square feet. After heavy rain the water rises to a height of 15 inches over the sill of the weir. Find the time required to lower the water to 2 inches. Coefficient of discharge = 0·62. *Ans.* 24·7 hours.

42. Water flows over a Vee notch in one of the vertical sides of a tank. The head over the apex of the notch is 1·28 feet when the supply to the tank is cut off. Find the time required to lower the head to 0·12 feet.

Area of water surface . . . . . 418 square feet.
Coefficient of discharge . . . . . 0·6

Ans. 42·2 minutes.

43. Find the loss of head and the horse-power required to force water through the tubes of a surface condenser. The water passes through half the number of tubes into a chamber, and from thence returns through the rest of the tubes. The ends of the tubes in all cases are flush with the tube-plates.

Number of tubes . . . . . 600.
Length of tubes . . . . . 8 feet.
Internal diameter of tubes . . . . . 0·75 inches.
Quantity of water passing per minute . . 500 gallons.

Choose your own values for the friction and the contraction coefficients. *Ans.* 0·07.

44. Two reservoirs with vertical walls are connected by means of a horizontal pipe having bell-mouthed ends. Find the time required to bring the water to a common level in each.

Area of reservoir A . . . . . 4000 square feet.
,, ,, B . . . . . 2500 ,, ,,
Level of water in A above datum . . 67 feet.
,, ,, B ,, ,, . . 42 ,,
Diameter of connecting main . . . . 2 feet 3 inches.
Length ,, ,, ,, ... 600 feet.

Ans. 1000 seconds.

45. Water flows down a steep slope. Find the velocity of flow when it reaches a point 60 feet below.

Width of channel . . . . . 15 feet.
Depth of water at start . . . . . 2 ,,
Initial velocity of water . . . . . 3 ,, per second.
Slope of bed, 1 vertical in 4 horizontal.
Coefficient of friction, 0·01.

You may neglect any variation in the hydraulic mean depth. It will be sufficient to calculate the velocity of flow at each 10 feet vertical fall. *Ans.* 56·3 feet per sec.

46. Water at a temperature of 80° F. flows through a pipe 0·5 inch diameter. Find the critical velocity when the velocity is gradually increasing. Find the quantity of water delivered per second when the head of water at the inlet is (i.) 2 inches, (ii.) 2 feet, (iii.) 20 feet above the outlet end. Neglecting the kinetic energy at discharge. Length of pipe 100 feet.

Ans. (i.) 0·00042 cubic feet per second, (ii.) 0·0013, (iii.) 0·0043.

47. Find the quantity of water flowing over a flat-topped weir, with no end contractions; the height of the approach stream is 14·2 inches over the sill. Length of weir 7 feet. *Ans.* 27·9 cubic feet per second.

48. Compare the time of lowering the level of the water in Question 19 with the time found by the approximate expression given on page 663, taking layers 6 inches in thickness and no water flowing into the tank.

49. A main 3 feet 6 inches in diameter and 8 miles in length conveys water to a city; the height of the water in the reservoir is 187 feet above

the outlet end of the pipe. The quantity of water delivered is insufficient for the requirements of the city; a second main of the same size is therefore laid alongside, starting from the city—it is coupled to the original main at a point 2 miles along by means of a suitable branch which offers no appreciable resistance. Find to what extent the branch main increases the flow. *Ans.* 11 per cent. increase.

Chapter XIX.

Hydraulic Motors and Machines.

1. Calculate the approximate hydraulic and total efficiencies of an overshot water-wheel required for a fall of 40 feet. If 30 cubic feet of water be delivered per second, find the useful horse-power of the wheel.

Ans. 78 per cent. hydraulic efficiency.
70 „ total „
95 H.P.

2. (Victoria, 1895.) Find the diameter of a ram necessary for an accumulator, loaded with 100 tons, in order that 50 H.P. may be transmitted from the accumulator through a pipe 2000 yards long and 4 inches in diameter with a loss of 2 H.P. *Ans.* 19·8 inches.

3. (Victoria, 1896.) The velocity of flow of water in a service pipe 48 feet long is 66 feet per second. If the stop-valve be closed so as to bring the water to rest uniformly in one-ninth of a second, find the (mean) increase of pressure near the valve, neglecting the resistance of the pipe. *Ans.* 384 lbs. square inch.

4. If the water in the last question had been brought suddenly to rest, *i.e.* in an infinitely small space of time, what would have been the resulting pressure. *Ans.* 4190 lbs. square inch.

5. A rotary motor is driven by water from a supply-pipe 200 feet in length. The diameter of the pipe is 3 inches, and the piston of the motor 4 inches; the stroke is 6 inches; length of connecting-rod, 1 foot; revolutions, 120 per minute. Calculate the inertia pressure at the end of the "out" stroke. *Ans.* 247 lbs. square inch.

6. Calculate the horse-power that can be obtained for one minute from an accumulator having a ram of 20 inches diameter, 23-feet stroke, loaded to a pressure of 750 lbs. per square inch. *Ans.* 164.

7. A hydraulic crane, having a velocity ratio of 8 to 1, is required to lift a load of 5 tons. Taking the efficiency of the chain gear at 80 per cent., and the loss of pressure by friction as 90 lbs. per square inch, find the size of ram required for a pressure in the mains of 700 lbs. per square inch. *Ans.* 15·3 inches.

8. Calculate the side pressure per square foot of projected area on the piers of a bridge standing in the middle of a river, velocity of stream 8 miles per hour, (i.) when the piers present a flat surface to the stream; (ii.) when they are chamfered off at an angle of 45°.

Ans. (i.) 134 lbs.; (ii.) 39 lbs.

9. A jet of water 1½ inch diameter, moving at 50 feet per second, impinges normally on a series of flat vanes moving at a velocity of 20 feet per second. Find the pressure exerted on the vanes. *Ans.* 35·7 lbs.

10. A jet of water, 2 inches in diameter, moving at a velocity of 60 feet per second, glides without shock on to a series of smooth curved vanes moving in a direction parallel to the jet with a velocity of 35 feet per

second. The last tip of the vane makes an angle of 60° with the first tip. Find the pressure exerted on the vanes. *Ans.* 31·7 lbs.

11. Find the total efficiency of a Pelton wheel working under the following conditions: Diameter of nozzle, 0·494 inch; diameter of brake wheel, 12 inches; net load on brake, 8·8 lbs.; revolutions per minute, 538; weight of water used per minute, 330 lbs. *Ans.* 66·4 per cent.

12. (I.C.E., February, 1898.) In an inward-flow turbine, the water enters the inlet circumference, 2 feet diameter, at 60 feet per second and at 10° to the tangent to the circumference. The velocity of flow through the wheel is 5 feet per second. The water leaves the inner circumference, 1 foot diameter, with a radial velocity of 5 feet per second. The peripheral velocity of the inlet surface of the wheel is 50 feet per second. Find the angles of the vanes (to the tangent) at the inlet and outlet surfaces.

Inlet, 49°; outlet, 11°.

13. A locomotive picks up water from a trough when running. Calculate the number of gallons of water delivered into the tender per second when running at 50 miles per hour, assuming that 40 per cent. of the head which would be available for delivery purposes is wasted in friction. The outlet of the delivery pipe is 11 feet above the surface of the water in the trough. Diameter of pipe 6 inches. At what speed will delivery commence?

Ans. 64·9 gallons per second. Delivery commences when the speed reaches 26·6 feet per second.

14. In a reaction turbine 10 per cent. of the head is wasted in friction in the guide blades, and 5 per cent. of the head is rejected in the discharge. The degree of reaction is 1·5. Design the blades for a turbine rod. The head of water at entry is 80 feet, and 4 cubic feet of water are supplied per second. Find (i.) the area of passages required; (ii.) the horse-power of the wheel, taking the mechanical efficiency apart from the hydraulic losses as 93 per cent.; (iii.) the speed of the wheel, the diameter of which is 4 feet.

Ans. Guide angle = 23·5°, angle of inlet tip of wheel blades = 35°, ditto outlet = 15°. (i.) 0·25 square feet, (ii.) 28·7, (iii.) 285 revolutions per minute.

15. The mean radius of the inlet passage of a parallel-flow impulse turbine wheel is 3 feet, the breadth of the passages is 8 inches, the pitch of the wheel blades is 6 inches and the thickness $\frac{1}{2}$ inch. The velocity of flow is $\frac{1}{8}\sqrt{2gH}$. The available head at the sluice is 200 feet. The loss of velocity due to friction in the guides and sluice is 5 per cent. Find (i.) the quantity of water passing through the turbine, (ii.) the hydraulic efficiency, (iii.) the gross horse-power, and (iv.) the speed.

Ans. (i.) 164 cubic feet per second, (ii.) efficiency 88 per cent., (iii.) horse-power 3300, (iv.) 173 revolutions per minute.

16. Find the maximum horse-power that can be transmitted through a water main 4 inches diameter, 2000 feet long; head 800 lbs. per square inch. *Ans.* 190.

17. Two cubic feet of water per second strike a fixed hollow cup; the jet is parallel to the inlet edge, which is at an angle of 45° to the horizontal, and the back of the outlet edge is at an angle of 30° to the horizontal. The speed of the jet is 50 feet per second. Find the pressure exerted (i.) in the initial direction of the jet, (ii.) horizontally, (iii.) vertically.

Ans. (i.) 245 lbs., (ii.) 305 lbs., (iii.) 40 lbs.

18. Water is supplied to a Pelton wheel from a reservoir situated 800 feet above the nozzle. The diameter of the pipe is 6 inches, the length 2810 feet, the diameter of the nozzle 1½ inches. Find the peripheral speed of the wheel to give the best results (ratio 2·2). If the efficiency of the wheel is 82 per cent., find the useful horse-power developed.

Ans. 84 feet-seconds; horse-power 112.

19. A jet of water plays on the apex of a stationary cone which causes the water to spread symmetrically. Find the axial pressure on the cone.

Diameter of jet	0·75 inch.
Quantity of water passing per second	18 lbs.
Apex angle	45°.

Ans. 4 lbs.

Chapter XX.

PUMPS.

1. Find the quantity of water delivered and the horse-power required to drive a single-acting pump working under the following conditions: diameter of pump-barrel, 2 feet; length of stroke, 6 feet; slip, 4 per cent.; head of water on pump, 50 feet, exclusive of friction; speed of flow in main, 3 feet per second; length of main, 1 mile; strokes of pump, 20 per minute; mechanical efficiency, 80 per cent.

Ans. 135,500 gallons per hour; 52·4 H.P.

2. Find the horse-power required to drive a feed-pump for supplying a 100 H.P. boiler working at 130 lbs. per square inch, reckoning 30 lbs. of water per horse-power hour. Mechanical efficiency of pump, 60 per cent.

Ans. 0·76.

3. (I.C.E., February, 1898.) A steam pump is to deliver 1000 gallons of water per minute against a pressure of 100 lbs. per square inch. Taking the efficiency of the pump to be 0·7, what indicated horse-power must be provided?

Ans. 100.

4. Find the speed of a centrifugal pump having radial vanes in order to lift the water to a height of 20 feet; the outside diameter of the vanes is 18 inches. Neglecting all sources of loss.

Ans. 470 revolutions per minute.

5. Find the speed of a centrifugal pump having curved vanes as in Fig. 726, with no volute, required to lift water to a height of 20 feet, the outside diameter of the vanes being 18 inches. $V_r = \frac{\sqrt{2gH}}{4}$, $\theta = 30°$, neglecting losses by friction.

Ans. 511 revolutions per minute.

6. What is the hydraulic efficiency, neglecting friction, of the pump in the last question?

Ans. 65 per cent.

7. In the case of the pump given in Question 5, calculate the horse-power absorbed by friction on the outside of the two discs. Taking the inner diameter as one-half the outer, and the resistance per square foot at 10 feet per second at 0·8 lb.

Ans. 1·3.

8. Find an expression for the gain in pressure which occurs in the whirlpool chamber of a centrifugal pump.

In a given instance the radial velocity of flow was 10 feet per second, and the velocity of whirl on leaving the impeller 65 feet per second.

Find the increase in pressure at the outer radius of the whirlpool chamber, which is 2·4 times the radius of the impeller. In the pressure conversion one-third of the energy is dissipated in eddies. *Ans.* 37 feet.

9. It is required to pump the water out of a circular brick-lined reservoir by means of a fixed pump on the bank; the pump delivers 10,000 gallons of water per minute. Find (*a*) the time taken to empty the pond, (*b*) the number of useful foot-tons of work done. The diameter of the reservoir at the bottom is 34 feet and 87 feet at the top; the depth of the water is 28 feet.

The delivery pipe of the pump is 37 feet above the bottom of the reservoir. *Ans.* (*a*) 53·4 minutes; (*b*) 48,700.

10. In a centrifugal pump the quantity Q delivered in cubic feet per second is given by the expression $Q = \frac{N}{90} - 9$, where N is the number of revolutions per minute. The pump has a suction head of 4 feet, the suction pipe is 6 inches diameter and 8 feet long, the frictional resistance of the foot valve is equal to the friction of 7 feet of pipe. Find the speed at which cavitation commences in the "eye" of the pump.

Ans. 1400 revolutions per minute.

11. A centrifugal pump is required to raise water through a total height of 300 feet. Find the relative power wasted in disc friction for (i.) a single stage, (ii.) a three-stage pump, the revolutions per minute being the same in both cases. *Ans.* Taking (i.) as unity, (ii.) 5·2.

12. Find the horse-power wasted in skin friction on the sides of a centrifugal pump runner 15 inches in diameter when lifting against a head of 170 feet. $V = 1{\cdot}24\sqrt{2gH}$. *Ans.* 24·9.

13. A pump runs steadily at 60 revolutions per minute. Diameter of plunger 8 inches, diameter of suction pipe 6 inches. Stroke 1 foot. Length of suction pipe 90 feet. Delivery pressure 20 lbs. per square inch. Find the coefficient of discharge when no cavitation takes place. (i.) For a long rod; (2) when the rod is 4 cranks long.

Ans. (i.) 1·15; (ii.) 1·18 by graphical solution.

14. Find the force in the piston rod of a double-acting pump when the piston is 6 inches from the middle of the stroke. Length of crank 9 inches. Speed 70 revolutions per minute. Long connecting rod. Water enters the pump under a head of 18 feet, and is discharged at a head of 60 feet above the pump. The suction pipe is short. Delivery pipe 113 feet long. Area of piston and main 18 square inches.

Ans. 1065 lbs., and − 408 lbs

INDEX OF SYMBOLS

The following symbols have been adopted throughout, except where specially noted :—

A, a (with or without suffixes), area ; *exceptions*, p. 15, A is used for angular acceleration ; A', see p. 546

B, used as a suffix in locomotive balancing problems, and always in connection with balance weights ; in hydraulics, breadth of weir

b. See p. 501

C, centrifugal force ; *exceptions*, p. 683, constant in Thrupp's formula ; compressive stress in "Strut" chapter ; see also pp. 657, 668, 676

C_a, chord of an arc

C_g. See p. 226

c, centres of locomotive cylinders, pp. 221–226 ; clearance in rivet-holes, pp. 407–420 ; clearance in cylinder, pp. 190, 191

$c.$ *of g.*, centre of gravity

$c.$ *of p.*, centre of pressure

D, diameter of ; coils (inches) of springs, pp. 586–592 ; pipes (feet) for all questions of flow of water ; shafts (inches) in torsion

d, diameter of pulleys in belting questions ; rivets, pipes (inches) ; least diameter of struts (same units as length) ; diameter of wire in helical springs (inches) ; bolt at bottom of thread, p. 384

dt, dv, etc. See Appendix

E, Young's Modulus of Elasticity

E_n. See p. 196

e, extension, percentage of, p. 383

F, force ; *exception*, Chap. IX., "Friction," it is used for frictional resistance

f, stress, intensity of ; *exception*, in friction of water problems. See p. 648 and 679

f_a, acceleration

f_c and f_t. See p. 422.

f_b, bearing pressure on rivets

f_r, tensile strength of rivets

f_s, shear stress, intensity of

f_t, tensile strength of plates in riveted joints, of walls in pipes ; *exception*, p. 400

f_{st}, resultant tensile stress due to shear and direct stress. See pp. 392, 584

f_{sc}, ditto, compressive

f_{sm}. See p. 395

G, coefficient of rigidity

G_a. See p. 549

g, acceleration of gravity

H, h, height in feet, head in hydraulics ; *exceptions*, in beam sections the height is in inches

h, loss of head in feet due to friction in pipes in all flow of water problems

H_o, depth of immersion of centre of pressure of an immersed area

H_o, depth of immersion of centre of gravity of an immersed area

$H.P.$, horse-power ; *exception*, p 192, high pressure

I, moment of inertia, or second moment, about an axis passing through the c. of g. of a surface
I_o, moment of inertia about an axis parallel to the above-mentioned axis
I_p, the polar moment of inertia, or the second polar moment
I.P., intermediate pressure (cylinder)

J, Joule's mechanical equivalent of heat, p. 13
j, a constant

K, coefficient of elasticity of volume; sometimes used instead of κ (kappa) for radius of gyration; *exceptions*, pp. 196, 240, 259, 298, 324, 408, 461, 664, 675, 681
K_a, coefficient of approach
K_c, ,, contraction, p. 644
K_d, ,, discharge, p. 644
K_v, ,, velocity, p. 644
K_r, ,, resistance, p. 645

L, or l, length; *exception*, p. 383
L_c. See p. 509

M, bending moment in beam questions; *exceptions*, p. 383, mass in dynamic questions; see p. 603 for rolling loads
M_o, momentum
M_t, twisting moment
M_{cc}. See p. 583
M_u. See p. 585
m. See pp. 196, 233, 307

N, revolutions per minute, often with suffixes, which are explained where used; *exceptions*, pp. 324, 387, 663
N_s, revolutions per second
N.A., neutral axis
n, ratio of length of connecting-rod to radius of crank in engine and pump problems (see p. 586 for helical springs, p. 603 for rolling loads, p. 659 for weir problems, p. 668 for Venturi water-meter, also see pp. 648, 676, 687, 688, 703); ratio of velocity of the jet to that of the surface, in the pressure of jets impinging on moving surfaces

O.H., polar distance of a vector polygon

P, pressure; buckling stress in lbs. per square inch in struts, see p. 556; *exception*, see special meaning in "Friction" chapter
p, acceleration pressure (see p. 182) in lbs. per square inch for engine balancing problems
P_e, mean effective pressure on a piston in lbs. per square inch

Q, quantity of water flowing in cubic feet per second

R, radius (feet); hydraulic mean radius (feet) in flow of water; *exception*, see p. 561
r. See p. 559
R_1, R_2, reactions of beam supports; *exception*, rise of arch, p. 624
R_B, radius of c. of g. of balance weight
R_c, radius of coupling crank of locomotive
R_o, perpendicular distance between the two parallel axes in moment of inertia, or second moment problems
r_c, radius of crank shaft journal
r_g, radius of gudgeon pin
r_p, radius of crank pin

S, stress; *exceptions*, side of square shaft (inches), p. 578; span of arch (feet), pp. 619–627; speed of shearing, pp. 314, 675; wetted surface in square feet, p. 679
s, space

T, t, time in seconds; *exceptions*, thickness of pipe (inches), p. 700; tension modulus of rupture, p. 562; tension in belts, pp. 351–358; thickness of plate in riveted joints, girder webs, etc.
T_a, T_b, etc., number of teeth in wheels a, b, etc., respectively

u. See p. 538

V, v, velocity (feet per second);

exceptions (feet per minute), p. 353, (miles per hour), pp. 593, 594
V_h and V_v, used for pressure turbines. See p. 730, Fig. 690
V_w, velocity of rim of wheel or belt in feet per second
v_r, velocity ratio
V_r, velocity of rejection in turbines. See p. 725

W, weight in pounds
W_B, W_ρ, W_r. See p. 218
W_c, W_{Bc}. See p. 224
W_r, weight of 1 foot of material 1 sq. inch in section, pp. 202, 354
W_w, weight of 1 cubic foot of water in lbs.
w, weight per square inch; *exceptions*, width of belt (inches), p. 353; unit strip in riveted joints, pp. 410–418; weight of unit column of water, *i.e.* 1 foot high, 1 sq. inch section in hydraulics

x, extension or compression of a body under strain; *exceptions*, eccentricity of loading in combined bending and direct stresses, p. 538; in dynamics of steam engine and vibration, see pp. 179, 259

Y. See p. 461
y, distance of most strained skin from the neutral axis in a beam section. See also p. 544

Z, modulus of the section of a beam, *i.e.* $\frac{\text{moment of inertia}}{y}$
Z_p, polar modulus of the section of a shaft

GREEK ALPHABET USED.

α (*alpha*), angle embraced by belt, 352; a constant in the strut formula, 558
δ (*delta*), deflection; *exception*, p. 306; in mathematical work, a small increment in the variable
Δ (*delta*). See p. 463
η (*eta*), efficiency in every case
θ (*theta*), angle, subtending arc, 22. 29; of balance weight, 222; between two successive tangents on a bent beam, 508; between the line of force and the direction of sliding in Chap. IX., *exception*, p. 299; subtended by bent bar, p. 544
θ_c, angle expressed in circular measure
κ (*kappa*), radius of gyration
μ (*mu*), coefficient of friction
π (*pi*), ratio of circumference to diameter of circle
ρ (*rho*), radius
Σ (*sigma*), symbol of summation
ϕ (*phi*), friction angle
ω (*omega*), angular velocity
Ω (*omega*), angular velocity

GENERAL INDEX.

Acceleration, def., 5 ; cam for constant, 149 ; curves, 140–142 ; of cam, 159 ; reciprocating parts, 181–188 ; vibrating bodies, 259 ; on inclined plane, 286 ; pressure due to, 180–191
Accumulator, 697
Accuracy, 790 ; of beam theory, 791
Addendum Circle (Wheel Teeth), 161
Air, in motion, 501 ; vessels, 743 ; weight, 593
Aluminium, strength, 371, 372, 382, 385, 391, 428
Approach, of balls in bearing, 306 ; velocity of, 658
Andrews, reference to hook theory, 547
Angle of friction, 285 ; repose, 285, sections, 446 ; of twist in shafts, 579
Angular velocity (ω), def., 4 ; bars in mechanism, 130, 135
Anticlastic curvature, 401
Antifriction metals, 326 ; wheels, 299
Arc of circle, 22 ; c. of g., 64, 66 ; ellipse, 22 ; parabola, 24
Arch, line of thrust, 618 ; masonry, 614 ; ribs, 619 ; temperature effects, 624
Area, bearings, 328 ; circle, 28 ; cone, 36 ; ellipse, 30 ; hyperbola, 38 ; irregular figures, 26–32 ; parallelogram, 24 ; parabolic segment, 32 ; spherical dent, 34 ; sphere, 36 ; trapezium, 26 ; triangle, 24–26
Arms, flywheels, 205 ; governor, 240 ; oscillation of, 264
Asphalte, 386
Assumptions of beam theory, 433
Astronomical clock governor, 238
Axis, neutral (N.A.), 437 ; reinforced concrete beams, 469 ; of rotation, 211
Axle, balancing, 211 ; friction, Chap. IX.
Axode, 122, 123

Bairstow, reference to, 636
Baker, Prof. I. O., reference to, 618
Balancing, axles, 211–218 ; locomotives, 218–230 ; weights, 227
Ball bearings, 302–310, 348, 359 ; approach of balls in, 306 ; cost of, 309 ; distribution of load on, 307 ; friction of, 308 ; safe load and speed, 309
Bar, curved, 544 ; moment of inertia, 98
Barker, A. H., reference to, 3, 18, 74, 142, 220
Barlow, theory of thick cylinders, 421, 424, 425
Barr, Prof. A., reference to, 599
Beams, Chap. XII. ; angles, 446 ; bending moments, 474–505 ; box, 444 ; breaking load, 469 ; built in, 504, 524–531 ; cast iron, 441 ; channel, 446 ; circular, 452 : concrete, 469 ; continuous, 531–535 ; curved, 544 ; deflection, 506–531 ; elastic limit, 467 ; experimental proof of theory, Appendix, 791 ; I, 444 ; inclined, 548 ; irregular loads, 496–501, 529 ; irregular sections, 454 ; model, 430 ; plate, 608 ; propped, 518, 531–535 ; rails, 454 ; shear, 457, 474–503 ; stiffness, 535 ; unsymmetrical sections, 439.

Bearings, area, 328 ; ball, 302–310, 348, 359 ; collar, 303, 324, 330 ; conical, 330 ; lubrication, 320–325 ; onion, 332 ; pivot, 329 ; roller, 300 ; seizing, 325 ; Schiele, 332 ; wear, 323–329 ; work absorbed, 329–333.
Bell, reference to, 619
Bell crank lever, 56
Belts, centrifugal action on, 354 ; creeping, 356 ; friction, 351–353 ; strength, 353 ; tension on, 354 ; transmission of power by, 352–355.
Bending moment, Chap. XII. ; on arched ribs, 622 ; combined with twisting moment, 582 ; on coupling and connecting-rods, 207–209 ; of curved bar, 544
Bends in pipes, 689
Benjamin, Prof., reference to, 205
Bevil train, epicyclic, 174
Boiler, riveted joints for, 407–420 ; shell, 402, 420
Braced structures, deflection of, 606
Bracing, Chap. XVII.
Brass, strength, 428
Bridges, Chap. XVII. ; arch, 614–627 ; floors, 444, 452, 455 ; loads, 602 ; plate girder, 608 ; rolling load, 603 ; suspension, 108
Brittleness, 362, 386, 392
Broad-crested weir, 659
Bronze, strength, 428
Brownlee, reference to, 653
Buckling of struts, Chap. XV.
Bulb section, 455
Bull metal, 372, 428
Burr and Falk, reference to, 606
Bursting pressure, 402, 420-426 ; of flywheels, 201–209

Cams, 146–160 ; acceleration of, 159 ; constant acceleration, 149 ; constant velocity, 146 ; grooved, 158 ; simple harmonic, 151 ; size of, 152 ; velocity ratio, 154
Cantilever, bending moment on, 475, 484–487, 504 ; deflection, 507–514, 518 ; stiffness, 535
Cast-iron beams (see Beams) ; columns, 562–570 ; in compression, 384, 385, 427 ; strength, 405, 427 ; strain, 364
Catenary, 109
Cavitation, 747, 761, 773
Cement, Portland, 387
Centre of gravity (c. of g.), arc of circle, 64, 66 ; balance weights, 227 ; cone, 72 ; definition, 17, 58 ; irregular figures, 62–70 ; locomotive, 74 ; parabolic segments, 66, 68, 70 ; parallelogram, 60 ; pyramid, 72 ; trapezium, 60 ; triangle, 60 ; wedge, 72
Centre of pressure, 640-643
Centre, virtual, 122, 126
Centrifugal force, action on belts, 354 ; definition, 17 ; on flywheel rims, 201 ; on governor balls, 231 ; on governor arms, 241 ; reciprocating parts, 181
Centrifugal head, 733, 763
Centrifugal pump, 758–779
Centrode, 123, 124
Chain, pump, 738 ; stress in suspension, 109
Change speed gears, 171
Channel, friction of water in, 684 ; strength as beam, 446
Charnock, Prof., reference to, 596
Chatley, Prof., reference to, 596
Checking results, Appendix, 790
Circle, arc, 22 ; area, 28 ; circumference, 22 ; moment of inertia, 88, 90, 96, 98 ; strength as beam section, 452 ; strength as shaft section, 576
Clearance (wheel teeth), 161
Coefficient of friction, 284, 286, 288, 301, 302, 313–318, 321, 328, 353, 359, 679
Coil friction, 351
Collar bearing, 303, 324, 330
Columns, Chap. XV. ; eccentrically loaded, 568–570
Combined, bending and direct stresses, Chap. XIV. ; shear or torsion and direct stresses, 392, 581
Compound, pendulum, 261 ; stresses, 595
Compression, cement, 387 ; strength of materials in, 384–388, 427–428 (see also struts, Chap. XV.) ; of water, 639
Concrete, reinforced, 469
Cone, moment of inertia of 104 ; surface of, 36 ; volume, 46

Connecting-rods, bending stresses in, 209; influence of short, 183–187; virtual centre, 126, 133–138
Conservation of energy, 13
Constrained motion, 119
Continuous beams, 531–535; flow of water, 675
Contraction of streams, 644–651; 655–660
Copper, extension, 368; fracture, 367; strength, 427; stress-strain diagram, 364, 371, 372, 385
Corrugated floors, 444, 452
Cost of ball bearings, 369; struts, 566
Couples, 11
Coupling-rods, stress in, 207
Crane, forces in, 111, 112; hook, 547
Crank, and connecting-rod, 126, 133–136; pin, 189; shafts, 580; shaft governor, 243, 254; webs, 230
Cranked tie-bar, 542
Creeping of belts, 356
Critical velocity of water, 676
Crossed-arm governor, 237, 252
Crushing. See Compression
Cunningham, reference to, 636
Cup leathers, friction of, 334
Curved bar, 544
Cycloid, 163
Cylinder, moment of inertia, 98, 100; strength under pressure, 402, 420–426; volume of, 40

Dalby, Prof., reference to, 218, 220
D'Arcy's formula for flow of water, 681
D'Auria pump, 752–753
Davey pump, 756, 757
Deflection of beams, Chap. XIII.; braced structures, 606; due to shear, 461; helical springs, 585–592; table of (beams), 523
De Laval steam turbine governor, 243; centrifugal pump, 771
Delta metal, 372, 428
Diagrams, bending moment and shear, Chap. XII.; indicator, 187, 746; stress-strains, 364–389; twisting moment, 191–194
Differential calculus. See Appendix, 782
Dimensions, 2, 20; importance of checking, Appendix, 781
Dines, reference to, 594
Discharge of pipes, etc., 675–690; weirs, Chap. XVIII.
Discrepancies, beam theory, 466–469; strut theory, 551
Distribution of load on roofs, 597
Doble, water-wheel, 714
Ductile materials, compression, 384; shear, 391; tension, 365–372
Dunkerley, Prof., reference to, 267, 269, 270, 274
Durley, R. J., reference to, 142

Eccentric loading, 542, 568–570
Eden, reference to, 636
Efficiency. Bolt and nut, 297; centrifugal pump, 761–779; definition, 13; hydraulic motors, Chap. XIX.; inclined plane, 293; machines, 334–351; pulleys, 336–340; riveted joints, 416; reversed, 335; screws and worms, 295, 343; steam engine, 349; table of, 350
Elasticity, belts, 356; definition, 362; modulus of, 372–376; table of, 427–428; transverse, 376–377, 399; of volume, 404
Elastic limit, artificial raising, 372; definition, 362
Ellipse, area, 30; moment of inertia, 90, 92
Energy, definition, 13; kinetic, 14; loss due to shock, 671
Engine, dynamics of, Chap. VI.
"*Engineer*" Journal, reference to, 201
"*Engineering*" Journal, reference to, 185, 310, 343, 608, 657, 678, 758, 779
Epicyclic trains, 172–178; bevil trains, 174
Epicycloid, 163
Equivalent, mechanical, of heat, 13
Euler strut formula, 553
Extension of test pieces, 366
Extensometer, 363

Factor of safety, struts, 565
Fairburn, Sir W., reference to, 613
Falling weight, stress due to, 630

Flange of girder, 610
Flat plate, 405
Flexure, points of contrary, 525
Flooring, strength of, 444, 452, 455, 456
Floor, reinforced concrete, 472
Flow of water, 675–690; broad-crested weir, 659; continuous, 675; critical velocity, 676; D'Arcy's formula, 681; down open channel on steep slope, 686; through pipe with restricted outlet, 688; Thrupp's formula, 683; weirs and orifices, 644–655
Fluctuation of energy, 195
Flywheel, 194; arms, 205; bursting speed, 205; gas engine, 199; governor, 243; moment of inertia, 100, 102; shearing and punching machines, 199; stress in rims and arms, 201–207; weight, 196; work stored in, 198
Focal distance, 512
Foot-pound, 11; poundal, 10; step, 303
Force, centrifugal, 17; definition, 7; gravity, 9; polygon, 106; pump, 740; in structures, Chaps. IV., XVII.; triangle, 16
Fractures of metals, 367
Framed structures, Chap. XVII.
Friction, Chap. IX.; angle, 285; on axle, 329; ball bearings, 302–310, 348, 359; belts, 351–353; bends, 689; bolt and nut, 297; body on horizontal plane, 284; ditto inclined, 285–294; coefficient, 284, 286, 288, 301, 302; 313–318, 321, 328, 353, 359, 679; coil, 351; cone, 285; cup leather, 334; dry surfaces, 286; gearing, 343; governors, 250; levers, 342; lubricated surfaces, 310–321; pivots, 329–333; pulleys, 336–340; rolling, 297–302; roller bearings, 300; slides and shafting, 344–349; temperature effects, 316; time effects, 319; toothed gearing, 342; velocity, effect of, 316; water, 679; wedge, 294; worms and screws, 295, 343
Froude, reference to, 667, 679
Funicular polygon, 107, 108
Galton, experiments on friction, 287
Gas-engine flywheels, 179
Gearing, friction, 342; toothed, 160–178
Gears, change speed, 151
Gibson, Prof., reference to, 669, 702
Girders, continuous, 531; modulus of section, 444; plate, 608; triangulated, 645; Warren, 607
Gordon, strut formula, 557–563
Governing of water motors, 716–720
Governors, 230; astronomical, 238; crankshaft, 243; dashpot for, 244, 256; friction, 250; Hartnell, 239, 252–254; inertia effects, 245; isochronous, 236; loaded, 232; McLaren, 243, 254; oscillations of arms, 264; parabolic, 237; Porter, 235, 251; power of, 257; sensitiveness, 247; Watt, 231
Gravity, cam, 149; centre of (c. of g.) (see Centre); acceleration of, 9
Grover, F., reference to, 160
Guest, shaft theory, 585
Guldinus, theorems of, 36, 42
Gun-metal, 364, 368, 371, 385, 427
Gyration, radius of, bar, 98; circle, 88, 90, 96, 98; cone, 104; cylinder, 98, 100; definition, 15; ellipse, 90, 92; flywheel, 100, 102; graphic method, 96; irregular surface, 94; parabola, 92, 94; parallelogram, 78–82, 96; sphere, 102, 104; square, 88; trapezium, 84, 86
Gyroscope, 277; precession of, 278–280
Gyroscopic action, Chap. VII., p. 277; on locomotive, 282; on motor car, 283

Hartnell, governor, 239, 252–254; springs, 587
Hartnell-McLaren governor, 243
Harmonic, cam, 151; motion, simple, 259
Hastie engine, 695
Head of water, eddies, 671; energy, 666; friction, 675–688; effect of sudden change, 671, 672;

pressure, 638; velocity due to, 643, 654
Heerwagen, reference to, 306
Height of governor, 231
Heiman, reference to, 313
Hele-Shaw, reference to, 169, 674
Helical springs, 585–592; open coiled, 590; in torsion, 589
Hemp ropes, strength, 359
Hertz, reference to, 306
Hicks, Prof., reference to, 10, 12
Hill, reference to paper on locomotives, 220
Hobson's patent flooring, 455
Hodgkinson, beam section, 441
Hodograph, 17
Hoffmann, ball bearing, 305; roller bearing, 301
Holes, rivet, 408
Hooks, 547
Horse-power (H.P.), 11; of belts, 353; of shafts, 580
Humpage, gear, 176
Husband and Harby, reference to, 596, 606
Hyatt roller bearing, 301
Hydraulics. See Chaps. XVIII., XIX., XX.
Hyperbola, area, 38
Hypocycloid, 163

Impulse, definition, 7; strokes stored in flywheel, 199
Inclined beam, 548; planes, 285–294
Incomplete structures, 608
Indiarubber, 405; tyres, 298; beam, 429
Indicator diagram, correction for inertia of reciprocating parts, 179–189
Inertia, definition, 14; effects on governor, 245; water pressure, due to, 698; of reciprocating parts, 179–189
Inertia, moment of, 14, 75–105; angles, 447; bar, 98; beam sections, 432; channel, 447; circle, 88, 96, 98, 453; cone, 104; cylinder, 98–100; ellipse, 90–92; flooring, 445, 453, 455, 456; flywheel, 100, 102; girders, 445; H section, 445; hydraulic-press table, 456; irregular surface, 94; joists, 445; parabola, 92–94; parallelogram, 78–82, 96; rails, 454; rectangle, 443; sphere, 102–104; square, 88, 447; tees, 447; trapezium, 84, 86, 451; triangle, 82, 84, 451
Influence lines, 606
Injector, hydraulic, 654
Inside cylinder locomotive, 220
Involute, 168
Iron, angle, 446; cast, 364, 384, 385, 405, 427; ditto beams, 441; columns and struts, 562–570; modulus of elasticity, 427; riveted joints, 407–420; strength, 427; stress-strain diagrams, 364, 371, 385; weight, 48; wrought, 391, 405
Isochronous governor, 236

Jet, pressure due to, 706
Joints (see Riveted joints), 407–420; in masonry arches, 617
Joist, rolled, 444
Journal, friction of, 329

Kennedy, Sir Alex., reference to, 119, 343, 383
Kernot, Prof., reference to, 594
Kinetic energy, 14
Knees, resistance of, in water pipes, 689

Lamé's theory of thick cylinders, 423
Lap joint. See Riveted joints.
Lattice girders. See Structures, Chap. XVII.
Leather, belts, 351–358; cup, friction of, 334; friction on iron, 287
Levers, 52–56; friction, 342
Lift, hydraulic, 694
Limit of elasticity, 362; table of, 427, 428
Link-motion, 145
Link polygon, 107
Load, beams, see Chap. XII.; bridges, 602; distribution on balls in bearing, 307; live, 627; rolling, 603; roofs, 596; safe for ball bearing, 309
Locomotives, coupling-rods, 207; balancing, 218–230; centre of

gravity, 74; gyroscopic action on, 282
Lodge, Mensuration, reference to, 23, 29
Longmans, Mensuration, reference to, 27
Lubrication, 310–325

Machine, definition, 119; efficiency of, 334; frames, 548; hydraulic, Chap. XIX.
Martin, H. M., reference to, 594, 608, 618
Masonry arches, 614–619
Mass, 9
Materials, strength of, 427, 428
Mather and Platt, centrifugal pump, 774–778
McLaren, engine, 192; governor, 243, 254
Mechanisms, Chap. V.
Mensuration, Chap. II.
Metals, strength, 427–428; weight, 48
Meter, Venturi water, 668
Metric equivalents, Chap. I.
Modulus, of beam sections, 433; circle, 453; flooring, 445, 453, 455; girder, 445; graphic solution, 434; press table, 456; rails, 454; rectangular, 443; square on edge, 447; tees, angles, 447; trapezium, 451; triangle, 451; unsymmetrical, 439; of shaft sections, 578; of elasticity, 372–377, 404, 427; of belts, 356
Moment of inertia, or second moment. See Inertia
Moments, 12, Chap. III.; bending, Chap. X.; twisting, 191, 194
Momentum, 7; moment of, 761
Morin, laws of friction, 286
Morley, A., reference to, 547, 636
Morris pulley block, 339
Motion, constrained and free, 119; plane, 120; screw, spheric and relative, 120
Motor cars, gyroscopic action on, 283
Mouthpieces, hydraulic, 649
Moving loads, 603, 627
Musgrave engine, 144

Notch, rectangular, 655; V, 656
Nut, friction of, 297

Oak, strength, 428; strut, 561, 567
Obliquity of connecting-rod, 183
Oil as lubricant, 310–328
Onion bearing, 332
Orifices, flow of water through, 644–654
Oscillating cylinder, 136, 693
Oscillation, Chap. VII., of governor arms, 264; of springs, 262; torsional, of shafts, 265
Outside cylinder locomotive, 223–225
Overlap riveted joints, 410

Pappus, surfaces, 36; volumes, 42
Parabola, area, 30; c. of g., 66–70; chain, 109; governor, 237; moment of inertia, 92–94
Paraboloid, volume, 46
Parallelogram, area, 24; c. of g., 60; forces, 16; moment of inertia, 78–82, 96
Parsons, pump, 758
Pearson, Prof. K., reference to, 7, 547, 557
Pelton wheel, 713
Pendulum, compound, 261; simple, 261
Permanent set, 362
Perry, reference to book on Calculus, 3, and Appendix, 782
Petroff, Prof., reference to, 313
Phosphor bronze, 428
Pillars. See Struts
Pipes, bursting, 420; flow of water in, 675–690; pressure due to water-ram, 699, 750
Pitch, circles, 161–169; rivets, 407–418
Pivots, conical, 330; flat, 329; Schiele, 332
Plane, inclined, 285–294; motion, 120
Plasticity, 362
Plate girders, 608–614
Plate springs, 519
Plates, strength of, and strain energy in, 405
Poisson's ratio, 396
Polygon forces, 106, 107; c. of g. of arc, 64
Poneclot water wheel, 720
Porter governor, 235, 251
Power, 11; transmission, by shafting, 580; water-main, 703

Precession of gyroscope, 278; rate of, 280
Pressure, acceleration, 179; bearings, 329; bursting, 402, 420–426; centre of, 640; energy, 666; friction, effect of, Chap. IX.; inertia, 180; jets, 706–712; water, 639; wind, 593
Prism, volume, 40
Pulleys, friction, 336–341; Morris block, 339
Pumps, Chap. XX.
Punched holes, 420
Punching-machine, flywheel, 199; frame, 548
Pyramid, c. of g., 72; volume, 48

Quick-return mechanism, 136

Radius of gyration (κ), 15. See Gyration
Rail sections, 454
Rankine, reference to, 25, 585; strut formula, 561
Reciprocating parts of engines, 179–191, 219
Rectangle, area, 24; beam, 443; c. of g., 60; moment of inertia and radius of gyration, 78, 82, 96; shaft section, 578
Reduction in area, 367
Redundant structures, 608
Reinforced concrete beams, 469
Relative motion, 120
Rennie experiments on friction, 287
Renold, Hans, reference to, 358
Repetition of stress, 627–636
Repose, angle of, 285
Resistance, bends, knees, etc., 690; friction, Chap. IX.; rolling, 297
Resolution, 16, 106
Reynolds, Prof. Osborne, reference to, 13, 298, 313, 314, 321, 668, 675
Ribs. See Arch
Rigg, hydraulic engine, 138
Ring, volume of anchor, 48
Riveted joints, 407–420, 610, 611
Roller bearings, 300; Hoffmann, 301; Hyatt, 301
Rolling load on bridges, 603
Roofs, 116; 596–601
Root circle (wheel teeth), 161
Ropes, transmission of power by, 355, 358
Rose, reference to, 636
Rotating parts of locomotives, 218
Running balance, 212

Schiele pivot, 332
Scoop wheel, 739
Screw, friction, 295, 343; motion, 120
Second moments, 52. See Moments of inertia
Seizing of bearings, 325
Sensitiveness of governors, 247
Shaft, friction, 344; efficiency of, 344–348; horse-power, 580; oscillation of (torsional), 265; spring of, 579; torsion, 573–585; whirling of, 267
Sharpe, Prof., reference to, 205
Shear, Chap. XII.; beams, 457–466, 474; diagrams, 478–503; deflection of beams due to, 461–466; and direct stress, 392, 581; modulus of elasticity in, 376, 398, 427–428; nature of, 389; plate girders, 609; rivets, 409; shafts, 573; strength, 391
Shearing machine, flywheel, 199; frame, 548
Sheer legs, 113
Shock, loss of energy due to, 671; pressure due to, 699
Simple harmonic motion, 259; ditto cam, 151
Simpson's rule for areas, 34; volumes, 42
Skefco ball bearing, 306
Slides, friction, 344
Slip of pumps, 744–749
Slope of beams, 508
Smith, Prof. R. H., reference to, 142, and Appendix, 782
Sommerfeld, reference to, 313, 314
Speed, 2; change gears, 171; flywheel rims, 202; pumps, 743; safe for ball bearing, 309
Sphere, moment of inertia, 102–104; surface, 36; thin, subject to internal pressure, 412; volume, 44
Spheric motion, 120
Spring, controlled governor arms, 264; helical, 585–592; oscillation of, 262; plate, 519; safe load, 587, 589; square section, 589; in torsion, 589; work stored in, 588; weight, 588

Square, beam section, 442, 446; moment of inertia, 88; shaft, 578; spring section, 589
Standing balance, 211
Stannah pendulum pump, 138
Stanton, Dr., reference to, 594, 636
Steam-engine, balancing, 215–230; connecting-rod, 181–187, 209; crank shafts, 581; crank pin, 134, 189; friction, 349; governors, 230–258; mechanism, 133; reciprocating parts, 179–194; twisting moment diagrams, 191
Steel, effect of carbon, 371; strength, etc., 427; wire, 377–382
Stiffeners, plate girders, 611–614
Stiffness of girders, 535
Strain, 340, 396–405
Stream line theory, 666
Strength of materials, 427–428
Stress, compound, 395; coupling rods, 207; definition, 360; diagrams, 364–389; due to falling weight, 630; flywheel arms, 205; flywheel rims, 201; oblique, 387; real and nominal, 369, 388
Stribeck, reference to, 307, 313
Stroud, Prof. W., reference to, 336
Structures, arched, 614–627; dead loads on, 604; forces in, 597–608; girders (plate), 608: incomplete, 608; redundant, 608; rolling loads, 603; weight, 614; wind pressure, 593–596
Struts, bending of, 551; eccentric loading, 568–570; end holding, 555; Euler's formula, 553; Gordon's formula, 557–563; Rankine's formula, 561; straight line formula, 565–567; table, 564
Sudden loads, 627, 630
Suspension bridge, 108

Table, strength of, 456
Tee section, 446, 559
Teeth of wheels, 160; friction of, 342
Temperature, effect on arched ribs, 624; effect on friction, 316; water, 637
Tensile strength, see table, 427, 428
Tension, 361; and bending, 538; vibration of rod, 265
Thomson, Prof. J., reference to, 656
Thorpe, R. H., reference to, 654, 696, 753
Three-legs, 114
Thrupp, formula for flow of water, 683; reference to, 677
Thrust bearing, 303, 304, 321, 324, 330–333
Thurston, reference to, 318
Tie-bar, cranked, 542
Time, to empty tanks, 660–665; of vibration, 260
Todhunter and Pearson, reference to, 557
Toothed gearing, 160–178; friction, 342
Torsion, Chap. XVII.; and bending, 581–585; helical spring, 589
Torsional oscillation of shafts, 265
Tower, B., experiments on friction, 317, 320, 321
Trapezium, area, 26; c. of g., 60–62; modulus of section, 86, 451
Trantwine, reference to, 616
Triangle, area, 24–26; c. of g., 60; moment of inertia, 82–84, 451
Tripod, 114
Trough flooring, 444
Turbines, 722–737
Turner and Brightmore, reference to, 679
Twisting-moment diagram, 191; shafts, 481

Units, 1, 19, and Appendix, 781
Unwin, Prof., reference to, 109, 203, 595, 636

Valve in pipe, pressure due to closing, 702
Vector polygon, 108
Velocity, approach, 658; critical, of water, 676; points in machines, 127; ratio of cam, 159; ratio of wheel trains, 161, 169; virtual, 121; water machines, 704; water in pipes, 675–690
Venturi water meter, 668
Vibration, Chap. VII.; governor arm, 264; shafts (torsional), 265; springs, 262; tension rod, 265; time of, 260
Virtual centre, 122; length of beams, 525; of struts, 555; slope, 684; velocity, 127
Volume, cone, 46; cylinder, 40;

definition, 20; paraboloid, 46; prism, 40; pyramid, 48; ring, 48; Simpson's method, 42; slice of sphere, 44; solid of revolution, 42; sphere, 44; tapered body, 48
Vortex, forced, 670; free, 669; turbine, 734

Walmisley, reference to, 596
Warner and Swazey governor, 238
Warren girder, 607
Warren, reference to, 606
Water, Brownlee's experiments, 653; centre of pressure, 640; compressibility, 639; flow due to head, 643; flow over notches, 655; flow under constant head, 658; friction, 675–690; hammer, 750; inertia, 698; injector, 654; jets, 706–712; motors, Chap. XIX.; orifices, 644–651; pipes of variable section, 665; power through main, 703; pressure, 637; ditto machines, 693; velocity machines, 704–735; weight, 637; wheels, 691–693, 720
Watt governor, 231
Wear of bearings, 323, 326
Webs of plate girder, 608–614
Wedge friction, 294
Weight, 9; materials, 48
Weir, broad crested, 659
Westinghouse, experiments on friction, 287
Wheels, anti-friction, 299; rolling resistance, 297
Whirling of shafts, 267
White metal for bearings, 326
Wicksteed, J. H., reference to, 366
Williott diagram, 606
Wilson-Hartnell governor, 239, 252
Wind, 593; on roofs, 597
Wire, 377–382
Wöhler experiments, 631–636
Work, 10, 11; in fracturing a bar, 382; stored in flywheel, 198; governors, 257; springs, 588
Worms, friction, 343
Worthington pump, 755

Young's Modulus of Elasticity, 372

THE END.

PRINTED IN GREAT BRITAIN BY WILLIAM CLOWES AND SONS, LIMITED, BECCLES.